Qatar Univ. Sci. J. (1996), 16(2): 179-182

TWO FORMS OF THE BÄCKLUND TRANSFORMATION OF A THIRD ORDER DIFFERENTIAL EQUATION

H. M. EL-SHERBINY* and M. B. MANSOUR**
*Faculty of Science, Math. Dept., Qatar University, Qatar
**Faculty of Science, Math. Dept., South Valley University, Qena Egypt

صيغتان لتحويلات باكلاند لمعادلة تفاضلية من الرتبة الثالثة

حامد مصطفى الشربيني و محمود بخيت منصور

نشتق في هذا البحث صيغتين صريحتين لتحويل باكلاند لمعادلة تفاضلية تشتتية وتعتمد طريقة الإشتقاق على إستخدام تحويل يربط بين المعادلة المعطاه ومعادلتها المعممة.

Key Words : Bäcklund transformation, Dispersive equation

ABSTRACT

Two explicit forms of Bäcklund transformation of third order dispersive differential equation are derived. The method of derivation depends on the use of a transformation that couples the equation with its modified one.

INTRODUCTION

The Bäcklund transformation of the KdV equation

$$\eta_t + \eta\eta_x + \eta_{xxx} = 0$$

(where $\eta\eta_x$ is the nonlinear effect and η_{xxx} is the dispersion effect) was given in [1]. The method of derivation depends on the use of Miura transformation [2]

$$\eta = u^2 + u_x$$

which couples the KdV equation with the modified equation

$$u_t + u^2 u_x + u_{xxx} = 0$$

In this work we generalize the method given in [4] to construct two forms of the Bäcklund transformation of an equation possessing more dispersion and nonlinear effects. We consider the nonlinear evolution eqaution [3]

$$\eta_t + \eta_x + \eta\eta_t - \eta\eta_x + \eta_{xxx} - 3\eta_{xxt} + 3\eta_{xtt} - \eta_{ttt} = 0 \quad (1.1)$$

where $\eta(x,t)$ is a real scalar field for every $(x,t) \in R^2$.

In this section we derive the transformation which couples (1.1) with its modified equation.

$$u_t + u_x + u^2 u_t - u^2 u_x + u_{xxx} - 3u_{xxt} + 3u_{xtt} - u_{ttt} = 0 \quad (1.2)$$

Theorem 1: If u is a solution of (1.2) then

$$\eta = u^2 + \sqrt{6(u_t - u_x)} \quad (1.3)$$

is the solution of (1.1).

Proof: From the definition of η one can easily calculate $\eta_t, \eta_x, \eta_{xxx}, \eta_{xxt}, \eta_{xtt}$ and η_{ttt}. Thus we have

$$\eta_t+\eta_x+\eta\eta_t-\eta\eta_x+\eta_{xxx}-3\eta_{xxt}+3\eta_{xtt}-\eta_{ttt}=$$
$$(2u+\frac{\partial}{\partial t}-\frac{\partial}{\partial x})(u_t+u_x+u^2u_t-u^2u_x+u_{xxx}-3u_{xxt}+3u_{xtt}-u_{ttt})$$

which proves the theorem.

Theorem 2: Equation (1.1) is invariant under the transformation

$$x\to(1-\frac{\lambda}{2})x-\frac{\lambda}{2}t,\ t\to\frac{\lambda}{2}x+(1+\frac{\lambda}{2})t,\ \eta\to\eta+\lambda \quad (1.4)$$

Proof : By direct calculation of the whole derivatives in (1.1) in terms of the new variable the proof follows immediately.

Now, using the result of the last theorem we could work with $\eta-\lambda(\lambda$ is arbitrary parameter) rather than η, thus we write

$$\eta=\eta+u^2+\sqrt{6(u_t-u_x)}$$

and then (1.2) becomes

$$u_t+u_x+(u^2+\lambda)u_t-(u^2+\lambda)u_x+u_{xxx}-3u_{xxt}+3u_{xtt}-u_{ttt}=0 \quad (1.5)$$

Explicit form of the Bäcklund Transformation of (1.1)

It is clear that if u is a solution of (1.2) then so is $-u$. This suggests that we introduce two functions

$$\eta=\lambda+u^2+\sqrt{6(u_t-u_x)}$$
$$\eta'=\lambda+u^2-\sqrt{6(u_t-u_x)}$$

for given u and λ. These two equations imply that

$$\eta-\eta'=2\sqrt{6(u_t-u_x)}\ ,\ \eta+\eta'=2(\lambda+u^2) \quad (2.1)$$

At this stage it is convenient to introduce the additional transformation

$$\eta=-(v_t-v_x) \quad \text{and} \quad \eta'=(v'_t-v'_x) \quad (2.2)$$

and it follows from (1.1) that v and v' an be taken to satisfy the equations

$$v_t-v_x-\frac{1}{2}(v_t-v_x)^2+v_{xxx}-3v_{xxt}+3v_{xtt}+v_{ttt}=0 \quad (2.3)$$

$$v'_t-v'_x-\frac{1}{2}(v'_t-v'_x)^2+v'_{xxx}-3v'_{xxt}+3v'_{xtt}+v'_{ttt}=0 \quad (2.4)$$

and equation, (2.1) become

$$v-v'=-2\sqrt{6u} \quad (2.5)$$

$$(v+v')_t-(v+v')_x=-2\lambda+\frac{1}{12}(v-v')^2 \quad (2.6)$$

respectively. Equation (2.6) constitutes one part of the Bäcklund transformation for v and v' which in turn generates the solution of (1.1) via (2.2).X

To find the other part of the Bäcklund transformation it is clear that by making use of (2.1), (2.2) and (2.5), equation (1.5) can be written as

$$(v-v')_t+(v-v')_x-\frac{1}{2}[(v_t-v_x)^2-(v'_t-v'_x)^2]+(v-v')_{xxx}$$
$$-3(v-v')_{xxt}+3(v-v')_{xtt}-(v-v')_{ttt}=0 \quad (2.7)$$

Equations (2.6) and (2.7) constitute the Bäcklund transformation of (1.1). However, we can write equation (2.7) in a more convenient form as follows: First operate on both sides of (2.6) with the operator $\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^2$ to obtain

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^3(v+v')=-\frac{1}{6}\left\{\left[\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)(v-v')\right]^2+ +(v-v')\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^2(v-v')\right\} \quad (2.8)$$

i. e

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)^3(v+v')=\frac{1}{6}\left\{\left[\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)(v-v')\right]^2+ +(v-v')\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^2(v-v')\right\} \quad (2.9)$$

On the the other hand equation (2.7) can be rewritten as

$$(v-v')_t-(v-v')_x-\frac{1}{2}[(v_t-v_x)^2-(v'_t-v'_x)^2]+ +\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^3v-\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^3v'=0 \quad (2.10)$$

Using equation (2.9), equation (2.10) reduces to

$$(v-v')_t-(v-v')_x-\frac{1}{2}[(v_t-v_x)^2-(v'_t-v'_x)^2]$$
$$+2v'_x+2v'_t-(v'_t-v'_x)^2+\frac{1}{6}\left[\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)(v-v')\right]^2$$
$$+\frac{1}{6}(v-v')\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^2(v-v')=0$$

The last equation can be simplified as

$$(v+v')_t+(v+v')_x=\frac{1}{3}[(v_t-v_x)^2+(v_t-v_x)(v'_t-v'_x)$$
$$+(v'_t-v'_x)^2]-\frac{1}{6}(v-v')[(v-v')_{xx}-2(v-v')_{xt}+(v-v')_{tt}] \tag{2.11}$$

Hence, we have proved the following theorem

Theorem 3: The Bäcklund transfomation of the nonlinear evolution equation

$$\eta_t+\eta_x-\eta\eta_x+\eta\eta_t+\eta_{xxx}-3\eta_{xxt}+3\eta_{xtt}-\eta_{ttt}=0 \tag{1.1}$$

is given by the system of equations:

$$(v+v')_t-(v+v')_x=-2\lambda-\frac{1}{12}(v-v')^2 \tag{2.6}$$

$$(v+v')_t+(v+v')_x=\frac{1}{3}[(v_t-v_x)^2+(v_t+v_x)(v'_t-v'_x)+$$
$$(v'_t-v'_x)^2]-\frac{1}{6}(v-v')[(v-v')_{xx}-2(v-v')_{xt}+(v-v')_{tt}] \tag{2.11}$$

where $\eta=v_x-v_t$, $\eta'=v'_x-v'_t$

Once we have derived an explicit form of the Bäcklund transformation for the given equation (1.1), we turn now to establish another form for the Bäcklund transformation by making use of the first one.

Theorem 4: Another form of the Backlund transformation of equation (1.1) is given in the form:

$$(v+v')_t-(v+v')_x=-2\lambda-\frac{1}{12}(v-v')^2$$
$$(v'-v)_t+(v'-v)_x=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)[-2(v_{xx}-2v_{xt}+v_{tt})$$
$$-\frac{1}{3}(v-v')(v_t-v_x)+\frac{2}{3}\lambda(v-v')]$$

(where $\eta=v_x-v_t$, $\eta'=v'_x-v'_t$)

Proof : Rewritting (2.11) in the form:

$$(v'-v)_t+(v'-v)_x=-2(v_t+v_x)+\frac{1}{3}[(v_t-v_x)^2$$
$$+(v_t-v_x)(v'_t+v'_x)+(v'_t+v'_x)^2] \tag{2.12}$$
$$-\frac{1}{6}(v-v')[(v-v')_{xx}-2(v-v')_{xt}+(v-v')_{tt}]$$

and making use of (2.3), we have

$$(v'-v)_t+(v-v')_x=2(v_{xxx}-3v_{xxt}+3v_{xtt}-v_{ttt})-\frac{1}{3}(v_t-v_x)$$
$$[(v_t-v_x)+(v'_t-v'_x)]-\frac{1}{3}[(v_t-v_x)-(v'_t-v'_x)][(v_t-v_x)$$
$$-(v'_t-v'_x)]-\frac{1}{6}(v-v')[(v-v')_{xx}-2(v-v')_{xt}+(v-v')_{tt}]$$

Using now the relation

$$(v+v')_t-(v+v')_x=2\lambda-\frac{1}{12}(v-v')^2$$

we have

$$(v'-v)_t+(v-v')_x=2(v_{xxx}-3v_{xxt}+3v_{xtt}-v_{ttt})-\frac{1}{3}(v_t-v_x)$$
$$[(v_t-v_x)+(v'_t-v'_x)]-\frac{1}{3}[(v_t-v_x)-(v'_t-v'_x)][-2\lambda-\frac{1}{6}(v-v)^2]-$$
$$-\frac{1}{6}(v-v')[(v-v')_{xx}-2(v-v')_{xt}+(v-v')_{tt}]$$

i.e.,

$$(v'-v)_t+(v-v')_x=2(v_{xxx}-3v_{xxt}+3v_{xtt}-v_{ttt})-\frac{1}{3}(v_t-v_x)^2$$
$$[(v_t-v_x)+(v'_t-v'_x)]+\frac{2}{3}\lambda[(v_t-v_x)+(v'_t-v'_x)]-\frac{1}{36}(v-v')[(v_t-v_x)-$$
$$(v'_t-v'_x)]+\frac{1}{6}(v-v')[(v-v')_{xx}-2(v-v')_{xt}+(v-v')_{tt}]$$

Operating on both sides of the equation

$$(v+v')_t-(v+v')_x=2\lambda-\frac{1}{12}(v-v')^2$$

by the operator $\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)$, we have

$$(v'+v)_{xx}-2(v'+v)_{xt}+(v'+v)_{tt}=$$
$$-\frac{1}{6}(v-v')[(v_t-v_x)-(v'_t-v'_x)] \tag{2.14}$$

Equations (2.13) and (2.14) together yield

$$(v'-v)_t+(v'-v)_x=2(v_{xxx}-3v_{xxt}+3v_{xtt}-v_{ttt})$$
$$-\frac{1}{3}(v_t-v_x)[(v_t-v_x)+(v'_t-v'_x)]+\frac{2}{3}\lambda[(v_t-v_x)+(v'_t-v'_x)]$$
$$-\frac{1}{3}(v-v')[(v_{xx}-2v_{xt}-v_{tt})] \tag{2.15}$$

Now, since

$$v_{xxx}-3v_{xxt}+3v_{xtt}-v_{ttt}=-\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)(v_{xx}-2v_{xt}+v_{tt})$$

Then (2.15) can be rewritten in the form

$$(v'-v)_t+(v'-v)_x=-\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)[-2(v_{xx}-2v_{xt}+v_{tt})$$
$$-\frac{1}{3}(v-v')(v_t-v_x)+\frac{2}{3}\lambda(v-v')]$$

and the theorem is proved.

CONCLUSION

Theorem 3 and theorem 4 guarantee the existence of two explicit forms of Bäcklund transformation of the given equation (1.1). The two forms are in fact equivalent in the sense that from one we obtain the other as it appears from theorem 4. On the other hand this equation is thought to have multi-soliton solution since many equations such as the KdV equaiton, sine Gordon equation and others which possess Bäcklund transformation proved to have multi-soliton solutions [4]. We are trying to sort out this problem and shall report on it in the near future.

REFERENCES

[1] **Wahlquist, H. D.** and **Estabrook, F. B., 1973**. "Bäcklund transformation for solutions of the Korteweg de Vries equation", Phys. Rev 31; 1386-1392.

[2] **Gardner, C. S., Green, J. M., Kruskal, M. D.** and **Miura, R. M., 1967**. "Method for solving the Korteweg de Vries equation", Phys. Rev. Lett., 19; 1095-1099.

[3] **El-Sherbiny, H. M., El-Ansary, N. F.** and **Abd-Allah, M. A., 1990**. "On a class of nonlinear differential equations-solitary waves and Baklund transformation", Proc. of Assiut first intern Conf. of Math. and Stat. (V), 81-90.

[4] **Ablowitz, M. J., Kaup, D. J., Newell, A. C.** and **Segur, H., 1973**. "Nonlinear evolution equations of physical significance", Phys. Rev. Lett., 31; 125-128.