ON FINITE GROUP ACTIONS ON THE
SOLID KLEIN BOTTLE

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ABSTRACT

In this paper we classify all G-actions on the solid Klein bottle when G = \(\mathbb{Z}_n\) and when G = \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\).

Let G be a group and M a topological space. An action of G on M is a map \(\Theta: G \times M \rightarrow M\) such that (i) \(\Theta(g, \Theta((h,x))) = \Theta((gh,x))\) for all \(g, h \in G\) and \(x \in M\), and (ii) \(\Theta((e,x)) = x\) for all \(x \in M\), where e is the identity of G. \(\Theta(g,x)\) is denoted by \(g(x)\). The action \(\Theta\) is called effective if it is injective. Two G-actions \(\Theta\) on M and \(\Phi\) on N are weakly conjugate if there exists a group automorphism \(A: G \rightarrow G\) and a homeomorphism \(t: M \rightarrow N\) (called the connected homeomorphism) such that \(t \Theta((g,x)) = \Phi(A(t(x)))\), i.e. \(tg(x) = A(g)(t(x))\). If \(A(g) = g\), then \(\Theta\) and \(\Phi\) are conjugate.

In this paper we consider the classification of the G-actions on the solid Klein bottle \(SK\). We give complete classifications when G = \(\mathbb{Z}_n\), the finite cyclic group, and when G = \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\). We extend the results of Natsheh (4).

Throughout the paper we work in the PL category (our results are valid for Diff-category without any changes). We divide the paper into three sections. In section 1 we prove theorem 1, the product theorem and state theorem 2, the involutions on SK. In section 2, we classify all \(\mathbb{Z}_n\)-actions on SK, up to weak conjugation. In section 3, we classify the \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\)-actions on SK.

Let G be an Abelian group acting effectively on a connected space M. Let \(g, h \in G\) and \(q:M \rightarrow M/g\) be the orbit map induced by g. Then there exists a homeomorphism \(h^*\) on \(M/g\) uniquely determined by h such that \(h^* = qq^*\). \(h^*\) is called the action on \(M/g\) induced by h.
Throughout the paper \( S^n, D^n, \) and \( P^n \) denote the \( n \)-sphere, the \( n \)-cell and the \( n \)-dimensional projective plane, respectively. \( M_b \) denotes a Mobius band. \( C(X) \) denotes the cone over the space \( X \). \( S^1 \) is viewed as the set of complex numbers \( x \) with norm 1. The closed unit interval is denoted by \( I \). \( T = S^1 \times S^1 \).

\[
D^2 = \{ rx : 0 \leq r \leq 1, \, x \in S^1 \}
\]

\[
SK = \mathbb{R}xD^2/\sim, \quad (s, rx) \sim (s+1, C(rx)), \text{ where } C(rx) = r\bar{x}.
\]

Section 1.

In this section we make use of recent results of Dunwoody (1) and Meeks and Scott (3); Moreover we write down theorem 2 which was proved in (4).

Theorem 1. Let \( G \) be a finite group acting effectively on the solid Klein bottle \( SK \). Then the action is conjugate to an action which preserves the product structure, i.e. for every \( g \in G \) \( g( [ s, rx ] ) = [ \alpha(s), \beta(rx) ] \), up to conjugation.

Proof. Let \( g \in G, M = SK \) and \( M' \) be a disjoint copy of \( M \) with a corresponding \( g' \) action, \( g' : M', \, g'(x') = (g(x))' \). Consider the double of \( M, \, 2M = S^1 \times S^1 \), the non-orientable two-sphere bundle over \( S^1 \) obtained from \( M \) and \( M' \) by identifying them along their boundary by the identity map. Then \( g \) and \( g' \) define an action \( g \) on \( 2M \) and hence \( G \)-acts on \( 2M \). By Dunwoody (1), there exists a two sphere \( S \) properly embedded in \( 2M \) which does not bound a 3-cell such that for every \( g \in G \) \( g(S) = S \text{ or } g(S) \sim S = \emptyset \). Now since each of \( M \) and \( M' \) are invariant under the \( G \)-action and \( S \sim M = D^2 \) it follows that for every \( g \in G, g(D^2) = D^2 \text{ or } g(D^2) \sim D^2 = \emptyset \). Now by Meeks and Scott (3) the result follows.

The following theorem may be found in (4). It is an easy consequence of theorem 1 and Kims result (2).

Theorem 2. Let \( h \) be an involution on \( SK \), then \( h \) is conjugate to exactly one of the following involutions with fixed point sets \( M^*_i \)

\[
\begin{align*}
1. \quad & h_1([s, rx]) = [s, rx] \\
& \text{Fix}(h_1) = S^1 \times I \\
& M^*_1 = S^1 \times D^2 \\
2. \quad & h_2([s, rx]) = [s, -rx] \\
& \text{Fix}(h_2) = M_b \\
& M^*_2 = SK \\
3. \quad & h_3([s, rx]) = [s, -rx] \\
& \text{Fix}(h_3) = S^1 \\
& M^*_3 = SK \\
4. \quad & h_4([s, rx]) = [1-s, r\bar{x}] \\
& \text{Fix}(h_4) = D^2 \cup I \\
& M^*_4 = D^3 \\
5. \quad & h_5([s, rx]) = [1-s, -rx] \\
& \text{Fix}(h_5) = C(P^3) \\
& M^*_5 = C(P^3) \\
& \text{h}'([s, rx]) = [1-s, r\bar{x}]
\end{align*}
\]
Remark. It is easy to see that $h_4$, $h'_5$, are conjugate to $h'_4$, $h'_5$, respectively by taking the connecting homeomorphism $t$: $SK \rightarrow SK$ $t(\left[ s, rx \right]) = \left[ s + \frac{1}{2}, rx \right]$.

Section 2.

In this section we classify all $\mathbb{Z}_n$-actions on $SK$.

Theorem 3. Let $h$ be a generator of a $\mathbb{Z}_n$-action on $SK$. Then $h$ is weakly conjugate to one of the following maps, with quotient spaces $M^*$.

1. $h_1(\left[ s, rx \right]) = \left[ s + \frac{i}{n}, rx \right], n$ is odd
   $\text{Fix}(h'_1) = \Phi, 0 < i < n$
   $M^* = SK$

2. $h_2(\left[ s, rx \right]) = \left[ s + \frac{i}{k}, rx \right], n = 2k$
   $h'_2(\left[ s, rx \right]) = \left[ s + 1, rx \right] = \left[ s, rx \right]$
   $\text{Fix}(h'_2) = S'x 1$
   $M^* = S' \times D''$

3. $h_3(\left[ s, rx \right]) = \left[ s + \frac{i}{k}, -rx \right], n = 2k \text{ k is even}$
   $h'_3(\left[ s, rx \right]) = \left[ s + 1, rx \right] = \left[ s, rx \right]$
   $\text{Fix}(h'_3) = S'x 1$
   $M^* = SK$

4. $h_4(\left[ s, rx \right]) = \left[ s + \frac{i}{k}, -rx \right], n = 2k \text{ k is odd}$
   $h'_4(\left[ s, rx \right]) = \left[ s + 1, -rx \right] = \left[ s, -rx \right]$
   $\text{Fix}(h'_4) = Mb$
   $M^* = SK$

5. $h_5(\left[ s, rx \right]) = \left[ s + \frac{i}{k}, -rx \right], n = 2k \text{ k is odd}$
   $h'_5(\left[ s, rx \right]) = \left[ s + 1, -rx \right] = \left[ s, -rx \right]$
   $\text{Fix}(h'_5) = S'$
   $M^* = SK$

6. $h_6(\left[ s, rx \right]) = \left[ 1-s, rx \right], n = 2$
   $\text{Fix}(h_6) = D'' U 1$
   $M^* = D''$

7. $h_7(\left[ s, rx \right]) = \left[ 1-s, -rx \right], n = 2$
   $\text{Fix}(h_7) = I U \{^*\}$
   $M^* = C(P')$

Proof. Let $h$ be a generator of a $\mathbb{Z}_n$-action on $SK$. It follows from theorem 1 that, up to conjugation $h$ is given by either

$h(\left[ s, rx \right]) = \left[ s + \frac{i}{m}, g(rx) \right]$,

where $m$ divides $n$, $g$ is a homeomorphism on $D''$ such that $Cg = gC$ and $g^n = C^n \cdot n$. or

$h(\left[ s, rx \right]) = \left[ 1-s, g(rx) \right]$
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where $n$ is even $n = 2k$, $g$ is a periodic map on $D^i$ with period $n$ or $k$, and $CgC = g$.

First let $n$ be odd, then $m$ is also odd and $g^n(rx) = rx$ or $rx$, hence $g(rx) = rx$, from which we have $h([s,rx]) = [s + \frac{1}{m},rx]$ and $h''([s,rx]) = [s + 1,rx] = [s,rx]$, and hence $n = m$. Therefore $h$ is given by $h_1$, up to weak conjugation.

Second let $n$ be even, $n = 2k$ and $h$ is given by

$$h([s,rx]) = [s + \frac{1}{m}, g(rx)]$$

we have the following cases:

Case 1. $h'([s,rx]) = [s,rx]$, up to conjugation. Then $SK/h' = S'xD^i$ and $Fix(h') = S'x I \subset S' x \partial D^i$. $h$ induces a periodic map $h':(S'xD^i,S'xI) \rightarrow (S'xD^i,S'xI)$ which preserves the product structure. Hence up to weak conjugation $h(s,rx) = (s + \frac{1}{k}, g(rx))$ where $g(rx) = rx, -rx, rx$ or $-rx$. Therefore, up to weak conjugation, $h$ is given by $h([s,rx]) = [s + \frac{1}{k}, g(rx)]$, where $g(rx) = rx, -rx, rx$ or $-rx$.

If $g(rx) = rx$, then up to weak conjugation $h$ is given by $h_2$. If $g(rx) = rx$, then $k$ is even and $h = h_2^{k-1}$, therefore $h$ is weakly conjugate to $h_2$. If $g(rx) = -rx$, then $k$ is even and $h = h_3$, up to weak conjugation. Finally if $g(rx) = -rx$, then $k$ is even and $h = h_3^{k+1}$, hence $h$ is weakly conjugate to $h_3$.

Case 2. $h'([s,rx]) = [-s,rx]$, up to conjugation. $SK/h' = SK$ and $Fix(h') = Mb$. $h(MB) = Mb$ and Mb is two-sided in $SK$, hence $h$ interchanges the two sides of $Mb$ and $k$ is odd. We finish this case as we did in Case 1 to conclude that $h$ is weakly conjugate to $h_4$.

Case 3. $h([s,rx]) = [-s,rx]$, up to conjugation. $SK/h' = SK$ and $Fix(h') = S'$ is a fiber contained in $(SK/h')$. $h$ induces $h:(SK/h',S') \rightarrow (SK/h',S')$, where $h$ has period $k$. We finish this case as in Case 1 to conclude that $h$ is weakly conjugate to $h_5$.

Third let $n$ be even, $n = 2k$ and $h$ is given by

$$h([s,rx]) = [1-s, g(rx)]$$

If $g(rx) = rx$, where $\omega$ is a primitive root of unity, then $\bar{g(rx)} = g(rx)$, hence $rx\bar{\omega} = rx \omega$ and $\bar{\omega} = \omega$ from which we have $\omega = 1$ or $-1$. Therefore $g(rx) = rx, -rx, rx$ or $-rx$ and $n = 2$. If $g(rx) = rx$, $h$ is given by $h_6$, up to conjugation. If $g(rx) = r\bar{x}$, then it is easy to check that $h$ is conjugate to $h_6$. Similarly if $g(rx) = -rx$ or $g(rx) = -rx$, then $h$ is conjugate to $h_7$.

Section 3.

In this section we classify the $Z_2 \oplus Z_2$-actions on $SK$.

Theorem 4. Let $Z_2 \oplus Z_2$-act effectively on $SK$, then the action is weakly conjugate to one of the following actions.
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1) \( G_1 = \{ e, h_1, h_2, h_3 \} \), 2) \( G_2 = \{ e, h_1, h_4, h'_4 \} \), 3) \( G_3 = \{ e, h_1, h_5, h'_5 \} \), 4) \( G_4 = \{ e, h_2, h_4, h_5 \} \) or 5) \( G_5 = \{ e, h_3, h_4, h_5 \} \). Where the \( h_i \)'s are the involutions on \( SK \) given in theorem 2.

Proof. Let \( h \) be a generator of a \( Z_2 \oplus Z_2 \)-action on \( SK \), then \( h \) is an involution. First let \( h \) be given by \( h \) up to conjugation. Let \( g \) be the second generator, then \( g \) is also an involution. If \( g = h_2 \) (or \( h_3 \)) then \( Z_2 \oplus Z_2 = G_1 \) up to weak conjugation. If \( g = h'_4 \), then \( Z_2 \oplus Z_2 = G_2 \) up to weak conjugation. If \( g = h_5 \), then \( Z_2 \oplus Z_2 = G_3 \) up to weak conjugation. Second if \( h = h_2 \), up to conjugation, then if \( g = h_1 \) or \( h_2 \) we get \( G_1 \). If \( g = h_4 \) then \( Z_2 \oplus Z_2 = G_4 \), up to weak conjugation. If \( g = h_5 \), then \( Z_2 \oplus Z_2 = G_4 \), up to weak conjugation where the connected homeomorphism \( t: SK \rightarrow SK \), \( t([s,rx]) = [s^{1/2},rx] \) makes this action and the preceding one weakly conjugate. Third if \( h = h_3 \), then for \( g = h_4 \) we have \( Z_2 \oplus Z_2 = G_5 \), up to weak conjugation.

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عن المجموعة المحدودة للأفعال في قنينة كلين

محمد عرفات النتشة

في هذا البحث تصنيف لجميع قنينية 
وعندما يكون
$G = Z_2 \oplus Z_2$.