On the numerical simulation of population dynamics with density-dependent migrations and the Allee effects

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Abstract. In this paper, the variational iteration method (VIM) and the Adomian decomposition method (ADM) are presented for the numerical simulation of the population dynamics model with density-dependent migrations and the Allee effects. The convergence of ADM is proved for the model problem. The results obtained by these methods are compared to the exact solution. It is found that these methods are always converges to the right solutions with high accuracy. Furthermore, VIM needs relative less computational work than ADM.

1. Introduction

Recently much attention has been devoted to various numerical methods which do not require discretization of space-time variables or linearization of the nonlinear differential equations, among which the variational iteration method (see [2], [6], [9]-[13], [20]-[23] and the reference cited therein) and the Adomian decomposition method (see [1], [3], [5], [7], [8], [14] and the reference cited therein) are widely used for this purpose. Many authors pointed out that the variational iteration method has merits over other methods and can overcome the difficulties arising in calculation of Adomian's polynomials in Adomian decomposition method (see [16], [17], [19] and the references therein). The aim of this paper is to develop VIM and ADM to simulate the solutions of the model of population dynamics with density-dependent migrations and the Allee effect [4], [18]. This model can be described by the transient non-linear advection-diffusion-reaction equation of the form:

$$\frac{\partial U}{\partial T} = -\frac{\partial}{\partial X}[\Theta(U)U - D\frac{\partial U}{\partial X}] + F(U)U \qquad X \in \Omega \ , \ \ T > 0 \, . \tag{1}$$

The unknown field $U = U(X,T)$ is the population density in $\Omega \subset \Re$ and T. U changes in space and time due to the non-linear velocity field $\Theta \equiv \Theta(U)$, the diffusion D and the intrinsic growth rate $F(U)$, which includes all local processes (such as birth, death and predation/harvesting).

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The model (1) specifies that the spatial distribution is affected by two physical processes, the advection and the isotropic diffusion of Fickian type [4], [18]. Here, we also consider a biological mechanism on the advection process in order to include the case when the species purposely migrates in some particular direction due to some chemical communication. These assumptions yield the following non-linear velocity field

$$\Theta(U) = \Theta_0 + \Theta_1 U \quad . \tag{2}$$

In this speed of migration model (2), Θ_0 is the density-independent migration velocity, which is known or might come from a hydrodynamic solver. The model (2) also assumes the existence of a density-dependent migration that varies linearly with the population density, where Θ_1 depends on the species taxis. We assume here, for simplicity, that the fluid is incompressible ($\mathrm{div}(\Theta_0) = 0$) and Θ_0, Θ_1 and the diffusion coefficient D are constants, yielding

$$\frac{\partial U}{\partial T} + (\Theta_0 + 2\Theta_1 U)\frac{\partial U}{\partial X} = D\frac{\partial^2 U}{\partial X^2} + F(U)U. \tag{3}$$

Consider the growth dynamics with Allee effects given by

$$F(U)U = \tilde{\alpha}\, U(U - K_0)(K - U) \quad , \tag{4}$$

Where K is the carrying capacity and K_0 is the measure of the Allee effects. When K is constant, it is convenient to use the dimensionless variable $u = U/K$ so that (4) is re-written as:

$$f(u) = \alpha\, u(u - \beta)(1 - u)\,, \tag{5}$$

where $\beta = K_0/K$ represents the strength of the Allee effects. The strong and the weak Allee effects occur when $0 < \beta < 1$ and $-1 < \beta < 0$, respectively. The parameter $\alpha = \alpha(\beta)$ is a normalization constant which is defined by a maximum growth rate, leading to a family of models. The qualitative results regarding the Allee effects and asymptotic rates of spread are independent from the choice of the normalization constant. With this assumption and using $t = T\alpha K^2$ and $x = X\sqrt{\frac{\alpha K^2}{D}}$, equation (3) can be written in the following dimensionless form:

$$u_t + (\theta_0 + \theta_1 u)u_x = u_{xx} - \beta u + (1+\beta)u^2 - u^3, \tag{6}$$

where we used the additional dimensionless parameters $\theta_0 = \frac{\Theta_0}{K\sqrt{\alpha D}}$ and $\theta_1 = \frac{2\Theta_1}{\sqrt{\alpha D}}$. Hence, the population densities have been re-scaled so that $u \in [0,1]$ in $t \in [0, T_{final}]$. Travelling wave solutions are considered so that the (6) is solved in an unbounded domain with the following conditions at infinity: For (the species is at its carrying capacity); for $x \to -\infty \Rightarrow u = 0$ (the species is absent), some initial condition. Under these boundary conditions, one can find in [4] and the references sited therein, the asymptotic stability analysis of the travelling wave for the scaled diffusion–reaction equation

$$u_t = u_{xx} + g(u). \tag{7}$$

The existence of wave fronts $u(x,t) = U(x-ct)$ was derived, relying on the properties of g. Here, $g(u) = -\beta u + (1+\beta) u^2 - u^3$ and has at least two distinct zeros, $g_1 = 1$ and $g_0 = 0$; if there exists a strong Allee effect, there still is another zero between g_0 and g_1 at which the percapita growth rate is positive. For more details on this model, see [4] and [18].

2. Implementation of VIM

In this section, VIM will apply to the following nonlinear partial differential equation of the form:

$$u_t + (\theta_0 + \theta_1 u)u_x = u_{xx} - \beta u + (1+\beta)u^2 - u^3, \tag{8}$$

subject to the initial condition $u(x,0) = f(x)$. First, we construct the correction functional:

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)[u_{n\tau} + (\theta_0 + \theta_1 \hat{u}_n)\hat{u}_{nx} - \hat{u}_{nxx} + \beta\hat{u}_n - (1+\beta)\hat{u}_n^2 + \hat{u}_n^3]\, d\tau, \tag{9}$$

where λ is a general Lagrange multiplier, $\hat{u}_n$, $\hat{u}_{nx}$, $\hat{u}_{nxx}$ denote restricted variations, i.e.

$$\delta\hat{u}_n = \delta\hat{u}_{nx} = \delta\hat{u}_{nxx} = 0.$$

Making the above correction functional stationary, we obtain the following stationary condition:

$$\dot{\lambda}(\tau) = 0, \qquad 1 + \lambda(\tau)|_{\tau = t} = 0,$$

The Lagrange multiplier, therefore, can be defined in the following form:

$$\lambda(\tau) = -1. \tag{10}$$

Substituting from (10) into (9) results the following iteration formula:

$$u_{n+1} = u_n - \int_0^t [u_{n\tau} + (\theta_0 + \theta_1 u_n)u_{nx} - u_{nxx} + \beta u_n - (1+\beta)u_n^2 + u_n^3]\, d\tau. \tag{11}$$

Now, if we start with the following initial approximation

$$u(x,0) = \frac{\beta \exp(\lambda_1\xi_1) + \exp(\lambda_2\xi_2)}{1 + \exp(\lambda_1\xi_1) + \exp(\lambda_2\xi_2)}, \tag{12}$$

where, $\xi_i = x + \varphi_i$, $i = 1,2$; $\lambda_1 = \beta/\sqrt{2}$ and $\lambda_2 = 1/\sqrt{2}$, and φ_1, φ_2 are arbitrary constants. Using the recurrence relation (11), we obtain the first components of the solution in the case ($\theta_0 = \theta_1 = 0$) in the following form:

$$u_0(x,t) = u(x,0),$$

$$u_1(x,t) = u_0(x,t) - \frac{1}{(1+\exp(\lambda_1\xi_1)+\exp(\lambda_2\xi_2))^3}[\, t\,[-\beta\exp(\lambda_1\xi_1)(-\beta+\lambda_1^2)+\exp(2\lambda_1\xi_1)\beta+$$
$$(\beta-\beta^2+\lambda_1^2)+\exp(2\lambda_1\xi_1+\lambda_2\xi_2)(-1+\beta)(\beta-\beta^2+(\lambda_1-\lambda_2)^2)+\exp(\lambda_2\xi_2)(\beta-\lambda_2^2)+$$
$$\exp(2\lambda_2\xi_2)(-1+\beta-\lambda_2^2)-\exp(\lambda_1\xi_1+2\lambda_2\xi_2)(-1+\beta)(-1+\beta+\lambda_1^2-2\lambda_1\lambda_2+\lambda_2^2)+$$
$$\exp(\lambda_1\xi_1+\lambda_2\xi_2)((-1+2\beta)\lambda_1^2+2(1+\beta)\lambda_1\lambda_2+(-2+\beta)\lambda_2^2)\,]\,],$$

and so on. The rest of components of the iterative formula (11) were obtained in the same manner using the Mathematica package. The exact solution of the equation (8) [in the case ($\theta_0 = \theta_1 = 0$)] under the initial condition (12) is given by: $u(x,t) = \dfrac{\beta\exp(\lambda_1\xi_1)+\exp(\lambda_2\xi_2)}{1+\exp(\lambda_1\xi_1)+\exp(\lambda_2\xi_2)}$, where, $\xi_i = x - \eta_i t + \varphi_i$, $i = 1, 2$; $\eta_i = \sqrt{2}(1+\beta) - 3\lambda_i$; $\lambda_1 = \beta/\sqrt{2}$ and $\lambda_2 = 1/\sqrt{2}$, and φ_1, φ_2 are arbitrary constants. Here, we set $\beta = 0.2$, $\varphi_1 = 100$ and $\varphi_2 = -100$.

The error behaviour for different time values are shown in figures 1-4 where the numerical results are obtained by using two terms only from the iterative formula (11). It is evident that the overall errors can be made smaller by adding new terms from the iteration formula.

3. Implementation of ADM
In this section, the ADM will apply to (8) and (12), so we rewrite (8) in the following form:

$$L_t\, u = u_{xx} - \theta_0 u_x - \beta u + N(u), \tag{13}$$

where $L_t = \dfrac{\partial}{\partial t}$ is linear operator, $N(u) = -\theta_1 u\, u_x + (1+\beta)u^2 - u^3$ is nonlinear operator.

By taken the inverse operator $L_t^{-1}(\cdot) = \int_0^t (\cdot)\,dt$ of (13), then the solution of (13) can be written in the form

$$u(x,t) = u(x,0) + L_t^{-1}[u_{xx} - \theta_0 u_x - \beta u + N(u)]. \tag{14}$$

The ADM assumes that the unknown solution $u(x,t)$ can be expressed by an infinite series of the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{15}$$

and the nonlinear operator term $N(u)$ can be decomposed by an infinite series of polynomials, given by:

$$N(u) = \sum_{n=0}^{\infty} A_n. \tag{16}$$

the components $u_n(x,t)$ will be determined recurrently and A_n are the Adomian's polynomials of u_0, u_1, u_2, … defined by:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{ds^n} N\left(\sum_{i=0}^{n} s^i u_i\right)\right]_{s=0}, \qquad n = 0,1,2,\ \dots \tag{17}$$

Substituting from (15), (16) in (14), we can obatain the subsequent components:

$$u_0(x,t) = u(x,0), \qquad u_{n+1}(x,t) = L_t^{-1}(u_{nxx} - \theta_0 u_{nx} - \beta u_n) + L_t^{-1}(A_n), \quad n \ge 0. \tag{18}$$

One can use the general form of formula (17) for A_n as follows:

$$A_0 = (1+\beta)u_0^2 - u_0^3 - \theta_1 u_0 u_{0x},$$

$$A_1 = 2(1+\beta)\, u_0 u_1 - 3u_0^2 u_1 - \theta_1 u_1 u_{0x} - -\theta_1 u_0 u_{1x},$$

$$A_2 = \frac{1}{2}(-6u_0 u_1^2 - 6u_0^2 u_2 + (1+\beta)(2u_1^2 + 4u_0 u_2) - \theta_1(2u_2 u_{0x} + 2u_1 u_{1x} + 2u_0 u_{2x})),$$

$$A_3 = \frac{1}{6}(-6u_1^3 - 36\, u_0 u_1 u_2 - 18\, u_0^2 u_3 + 12(1+\beta)(u_1 u_2 + u_0 u_3) - 6\theta_1(u_3 u_{0x} + u_2 u_{1x} + u_1 u_{2x} + u_0 u_{3x})),$$

For numerical comparisons purpose, based on the ADM, we constructed the solution $u(x,t)$ as:

$$\lim_{n\to\infty} \Phi_n(x,t) = u(x,t), \text{ where } \Phi_n(x,t) = \sum_{m=0}^{n-1} u_m(x,t), \quad n \ge 0. \tag{19}$$

To obtain the components of the solution, we start by substituting the initial condition (12) in (18):

$$u_0(x,t) = u(x,0),$$

$$\begin{aligned}
u_1(x,t) = {} & \frac{1}{(1+\exp(\lambda_1\xi_1)+\exp(\lambda_2\xi_2))^3}[\,t\,[\beta\exp(\lambda_1\xi_1)(-\beta+\lambda_1^2) - \exp(2\lambda_1\xi_1)\beta(\beta-\beta^2+\lambda_1^2) \\
& - \exp(2\lambda_1\xi_1+\lambda_2\xi_2)(-1+\beta)(\beta-\beta^2+(\lambda_1-\lambda_2)^2) + \exp(\lambda_2\xi_2)(\beta-\lambda_2^2) - \exp(2\lambda_2\xi_2)(-1+\beta+\lambda_2^2) \\
& + \exp(\lambda_1\xi_1+2\lambda_2\xi_2)(-1+\beta)(-1+\beta+\lambda_1^2-2\lambda_1\lambda_2+\lambda_2^2) + \exp(\lambda_1\xi_1+\lambda_2\xi_2)((-1+2\beta)\lambda_1^2 \\
& - 2(1+\beta)\lambda_1\lambda_2 - (-2+\beta)\lambda_2^2)\,]],
\end{aligned}$$

and so on the other terms can be obtained in the case ($\theta_0 = \theta_1 = 0$).

3.1. Convergence Analysis of the ADM

In this section, we will prove the convergence of ADM applied to equation (8). Let us define the Hilbert space $H = L^2((\alpha(\beta)\times[0,T])$, as a set of all applications

$$u : (\alpha,\beta)\times[0,T] \to R \quad \text{with} \quad \int_{(\alpha,\beta)\times[0,T]} u^2(x,s)\,ds\,d\tau < +\infty.$$

Consider (8) with the notation $L(u) = \frac{\partial u}{\partial t}$, then we can write (8) in the following operator form:

$$L(u) = u_{xx} - \theta_0 u_x - \theta_1 u u_x - \beta u + (1+\beta)u^2 - u^3 . \tag{20}$$

Theorem: (Sufficient conditions of convergence)
The ADM applied to the nonlinear equation (20) is converges towards a particular solution if the following two hypotheses are satisfied:

(H1) : $(L(u) - L(v), u - v) \geq m\|u - v\|^2$, $m > 0$, $\forall\, u, v \in H$;

(H2) : $\exists$ $C(K) > 0, K > 0$ such that $\forall$ $u, v \in H$ with $\|u\| \leq K$, $\|v\| \leq K$,

we have $(L(u)-L(v), w) \leq C(K)\, \|u-v\|\, \|w\|$ $\forall\, w \in H$.

Proof : To verify (H1) for the operator $L(u)$, we have

$$L(u) - L(v) = \frac{\partial^2}{\partial x^2}(u\text{-}v) - \theta_0 \frac{\partial}{\partial x}(u\text{-}v) - \frac{1}{2}\theta_1 \frac{\partial}{\partial x}(u^2 - v^2) - \beta(u\text{-}v) + (1+\beta)(u^2 - v^2) - (u^3 - v^3) .$$

Then we claim:

$$(L(u) - L(v), u - v) = (\frac{\partial^2}{\partial x^2}(u - v), u - v) - \theta_0 (\frac{\partial}{\partial x}(u - v), u - v) - \frac{1}{2}\theta_1 (\frac{\partial}{\partial x}(u^2 - v^2), u - v) - \beta(u - v, u - v) + (1+\beta)(u^2 - v^2, u - v) - (u^3 - v^3, u - v). \tag{21}$$

Since $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ are differential operators in H, then there exist constants δ_1 and δ_2:

$$(-\frac{\partial}{\partial x}(u - v), u - v) \geq \delta_1 \|u - v\| \|u - v\| = \delta_1 \|u - v\|^2 , \tag{22}$$

$$(\frac{\partial^2}{\partial x^2}(u - v), u - v) \leq \delta_2 \|u - v\| \|u - v\| \leq \delta_2 \|u - v\|^2 , \tag{23}$$

and $$(\frac{\partial}{\partial x}(u^2 - v^2), u - v) \leq \delta_1 \left\|u^2 - v^2\right\| \|u - v\|,$$

this according to Schwartz inequality. Now, by using the mean value theorem and the above relation we get: $(\frac{\partial}{\partial x}(u^2 - v^2), u - v) \leq \delta_1 \left\|u^2 - v^2\right\| \|u - v\| = 2\delta_1 \eta^2 \|u - v\|^2$,

where, $u < \eta < v$ and $\|u\|, \|v\| \leq K$. Therefore, $$(-\frac{\partial}{\partial x}(u^2 - v^2), u - v) \geq 2\delta_1 K^2 \|u - v\|^2 \tag{24}$$

Also, we have $$(u - v, u - v) \leq \|u - v\| \|u - v\| = \|u - v\|^2 , \tag{25}$$

$$(u^2 - v^2, u - v) \leq \left\|u^2 - v^2\right\| \|u - v\| = 2\eta^2 \|u - v\|^2 = 2K \|u - v\|^2 , \tag{26}$$

$$(u^3 - v^3, u - v) \le \left\| u^3 - v^3 \right\| \left\| u - v \right\| = 3\eta^3 \left\| u - v \right\|^2 = 3K^2 \left\| u - v \right\|^2 , \tag{27}$$

substituting from (22)-(27) into (21), we get

$$(L(u) - L(v), u - v) \ge (\delta_2 + \theta_0\delta_1 + 0.5\theta_1\delta_1 K + \beta + 2(1+\beta)K + 3K^2) \left\| u - v \right\|^2 = m \left\| u - v \right\|^2 ,$$

where, $m = \delta_2 + \theta_0\delta_1 + 0.5\theta_1\delta_1 K + \beta + 2(1+\beta)K + 3K^2$. Hence, we verified (H1).
To verify (H2) for the operator $L(u)$, we have

$$\begin{aligned}(L(u) - L(v), w) = {} & (\frac{\partial^2}{\partial x^2}(u\text{-}v), w) - \theta_0 (\frac{\partial}{\partial x}(u\text{-}v), w) - \frac{1}{2}\theta_1 (\frac{\partial}{\partial x}(u^2 - v^2) , w) \\ & - \beta(u\text{-}v , w) + (1+\beta)(u^2 - v^2, w) - (u^3 - v^3, w),\end{aligned} \tag{28}$$

therefore, $(L(u) - L(v), w) \le (1 + \theta_0 + 0.5\theta_1 K + \beta + 2(1+\beta)K + 3K^2) \left\| u - v \right\| \left\| w \right\| = C(K) \left\| u - v \right\| \left\| w \right\|$,

where, $C(K) = 1 + \theta_0 + 0.5\,\theta_1 K + \beta + 2(1+\beta)K + 3\,K^2$. Hence, we verified (H2).

4. Special Cases of the Model

***Case I**: No Migration:* $\theta_0 = \theta_1 = 0$ and (6) reduces to:

$$u_t = u_{xx} - \beta\, u + (1+\beta)\, u^2 - u^3. \tag{29}$$

The exact solution of (29) is $u(x,t) = \dfrac{\beta \exp(\lambda_1 \xi_1) + \exp(\lambda_2 \xi_2)}{1 + \exp(\lambda_1 \xi_1) + \exp(\lambda_2 \xi_2)}$, where, $\xi_i = x - \eta_i t + \varphi_i$,
$i = 1, 2$; $\eta_i = \sqrt{2}\,(1+\beta) - 3\lambda_i$; $\lambda_1 = \beta/\sqrt{2}$ and $\lambda_2 = 1/\sqrt{2}$, and φ_1, φ_2 are arbitrary constants.

***Case II:** Density-Independent Migration:*
In the case that the speed of the species migration does not depend on the population density e.g., when drifting with the wind, the dynamics of the population are described by the following equation:

$$u_t + \theta_0 u_x = u_{xx} - \beta\, u + (1+\beta)\, u^2 - u^3, \tag{30}$$

where θ_0 is the speed of advection.
Considering traveling wave coordinates, $(x,t) \to (z,t)$ where $z = x - \theta_0 t$, so that $u = \hat{u}(z,t)$, from (30) we obtain

$$\hat{u}_t = \hat{u}_{xx} - \beta\, \hat{u} + (1+\beta)\, \hat{u}^2 - \hat{u}^3. \tag{31}$$

Equation (31) coincides with (29) and thus the exact solution of (29) gives also an exact solution of (31) with the obvious change $x \to z$.

Case III: *Density-Dependent Migration:*

In this section, we consider the case when the density-independent advection caused by environmental factors is absent and migration takes place due to biological mechanisms which are assumed to be density-dependent. Then $\theta_0 = 0$ and from (6) we arrive at the following equation:

$$u_t + \theta_1 u u_x = u_{xx} - \beta u + (1+\beta) u^2 - u^3, \tag{31}$$

The exact solution of (32) is given by:

$$u(x, t) = \frac{\beta \exp(\omega_1 \psi_1) + \exp(\omega_2 \psi_2)}{1 + \exp(\omega_1 \psi_1) + \exp(\omega_2 \psi_2)},$$

where, $\psi_i = x - q_i t + \varepsilon_i, q_i = (1+\beta)\nu - (3 + \theta_1 \nu)\omega_i$; $i = 1, 2$; $\omega_1 = \beta / \nu$, $\omega_2 = 2/\nu$ such that $\nu = 0.5(\theta_1 + \sqrt{\theta_1^2 + 8})$ and φ_1, φ_2 are arbitrary constants.

Case IV: *General Case:*

In a general case, migrations can take place due to both density-dependent and density-independent factors. The dynamics of a given population are then described by full (6) where now $\theta_0 \neq 0$ and $\theta_1 \neq 0$. The exact solution in this case exact solution:

$$u(x, t) = \frac{\beta \exp[\omega_1 (x - (q_1 + \theta_0)t + \varepsilon_1)] + \exp[\omega_2 (x - (q_2 + \theta_0)t + \varepsilon_2)]}{1 + \exp[\omega_1 (x - (q_1 + \theta_0)t + \varepsilon_1)] + \exp[\omega_2 (x - (q_2 + \theta_0)t + \varepsilon_2)]}.$$

where the notations are the same as in (32).

The figures 1-4, simulate the error between the exact solution and the both methods approximate solution of the above four cases respectively.

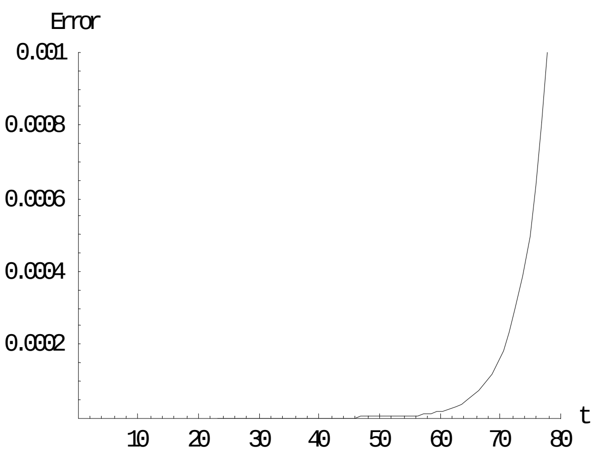

Figure 1: (Case I) The error at $x_0 = 20$ and $\theta_0 = \theta_1 = 0$.

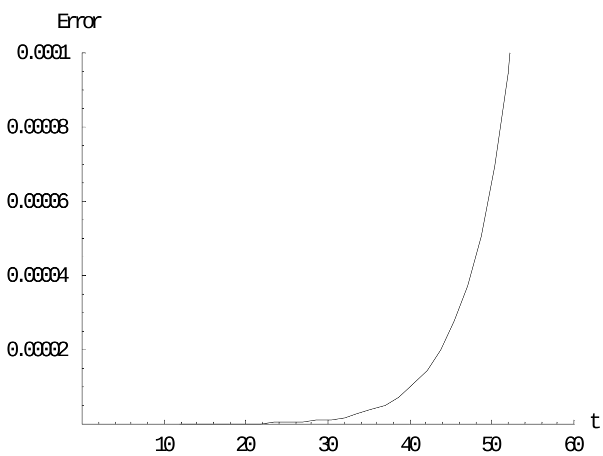

Figure 2: (Case II) The error at $x_0 = 65$ and $\theta_0 = 0.1$, $\theta_1 = 0$.

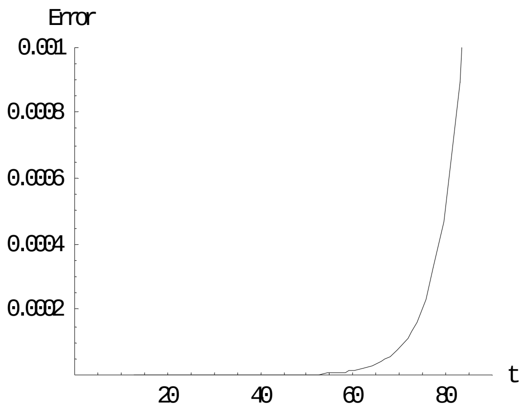

Figure 3: (Case III) The error at $x_0 = 50$ and $\theta_0 = 0$, $\theta_1 = 0.1$.

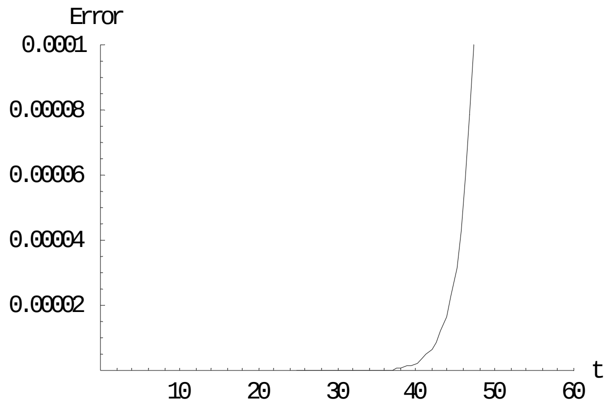

Figure 4: (Case IV) The error at $x_0 = 50$ and $\theta_0 = -\theta_1 = -0.1$.

5. Conclusion

In this paper, VIM and ADM are applied to solve the model of population dynamics with density-dependent migrations and the Allee effects, the methods need much less computational work compared with traditional methods. We achieved a very good approximation with the actual solution of the model by using two terms of the iteration scheme derived above in the ADM and VIM. It is evident that the overall results come very close to the exact solution even using only few terms of the iteration formula. Errors can be made smaller by taking new terms of the iteration formulas. It is found that these methods are always converges to the right solutions with high accuracy. We found that the variational iteration method can overcome the difficulties arising in calculation of Adomian's polynomials in Adomian decomposition method. Furthermore, VIM needs relative less computational work than ADM.

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