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# Bayesian inference for Pr $(Y < X)$ in the exponential distribution based on records

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ABSTRACT

We consider Bayesian estimation of the stress–strength reliability based on record values. The estimators are derived under the squared error loss function in the one parameter as well as two-parameter exponential distributions. The Bayes estimators are derived, in some cases in closed form, and their performance is investigated in terms of their bias and mean squared errors and compared with the maximum likelihood estimators. An illustrative example is given.



## 1. Introduction

Let $X_1, X_2, \ldots$ be an infinite sequence of *iid* random variables. An observation $X_j$ is called an upper record if it exceeds all previous observations. In this paper, we shall consider Bayesian inference for the stress–strength reliability $\Pr(Y < X)$ based on upper records (or simply records) when $X$ and $Y$ have exponential distributions. Here the random variable $Y$ denotes the strength while $X$ denotes the stress. In reliability engineering the system is still functioning as far as the stress does not exceed strength. The probability of this event is called the stress–strength reliability. Even though the name comes from engineering, the application of this model extends to other fields. For example in medical science when $Y$ and $X$ denote the lifetimes of patients under the control and treatment groups respectively, this probability is used as a measure of treatment effectiveness. Other applications and motivations for this model may be found in [1]. Baklizi [2] has considered this model and developed classical inference procedures for the stress–strength reliability in the one and two-parameter exponential distribution. In a subsequent publication, Baklizi [3] has considered classical and Bayesian inference for this model in the generalized exponential distribution. In this paper we shall consider Bayesian inference for $\Pr(Y < X)$ in the one and two-parameter exponential distribution assuming either common location parameters, common scale parameters or unrestricted scale and location parameters. We have derived the Bayes estimators of the stress–strength reliability in all these cases and compared them with the maximum likelihood estimators in terms of bias and mean squared error (MSE) using simulation under a variety of experimental conditions.

In section we consider the one parameter case while the two parameter case is treated in Section 3. In Section 4 we describe a detailed simulation study carried out to investigate and compare the performance of the derived Bayes estimators with the maximum likelihood estimator. An example is given in Section 5. Section 6 concludes the paper.

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## 2. Inference about Pr (Y<X) in the one parameter exponential distribution

There has been a lot of interest in the Bayesian estimation of the stress–strength reliability in the one parameter exponential distribution. Enis and Geisser [4] considered this problem with conjugate gamma priors for the parameters of the stress and strength distributions. The case of non-informative Jeffreys prior is treated in [1]. Jeevanad and Nair [5] have considered Bayesian estimation of the stress–strength reliability in the presence of spurious observations. Thompson and Basu [6] have derived the reference and non- informative prior distributions for the estimation of the stress–strength reliability.

All these contributions deal with ordinary random samples from the parent populations. In this section we consider estimation of the exponential stress–strength model based on record values. The probability density function of the one parameter exponential distribution is given by;

$$f(x|\sigma) = \frac{1}{\sigma} e^{-x/\sigma}, x > 0, \quad \sigma > 0. \tag{1}$$

Let $r_0, r_1, \ldots, r_n$ be record values observed from the distribution of $X \sim Exp(\sigma_1)$. Suppose also that $s_0, s_1, \ldots, s_m$ are the records observed from the distribution of $Y \sim Exp(\sigma_2)$ independently from the first sample. The stress–strength reliability can be shown to be equal to $\frac{\sigma_1}{\sigma_1+\sigma_2}$ [1]. The likelihood functions of $\sigma_1$ and $\sigma_2$ given $\underline{r} = (r_0, \ldots r_n)$ and $\underline{s} = (s_0, \ldots s_m)$ respectively are given by Arnold et al., [7];

$$L_1(\sigma_1|\underline{r}) = \frac{1}{\sigma_1^{n+1}} e^{-r_n/\sigma_1}, \quad 0 < r_0 < r_1 < \ldots < r_n, \tag{2}$$

$$L_2(\sigma_2|\underline{s}) = \frac{1}{\sigma_2^{m+1}} e^{-s_m/\sigma_2}, \quad 0 < s_0 < s_1 < \ldots < s_m, \tag{3}$$

The non-informative prior distributions for the scale parameters $\sigma_1$ and $\sigma_2$ are given respectively by (Carlin and Louis, 2000);

$$\pi_1(\sigma_1) = \frac{1}{\sigma_1}, \quad \sigma_1 > 0, \quad \pi_2(\sigma_2) = \frac{1}{\sigma_2}, \quad \sigma_2 > 0. \tag{4}$$

Hence the posterior distribution of $\sigma_1$ given $\underline{r}$ is given by;

$$\Pi_1^*(\sigma_1|\underline{r}) = \frac{(r_n)^{(n+1)} e^{-r_n/\sigma_1}}{\Gamma(n+1)\sigma_1^{(n+2)}}, \quad 0 < r_0 < r_1 < \ldots < r_n. \tag{5}$$

Similarly;

$$\Pi_2^*(\sigma_2|\underline{s}) = \frac{(s_m)^{(m+1)} e^{-s_m/\sigma_2}}{\Gamma(m+1)\sigma_2^{(m+2)}}, \quad 0 < s_0 < s_1 < \ldots < s_m. \tag{6}$$

The Bayes estimator is the posterior mean of $\theta = \Pr(Y < X)$ given by $E(\theta) = E\left(\frac{\sigma_1}{\sigma_1+\sigma_2}\right)$. It may be approached as follows; from (5) and (6) we see that $\sigma_1$ and $\sigma_2$ have inverted gamma posterior distributions. Using the relation between the gamma and inverted gamma distributions we have $r_n/\sigma_1 \sim Gamma(n,1)$ and $s_m/\sigma_2 \sim Gamma(m,1)$. It follows that $2r_n/\sigma_1 \sim \chi^2_{2n}$ and $2s_m/\sigma_2 \sim \chi^2_{2m}$. Assuming that $\sigma_1$ and $\sigma_2$ are independent, the two chi-square random variables are independent, therefore $\frac{m\sigma_2 r_n}{n\sigma_1 s_m} \sim F_{2n,2m}$. Note that the stress–strength reliability can be written as;

$$\theta = \frac{\sigma_1}{\sigma_1+\sigma_2} = \frac{1}{1+\sigma_2/\sigma_1} = \frac{1}{1+(ns_m/mr_n)m\sigma_2 r_n/n\sigma_1 s_m}.$$

This shows that the distribution of $\theta$ is that of $\frac{1}{1+(ns_m/mr_n)W_1}$ where $W_1 \sim F_{2n,2m}$. This fact can be used to find the posterior mean, perhaps by numerical methods. An approximate, closed form, expression for the posterior mean of $\theta$ can be found using the formulae given in [8]. Applying these formulae we obtained;

$$E(\theta) \cong \left(1+\frac{n^2 s_m}{m(n-1)r_n}\right)^{-1} + \frac{n^2(n+m-1)}{m(n-1)(n-2)}\left(\frac{ns_m}{mr_n}\right)^2\left(1+\frac{n^2 s_m}{m(n-1)r_n}\right)^{-3}. \tag{7}$$

If one is interested in an equal tailed $(1-\alpha)$ interval for $\theta$ then the bounds of the intervals are;

$$\left\{\frac{1}{1+(ns_m/mr_n)F_{1-\alpha/2,2n,2m}}, \frac{1}{1+(ns_m/mr_n)F_{\alpha/2,2n,2m}}\right\}, \tag{8}$$

where $F_{\gamma,2n,2m}$ is the $\gamma^{th}$ quantile of the $F_{2n,2m}$ distribution.

Sometimes interest is in the use of the conjugate family of prior distributions for $\sigma_1$ and $\sigma_2$. It is clear that the conjugate family of distributions is the inverted Gamma family of distributions. The conjugate prior distribution for $\sigma_i, \ i = 1,2$ is therefore given by;

$$\pi_i(\sigma_i) = \frac{\beta_i^{\alpha_i} e^{-\beta_i/\sigma_i}}{\Gamma(\alpha_i)\sigma_i^{\alpha_i+1}}, \quad \sigma_i > 0, \tag{9}$$

where $\alpha_i$ and $\beta_i$ are parameters of the prior distribution of $\sigma_i$. It follows that the posterior distribution of $\sigma_1$ given $\underline{r}$ is

$$\Pi_1^*(\sigma_1|\underline{r}) = \frac{(\beta_1 + r_n)^{\alpha_1+n+1} e^{-\frac{(\beta_1+r_n)}{\sigma_1}}}{\Gamma(\alpha_1+n+1)\sigma_1^{(\alpha_1+n+2)}}, \quad \sigma_1 > 0. \tag{10}$$

Similarly the posterior distribution of $\sigma_2$ is given by

$$\Pi_2^*(\sigma_2|\underline{s}) = \frac{(\beta_2 + s_m)^{\alpha_2+m+1} e^{-\frac{(\beta_2+s_m)}{\sigma_2}}}{\Gamma(\alpha_2+m+1)\sigma_2^{(\alpha_2+m+2)}}, \quad \sigma_2 > 0. \tag{11}$$

Note that

$$(r_n + \beta_1)/\sigma_1|\underline{r} \sim Gamma(n+\alpha_1, 1), \tag{12}$$

$$(s_m + \beta_2)/\sigma_2|\underline{s} \sim Gamma(m+\alpha_2, 1). \tag{13}$$

It follows that $2(r_n + \beta_1)/\sigma_1 \sim \chi^2_{2(n+\alpha_1)}$ and $2(s_m + \beta_2)/\sigma_2 \sim \chi^2_{2(m+\alpha_2)}$. Therefore, and since $\sigma_1$ and $\sigma_2$ are assumed independent; $\frac{(m+\alpha_2)(r_n+\beta_1)\sigma_2}{(n+\alpha_1)(s_m+\beta_2)\sigma_1} \sim F_{2(n+\alpha_1),2(m+\alpha_2)}$. In this case we can write;

$$\theta = \frac{\sigma_1}{\sigma_1+\sigma_2} = \frac{1}{1+\sigma_2/\sigma_1} = \frac{1}{1 + \frac{(n+\alpha_1)(s_m+\beta_2)}{(m+\alpha_2)(r_n+\beta_1)} \frac{(m+\alpha_2)(r_n+\beta_1)\sigma_2}{(n+\alpha_1)(s_m+\beta_2)\sigma_1}}.$$

This shows that the distribution of $\theta$ is that of $\frac{1}{1+\frac{(n+\alpha_1)(s_m+\beta_2)}{(m+\alpha_2)(r_n+\beta_1)}W_2}$ where $W_2 \sim F_{2(n+\upsilon_1),2(m+\upsilon_2)}$. An equal tailed Bayesian interval for $\theta$ can be obtained as in (8). Let $k = \left(1 + \frac{(n+\alpha_1)^2(s_m+\beta_2)}{(m+\alpha_2)(n+\alpha_1-1)(r_n+\beta_1)}\right)^{-1}$. An approximate, closed form, expression for the posterior mean of $\theta$ using the formulae in [8] is given by;

$$E(\theta) \cong k + \frac{(n+\alpha_1)^2(n+\alpha_1+m+\alpha_2-1)}{(m+\alpha_2)(n+\alpha_1-1)(n+\alpha_1-2)}\left(\frac{(n+\alpha_1)(s_m+\beta_2)}{(m+\alpha_2)(r_n+\beta_1)}\right)^2 k^3. \tag{14}$$

## 3. Inference for Pr (X <Y) in the two-parameter exponential distribution

The probability density function of the two-parameter exponential distribution is given by;

$$f(x|\mu,\sigma) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, x > \mu, \quad -\infty < \mu < \infty, \quad \sigma > 0. \tag{15}$$

Let $r_0, r_1, \ldots, r_n$ be record values observed from $Exp(\mu_1, \sigma_1)$, and $s_0, s_1, \ldots, s_m$ be the records observed from $Exp(\mu_2, \sigma_2)$ independently from the first sample. Using standard arguments, one can show that [9,10];

$$\theta = \Pr(Y < X) = \begin{cases} \frac{\sigma_1/\sigma_2}{\sigma_1/\sigma_2+1} e^{-(\mu_2-\mu_1)/\sigma_1}, & \mu_2 - \mu_1 \geqslant 0 \\ 1 - \frac{1}{\sigma_1/\sigma_2+1} e^{-(\mu_1-\mu_2)/\sigma_2}, & \mu_2 - \mu_1 < 0 \end{cases}. \tag{16}$$

In the next subsections we shall consider the cases when the parameters are unrestricted and when either the location parameters are equal or the scale parameters are equal.

### 3.1. The case of unrestricted scale and location parameters

When there are no restrictions on the scale or location parameters and with the independence between samples assumption, the likelihood function of $(\mu_1, \mu_2, \sigma_1, \sigma_2)$ is given by;

$$L(\mu_1, \mu_2, \sigma_1, \sigma_2|\underline{r}, \underline{s}) = \frac{1}{\sigma^{n+m+2}} e^{-(r_n-\mu_1)/\sigma_1 - (s_m-\mu_2)/\sigma_2}, \mu_1 < r_0, \mu_2 < s_0, \sigma_1, \sigma_2 > 0. \tag{17}$$

The non-informative prior distributions of the location parameters are;

$$\pi(\mu_i) = 1, \quad -\infty < \mu_i < \infty. \tag{18}$$

For the corresponding scale parameters we take the conjugate priors;

$$\pi_i(\sigma_i|\mu) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)\sigma_i^{\alpha_i+1}} e^{-\beta_i/\sigma_i}, \sigma_i > 0, \quad i = 1, 2. \tag{19}$$

The posterior distribution of $(\mu_1, \mu_2, \sigma_1, \sigma_2)$ is therefore given by;

$$\Pi^*(\mu_1, \mu_2, \sigma_1, \sigma_2|\underline{r}, \underline{s}) = c_1 \frac{1}{\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu_1+\beta_1)/\sigma_1} \frac{1}{\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu_2+\beta_2)/\sigma_2} \quad \mu_1 < r_0, \mu_2 < s_0, \quad \sigma_1, \sigma_2 > 0. \tag{20}$$

The proportionality constant is given by;

$$\begin{aligned} c_1^{-1} &= \int_{-\infty}^{s_0}\int_{-\infty}^{r_0}\int_0^{\infty}\int_0^{\infty} \frac{1}{\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu_1+\beta_1)/\sigma_1} \frac{1}{\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu_2+\beta_2)/\sigma_2} d\sigma_1 d\sigma_2 d\mu_1 d\mu_2 \\ &= \int_{-\infty}^{r_0}\int_0^{\infty} \frac{1}{\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu_1+\beta_1)/\sigma_1} d\sigma_1 d\mu_1 \int_{-\infty}^{s_0}\int_0^{\infty} \frac{1}{\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu_2+\beta_2)/\sigma_2} d\sigma_2 d\mu_2 \\ &= \int_{-\infty}^{r_0} \frac{\Gamma(n+\alpha_1+1)}{(r_n-\mu_1+\beta_1)^{n+\alpha_1+1}} d\mu_1 \int_{-\infty}^{s_0} \frac{\Gamma(m+\alpha_2+1)}{(s_m-\mu_2+\beta_2)^{m+\alpha_2+1}} d\mu_2 = \frac{\Gamma(n+\alpha_1)}{(r_n-r_0+\beta_1)^{n+\alpha_1}} \frac{\Gamma(m+\alpha_2)}{(s_m-s_0+\beta_2)^{m+\alpha_2}}. \end{aligned} \tag{21}$$

Therefore the posterior distribution can be written as;

$$\Pi^*(\mu_1, \mu_2, \sigma_1, \sigma_2|\underline{r}, \underline{s}) = \frac{(r_n-r_0+\beta_1)^{n+\alpha_1}}{\Gamma(n+\alpha_1)\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu_1+\beta_1)/\sigma_1} \frac{(s_m-s_0+\beta_2)^{m+\alpha_2}}{\Gamma(m+\alpha_2)\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu_2+\beta_2)/\sigma_2} \quad \mu_1 < r_0, \mu_2 < s_0, \quad \sigma_1, \sigma_2 > 0. \tag{22}$$

The stress–strength reliability is given by;

$$\theta = \Pr(Y < X) = \begin{cases} \frac{\sigma_1/\sigma_2}{\sigma_1/\sigma_2+1} e^{-(\mu_2-\mu_1)/\sigma_1}, & \mu_2 - \mu_1 \geqslant 0 \\ 1 - \frac{1}{\sigma_1/\sigma_2+1} e^{-(\mu_1-\mu_2)/\sigma_2}, & \mu_2 - \mu_1 < 0 \end{cases},$$

which can be written as;

$$\theta = \frac{\sigma_1}{\sigma_1+\sigma_2} e^{-(\mu_2-\mu_1)/\sigma_1} I_{\mu_2 \geqslant \mu_1} + \left(1 - \frac{\sigma_2}{\sigma_1+\sigma_2} e^{-(\mu_1-\mu_2)/\sigma_2}\right) I_{\mu_2<\mu_1} \tag{23}$$

The Bayes estimator of the stress–strength reliability is given by the posterior expectation;

$$\begin{aligned} E(\theta) = c_1^{-1} &\int_{-\infty}^{s_0}\int_{-\infty}^{r_0}\int_0^{\infty}\int_0^{\infty} \frac{1}{\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu_1+\beta_1)/\sigma_1} \frac{1}{\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu_2+\beta_2)/\sigma_2} \\ &\times \left\{ \frac{\sigma_1}{\sigma_1+\sigma_2} e^{-(\mu_2-\mu_1)/\sigma_1} I_{\mu_2\geqslant\mu_1} + \left(1 - \frac{\sigma_2}{\sigma_1+\sigma_2} e^{-(\mu_1-\mu_2)/\sigma_2}\right) I_{\mu_2<\mu_1} \right\} d\mu_1 d\mu_2 d\sigma_1 d\sigma_2. \end{aligned} \tag{24}$$

It seems that the above integrals do not have closed form, however they can be computed numerically. Alternatively we can use Monte Carlo methods as follows; the posterior distribution can be written as the product of the following two independent posteriors;

$$\Pi_1^*(\mu_1, \sigma_1|\underline{r}) = \frac{(n+\alpha_1)(r_n-r_0+\beta_1)^{n+\alpha_1}}{\Gamma(n+\alpha_1+1)\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu_1+\beta_1)/\sigma_1}, \quad \mu < r_0, \sigma_1 > 0,$$

$$\Pi_2^*(\mu_2, \sigma_2|\underline{s}) = \frac{(m+\alpha_2)(s_m-s_0+\beta_2)^{m+\alpha_2}}{\Gamma(m+\alpha_2+1)\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu_2+\beta_2)/\sigma_2}, \quad \mu < s_0, \sigma_2 > 0.$$

It follows that the conditional distributions of $\sigma_i$ given $\mu_i$ are given by;

$$\Pi_1^*(\sigma_1|\mu_1, \underline{r}) = \frac{1}{\Gamma(n+\alpha_1+1)\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu_1+\beta_1)/\sigma_1}, \quad \mu < r_0, \quad \sigma_1 > 0, \tag{25}$$

$$\Pi_2^*(\sigma_2|\mu_2, \underline{s}) = \frac{1}{\Gamma(m+\alpha_2+1)\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu_2+\beta_2)/\sigma_2}, \quad \mu < s_0, \sigma_2 > 0. \tag{26}$$

With marginal distributions of $\mu_i$ given by;

$$\Pi_1^*(\mu_1|\underline{r}) = \frac{(n+\alpha_1)(r_n-r_0+\beta_1)^{n+\alpha_1}}{(r_n-\mu_1+\beta_1)^{n+\alpha_1+1}}, \quad \mu_1 < r_0, \tag{27}$$

$$\Pi_2^*(\mu_2|\underline{s}) = \frac{(m+\alpha_2)(s_m - s_0 + \beta_2)^{m+\alpha_2}}{(s_m - \mu_2 + \beta_2)^{m+\alpha_2+1}}, \quad \mu_2 < s_0. \tag{28}$$

Therefore we can directly generate $\mu_{1,1}$ and $\mu_{2,1}$ using the marginal posteriors given in (27) and (28) respectively. Using these values we can generate $\sigma_{1,1}$ and $\sigma_{2,1}$ directly using the conditional posteriors given in (25) and (26). Using these values we can calculate $\theta_1$ using (23). The posterior mean of $\theta$ can be estimated accurately by repeating this procedure a sufficient number of times, say $M$, and then using the mean of $\theta_1, \ldots, \theta_M$ as an estimate of $\theta$. This procedure will be used in the simulation study later to investigate and compare the performance of the Bayes estimator with the maximum likelihood estimator obtained by inserting the MLEs of $(\mu_1, \mu_2, \sigma_1, \sigma_2)$ in (23) and utilizing the invariance property of maximum likelihood estimators. These MLEs are given by $\hat{\mu}_1 = r_0,\ \hat{\mu}_2 = s_0,\ \hat{\sigma}_1 = (r_n - r_0)/n + 1$ and $\hat{\sigma}_2 = (s_m - s_0)/m + 1$ [7].

### 3.2. The case of common location parameters

Bai and Hong [11] discussed point and interval estimation of $\theta = \Pr(Y < X)$. They derived two types of approximate intervals. Later, Baklizi [12] considered several types of asymptotic, approximate and bootstrap intervals and compared them using simulation techniques. Baklizi and El-Masri [13] considered shrinkage estimation of the stress–strength reliability in the two-parameter exponential distribution with common location while [14] constructed tests and confidence intervals based on the generalized variable approach. These efforts are all based on ordinary simple random samples from the stress and strength variables. Here we will discuss inference based on record values. Assuming that $\mu_1 = \mu_2 = \mu$, the expression for $\theta$ in this case reduces from (16) to;

$$\theta = \Pr(Y < X) = \frac{\sigma_1}{\sigma_1 + \sigma_2}. \tag{29}$$

Let $r_0, r_1, \ldots, r_n$ be upper record values observed from $Exp(\mu, \sigma_1)$ and let $s_0, s_1, \ldots, s_m$ be the upper records observed from $Exp(\mu, \sigma_2)$ independently from the first sample. The likelihood function of $(\mu, \sigma_1, \sigma_2)$ is given by;

$$L(\mu, \sigma_1, \sigma_2|\underline{r}, \underline{s}) = \frac{1}{\sigma_1^{n+1}} e^{-(r_n-\mu)/\sigma_1} \frac{1}{\sigma_2^{m+1}} e^{-(s_m-\mu)/\sigma_2}, \min(r_0, s_0) > \mu, \quad \sigma_1, \sigma_2 > 0. \tag{30}$$

The non-informative prior distribution of the location parameter is given by;

$$\pi(\mu) = 1, \quad -\infty < \mu < \infty. \tag{31}$$

The conjugate prior distributions of the scale parameters are;

$$\pi_i(\sigma_i|\mu) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)\sigma_i^{\alpha_i+1}} e^{-\beta_i/\sigma_i}, \quad \sigma_i > 0, \quad i = 1, 2. \tag{32}$$

Using standard Bayesian arguments, the posterior distribution of $(\mu, \sigma_1, \sigma_2)$ is given by;

$$\Pi^*(\mu, \sigma_1, \sigma_2|\underline{r}, \underline{s}) = c_2 \frac{1}{\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu+\beta_1)/\sigma_1} \frac{1}{\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu+\beta_2)/\sigma_2}, \tag{33}$$

where

$$\begin{aligned} c_2^{-1} &= \int_{-\infty}^{\min(r_0,s_0)} \int_0^\infty \int_0^\infty \frac{1}{\sigma_1^{n+\alpha_1+2}} e^{-(r_n-\mu+\beta_1)/\sigma_1} \frac{1}{\sigma_2^{m+\alpha_2+2}} e^{-(s_m-\mu+\beta_2)/\sigma 2} d\sigma_1 d\sigma_2 d\mu \\ &= \int_{-\infty}^{\min(r_0,s_0)} \frac{\Gamma(n+\alpha_1+1)}{(r_n-\mu+\beta_1)^{(n+\alpha_1+1)}} \frac{\Gamma(m+\alpha_2+1)}{(s_m-\mu+\beta_2)^{(m+\alpha_2+1)}} d\mu. \end{aligned} \tag{34}$$

The Bayes estimator of the stress–strength reliability is given by;

$$E(\theta) = E\left(\frac{\sigma_1}{\sigma_1 + \sigma_2}\right). \tag{35}$$

It appears that a closed form expression for this posterior expectation is not possible, therefore we need to approximate or estimate it. A possibility is to estimate it using Monte Carlo methods. To generate observations from this complicated posterior we consider the following approach. The conditional posterior probability distributions of the scale parameters $\sigma_1$ and $\sigma_2$ given the common location parameter $\mu$ and the data are given by;

$$\Pi_1^*(\sigma_1|\mu, \underline{r}, \underline{s}) = \frac{(\beta_1 + r_n - \mu)^{n+\alpha_1+1} e^{-(\beta_1+r_n-\mu)/\sigma_1}}{\Gamma(n+\alpha_1+1)\sigma_1^{\alpha_1+n+2}}, \quad \sigma_1 > 0, \tag{36}$$

$$\Pi_2^*(\sigma_2|\mu, \underline{r}, \underline{s}) = \frac{(\beta_2 + s_m - \mu)^{\alpha_2+m+1} e^{-(\beta_2+s_m-\mu)/\sigma 2}}{\Gamma(m+\alpha_2+1)\sigma_2^{\alpha_2+m+2}}, \quad \sigma_2 > 0. \tag{37}$$

It follows that the posterior distribution of $\mu$ satisfies

$$\Pi^*(\mu|\underline{r},\underline{s}) \propto (\beta_1 + r_n - \mu)^{-(n+\alpha_1+1)}(\beta_2 + s_m - \mu)^{-(m+\alpha_2+1)}, \quad \mu < \min(r_0, s_0). \tag{38}$$

The Bayes estimates may be obtained (estimated) using the following Monte Carlo algorithm;

(1) Generate $\mu_1$ from $\Pi^*(\mu|\underline{r},\underline{s})$ given by (38) using some version of the Metropolis algorithm.
(2) Generate $\sigma_{1,1}$ and $\sigma_{2,1}$ from $\Pi_1^*(\sigma_1|\mu_1,\underline{r},\underline{s})$ and $\Pi_2^*(\sigma_2|\mu_1,\underline{r},\underline{s})$ given by (36) and (37) respectively and calculate $\theta_1 = \frac{\sigma_{1,1}}{\sigma_{1,1}+\sigma_{2,1}}$.
(3) Repeat steps 1 and 2 $(M)$ times to get $\theta_1,\ldots,\theta_M$.
(4) Calculate $\hat{E}(\theta|\underline{r},\underline{s}) = \sum_{i=1}^{M}\theta_i/M = \bar{\theta}$.

If we consider the non-informative prior distributions for the scale parameters given by;

$$\pi_i(\sigma_i) = \frac{1}{\sigma_i}, \quad \sigma_i > 0, \quad i = 1,2. \tag{39}$$

The posterior distribution will be obtained in a similar manner as before, specifically;

$$\Pi^*(\mu,\sigma_1,\sigma_2|\underline{r},\underline{s}) \propto \frac{1}{\sigma_1^{n+2}}e^{-(r_n-\mu)/\sigma_1}\frac{1}{\sigma_2^{m+2}}e^{-(s_m-\mu)/\sigma_2}. \tag{40}$$

The conditional posterior probability distributions of the scale parameters $\sigma_1$ and $\sigma_2$ given the common location parameter $\mu$ and the data are given by;

$$\Pi_1^*(\sigma_1|\mu,\underline{r},\underline{s}) = \frac{(r_n-\mu)^{n+1}e^{-(r_n-\mu)/\sigma_1}}{\Gamma(n+1)\sigma_1^{n+2}}, \quad \sigma_1 > 0, \tag{41}$$

$$\Pi_2^*(\sigma_2|\mu,\underline{r},\underline{s}) = \frac{(s_m-\mu)^{m+1}e^{-(s_m-\mu)/\sigma 2}}{\Gamma(m+1)\sigma_2^{m+2}}, \quad \sigma_2 > 0. \tag{42}$$

It follows that the posterior distribution of $\mu$ satisfies

$$\Pi^*(\mu|\underline{r},\underline{s}) \propto (r_n - \mu)^{-(n+1)}(s_m - \mu)^{-(m+1)}, \quad \mu < \min(r_0, s_0). \tag{43}$$

Approximate Bayes estimates may be obtained using the same Monte Carlo algorithm given before. The proportionality constant in (43) is very complicated and difficult to obtain. In order to generate values of the $\mu$ from (43), we need to use an algorithm which did not require the knowledge of the proportionality constant. The "asymmetric" proposal distribution used here is;

$$q(\mu) = n(r_n - \mu)^{-(n+1)}, \quad \mu < r_n.$$

This distribution is similar in shape to the target distribution. It has heavier tails than the target distribution and is easy to generate samples from it. Note that the proposal distribution does not depend the current value of $\mu^{(t)}$ hence we have the "Independence Sampler". The algorithm is as follows;

(1) Set $t = 0$. Choose a starting value $\mu^{(0)}$, which is an arbitrary point satisfying $\Pi^*(\mu^{(0)}|\underline{r},\underline{s}) > 0$.
(2) Generate $\mu_{new}$ using the proposal distribution $q(\mu)$.
(3) Calculate the quantity $r = \min\left(\frac{\Pi^*(\mu_{new}|\underline{r},\underline{s})/q(\mu_{new})}{\Pi^*(\mu^{(t)}|\underline{r},\underline{s})/q(\mu^{(t)})}, 1\right)$.
(4) Generate $u$ from the uniform distribution $U(0,1)$.
(5) Set $\mu^{(t+1)} = \mu_{new}$ if $u < r$, otherwise set $\mu^{(t+1)} = \mu^{(t)}$.
(6) Repeat steps 2 – 5 until we obtain the desired number of samples.

In the simulations we will investigate and compare this estimator with the maximum likelihood estimator obtained by inserting the MLEs of $(\mu,\sigma_1,\sigma_2)$ in 29. These MLEs are given by; $\hat{\mu} = \min(r_0,s_0)$, $\hat{\sigma}_1 = (r_n - \hat{\mu})/n + 1$ and $\hat{\sigma}_2 = (s_m - \hat{\mu})/m + 1$.

### *3.3. The case of a common scale parameter*

Gupta and Gupta [15] obtained the maximum likelihood estimator (MLE), the minimum variance unbiased estimator (MVUE), and a Bayes estimator of $\theta$ in case of different location parameters and a common scale parameter. Ivshin [16] considered this problem with known scale parameters and obtained the MVUE of the stress–strength reliability and the MVUE of the variance of the reliability estimator. The above authors considered simple random samples, here we will consider the case of records.

Assume that $\sigma_1 = \sigma_2 = \sigma$, the expression for $\theta = \Pr(X < Y)$ reduces to;

$$\theta = \Pr(X < Y) = \begin{cases} 1 - \frac{1}{2}e^{-(\mu_2-\mu_1)/\sigma}, & \mu_2 - \mu_1 \geqslant 0 \\ \frac{1}{2}e^{-(\mu_1-\mu_2)/\sigma}, & \mu_2 - \mu_1 < 0 \end{cases}. \tag{44}$$

Let $r_0, r_1, \ldots, r_n$ be the record values observed from $Exp(\mu_1, \sigma)$, the two-parameter exponential distribution. Suppose that $s_0, s_1, \ldots, s_m$ are the records observed from $Exp(\mu_2, \sigma)$ independently from the first sample. The likelihood function of $(\mu_1, \mu_2, \sigma)$ is given by;

$$L(\mu_1, \mu_2, \sigma | \underline{r}, \underline{s}) = \frac{1}{\sigma^{n+m+2}} e^{-(r_n - \mu_1 + s_m - \mu_2)/\sigma}, \mu_1 < r_0, \mu_2 < s_0, \quad \sigma > 0. \tag{45}$$

Assuming constant prior distributions of the location parameters;

$$\pi_i(\mu_i) = 1, \quad -\infty < \mu_i < \infty, \quad i = 1, 2. \tag{46}$$

The conjugate prior distribution of the scale parameter is given by;

$$\pi(\sigma | \mu_1, \mu_2) = \frac{\beta^\alpha}{\Gamma(\alpha)\sigma^{\alpha+1}} e^{-\beta/\sigma}, \quad \sigma > 0. \tag{47}$$

This gives

$$\pi(\mu_1, \mu_2, \sigma) = \frac{\beta^\alpha}{\Gamma(\alpha)\sigma^{\alpha+1}} e^{-\beta/\sigma}, -\infty < \mu_1, \mu_2 < \infty, \quad \sigma > 0. \tag{48}$$

Combining the likelihood function with this prior distribution we obtain the posterior probability distribution of $(\mu_1, \mu_2, \sigma)$ as;

$$\Pi^*(\mu_1, \mu_2, \sigma | \underline{r}, \underline{s}) \propto \frac{e^{-(\beta + r_n - \mu_1 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+3}}, \quad \mu_1 < r_0, \quad \mu_2 < s_0, \quad \sigma > 0. \tag{49}$$

The constant of proportionality is given by;

$$c_3^{-1} = \int_0^\infty \int_{-\infty}^{s_0} \int_{-\infty}^{r_0} \frac{e^{-(\beta + r_n - \mu_1 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+3}} d\mu_1 d\mu_2 d\sigma = \int_0^\infty \int_{-\infty}^{s_0} \frac{e^{-(\beta + r_n - r_0 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+2}} d\mu_2 d\sigma = \int_0^\infty \frac{e^{-(\beta + r_n - r_0 + s_m - s_0)/\sigma}}{\sigma^{\alpha+n+m+1}} d\sigma$$
$$= \frac{\Gamma(\alpha + n + m)}{(\beta + r_n - r_0 + s_m - s_0)^{(\alpha+n+m)}}.$$

Therefore

$$\Pi(\mu_1, \mu_2, \sigma | \underline{r}, \underline{s}) = \frac{(\beta + r_n - r_0 + s_m - s_0)^{(\alpha+n+m)}}{\Gamma(\alpha + n + m)} \frac{e^{-(\beta + r_n - \mu_1 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+3}}, \quad \mu_1 < r_0, \quad \mu_2 < s_0, \quad \sigma > 0. \tag{50}$$

The Bayes estimator $\hat{\theta}$ is the posterior expectation of $\theta = (1 - \frac{1}{2}e^{-(\mu_2-\mu_1)/\sigma}) I_{\mu_2 > \mu_1} + \frac{1}{2}e^{-(\mu_1-\mu_2)/\sigma} I_{\mu_2 < \mu_1}$. This expectation is given by;

$$\hat{\theta} = \int_0^\infty \int_{-\infty}^{s_0} \int_{-\infty}^{r_0} \left( \left(1 - \frac{1}{2}e^{-(\mu_2 - \mu_1/\sigma)}\right) I_{\mu_2 > \mu_1} + \frac{1}{2}e^{-(\mu_1-\mu_2)/\sigma} I_{\mu_2 < \mu_1} \right) c_3 \frac{e^{-(\beta + r_n - \mu_1 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+3}} d\mu_1 d\mu_2 d\sigma, \tag{51}$$

Note that this expression can be written as;

$$\hat{\theta} = \int_0^\infty \int_{-\infty}^{s_0} \int_{-\infty}^{r_0} I_{\mu_2 > \mu_1} c_3 \frac{e^{-(\beta + r_n - \mu_1 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+3}} d\mu_1 d\mu_2 d\sigma - \int_0^\infty \int_{-\infty}^{s_0} \int_{-\infty}^{r_0} \frac{1}{2} e^{-(\mu_2-\mu_1)/\sigma} I_{\mu_2 > \mu_1} c_3 \frac{e^{-(\beta + r_n - \mu_1 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+3}} d\mu_1 d\mu_2 d\sigma$$
$$+ \int_0^\infty \int_{-\infty}^{s_0} \int_{-\infty}^{r_0} \frac{1}{2} e^{-(\mu_1-\mu_2)/\sigma} I_{\mu_2 < \mu_1} c_3 \frac{e^{-(\beta + r_n - \mu_1 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+3}} d\mu_1 d\mu_2 d\sigma$$
$$= c_3 \left( I_1 - \frac{1}{2}(I_2 - I_3) \right),$$

where;

$$I_1 = \int_0^\infty \int_{-\infty}^{s_0} \int_{-\infty}^{r_0} I_{\mu_2 > \mu_1} \frac{e^{-(\beta + r_n - \mu_1 + s_m - \mu_2)/\sigma}}{\sigma^{\alpha+n+m+3}} d\mu_1 d\mu_2 d\sigma, \quad I_2 = \int_0^\infty \int_{-\infty}^{s_0} \int_{-\infty}^{r_0} I_{\mu_2 > \mu_1} \frac{e^{-(\beta + r_n - 2\mu_1 + s_m)/\sigma}}{\sigma^{\alpha+n+m+3}} d\mu_1 d\mu_2 d\sigma,$$

$$I_3 = \int_0^\infty \int_{-\infty}^{s_0} \int_{-\infty}^{r_0} I_{\mu_2 < \mu_1} \frac{e^{-(\beta + r_n - 2\mu_2 + s_m)/\sigma}}{\sigma^{\alpha+n+m+3}} d\mu_1 d\mu_2 d\sigma.$$

We have two cases to consider;

*Case 1*: $r_0 < s_0$.

The integrals are obtained as follows;

$$I_1=\int_0^\infty\int_{-\infty}^{r_0}\int_{\mu_1}^{s_0}\frac{e^{-\left(\beta+r_n-\mu_1+s_m-\mu_2\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_2 d\mu_1 d\sigma=\int_0^\infty\int_{-\infty}^{r_0}\frac{e^{-\left(\beta+r_n-\mu_1+s_m-s_0\right)/\sigma}}{\sigma^{\alpha+n+m+2}}-\frac{e^{-\left(\beta+r_n-2\mu_1+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+2}}d\mu_1 d\sigma$$
$$=\int_0^\infty\frac{e^{-(\beta+r_n-r_0+s_m-s_0)/\sigma}}{\sigma^{\alpha+n+m+1}}-\frac{e^{-(\beta+r_n-2r_0+s_m)/\sigma}}{2\sigma^{\alpha+n+m+1}}d\sigma=\frac{\Gamma(\alpha+n+m)}{(\beta+r_n-r_0+s_m-s_0)^{\alpha+n+m}}-\frac{\Gamma(\alpha+n+m)}{2(\beta+r_n-2r_0+s_m)^{\alpha+n+m}},$$

$$I_2=\int_0^\infty\int_{-\infty}^{r_0}\int_{\mu_1}^{s_0}\frac{e^{-\left(\beta+r_n-2\mu_1+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_2 d\mu_1 d\sigma=\int_0^\infty\int_{-\infty}^{r_0}(s_0-\mu_1)\frac{e^{-\left(\beta+r_n-2\mu_1+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_1 d\sigma$$
$$=\int_0^\infty(s_0-r_0)\frac{e^{-(\beta+r_n-2r_0+s_m)/\sigma}}{2\sigma^{\alpha+n+m+2}}+\frac{e^{-(\beta+r_n-2r_0+s_m)/\sigma}}{4\sigma^{\alpha+n+m+1}}d\sigma=\frac{(s_0-r_0)\Gamma(\alpha+n+m+1)}{2(\beta+r_n-2r_0+s_m)^{\alpha+n+m+1}}+\frac{\Gamma(\alpha+n+m)}{4(\beta+r_n-2r_0+s_m)^{\alpha+n+m}}.$$

$$I_3=\int_0^\infty\int_{-\infty}^{r_0}\int_{-\infty}^{\mu_1}\frac{e^{-\left(\beta+r_n-2\mu_2+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_2 d\mu_1 d\sigma=\int_0^\infty\int_{-\infty}^{r_0}\frac{e^{-\left(\beta+r_n-2\mu_1+s_m\right)/\sigma}}{2\sigma^{\alpha+n+m+2}}d\mu_1 d\sigma=\int_0^\infty\frac{e^{-(\beta+r_n-2r_0+s_m)/\sigma}}{4\sigma^{\alpha+n+m+1}}d\sigma$$
$$=\frac{\Gamma(\alpha+n+m)}{4(\beta+r_n-2r_0+s_m)^{(\alpha+n+m)}}.$$

Therefore;

$$\hat{\theta}=\frac{(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m}}{\Gamma(\alpha+n+m)}\left\{\begin{array}{l}\frac{\Gamma(\alpha+n+m)}{(\beta+r_n-r_0+s_m-s_0)^{\alpha+n+m}}-\frac{\Gamma(\alpha+n+m)}{2(\beta+r_n-2r_0+s_m)^{\alpha+n+m}}\\-\frac{1}{2}\left(\frac{(s_0-r_0)\Gamma(\alpha+n+m+1)}{2(\beta+r_n-2r_0+s_m)^{\alpha+n+m+1}}\right)\end{array}\right\}$$
$$=1-\frac{(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{2(\beta+r_n-2r_0+s_m)^{\alpha+n+m}}-\frac{(\alpha+n+m)(s_0-r_0)(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{4(\beta+r_n-2r_0+s_m)^{\alpha+n+m+1}}. \quad (52)$$

*Case 2*: $s_0<r_0$

$$I_1=\int_0^\infty\int_{-\infty}^{s_0}\int_{\mu_1}^{s_0}\frac{e^{-\left(\beta+r_n-\mu_1+s_m-\mu_2\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_2 d\mu_1 d\sigma=\int_0^\infty\int_{-\infty}^{s_0}\frac{e^{-\left(\beta+r_n-\mu_1+s_m-s_0\right)/\sigma}}{\sigma^{\alpha+n+m+2}}-\frac{e^{-\left(\beta+r_n-2\mu_1+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+2}}d\mu_1 d\sigma$$
$$=\int_0^\infty\left(\frac{e^{-(\beta+r_n-2s_0+s_m)/\sigma}}{\sigma^{\alpha+n+m+1}}-\frac{e^{-(\beta+r_n-2s_0+s_m)/\sigma}}{2\sigma^{\alpha+n+m+1}}\right)d\sigma=\int_0^\infty\frac{e^{-(\beta+r_n-2s_0+s_m)/\sigma}}{2\sigma^{\alpha+n+m+1}}d\sigma=\frac{\Gamma(\alpha+n+m)}{2(\beta+r_n-2s_0+s_m)^{\alpha+n+m}},$$

$$I_2=\int_0^\infty\int_{-\infty}^{s_0}\int_{\mu_1}^{s_0}\frac{e^{-\left(\beta+r_n-2\mu_1+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_2 d\mu_1 d\sigma=\int_0^\infty\int_{-\infty}^{s_0}(s_0-\mu_1)\frac{e^{-\left(\beta+r_n-2\mu_1+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_1 d\sigma$$
$$=\int_0^\infty\int_{-\infty}^{s_0}\frac{e^{-\left(\beta+r_n-2\mu_1+s_m\right)/\sigma}}{2\sigma^{\alpha+n+m+2}}d\mu_1 d\sigma=\int_0^\infty\frac{e^{-(\beta+r_n-2s_0+s_m)/\sigma}}{4\sigma^{\alpha+n+m+1}}d\sigma=\frac{\Gamma(\alpha+n+m)}{4(\beta+r_n-2s_0+s_m)^{\alpha+n+m}},$$

$$I_3=\int_0^\infty\int_{-\infty}^{s_0}\int_{\mu_2}^{r_0}\frac{e^{-\left(\beta+r_n-2\mu_2+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_1 d\mu_2 d\sigma=\int_0^\infty\int_{-\infty}^{s_0}(r_0-\mu_2)\frac{e^{-\left(\beta+r_n-2\mu_2+s_m\right)/\sigma}}{\sigma^{\alpha+n+m+3}}d\mu_2 d\sigma$$
$$=\int_0^\infty(r_0-s_0)\frac{e^{-(\beta+r_n-2s_0+s_m)/\sigma}}{2\sigma^{\alpha+n+m+2}}d\sigma+\int_0^\infty\int_{-\infty}^{s_0}\frac{e^{-\left(\beta+r_n-2\mu_2+s_m\right)/\sigma}}{2\sigma^{\alpha+n+m+2}}d\mu_2 d\sigma$$
$$=\int_0^\infty(r_0-s_0)\frac{e^{-(\beta+r_n-2s_0+s_m)/\sigma}}{2\sigma^{\alpha+n+m+2}}+\frac{e^{-(\beta+r_n-2s_0+s_m)/\sigma}}{4\sigma^{\alpha+n+m+1}}d\sigma$$
$$=(r_0-s_0)\frac{\Gamma(\alpha+n+m+1)}{2(\beta+r_n-2s_0+s_m)^{\alpha+n+m+1}}+\frac{\Gamma(\alpha+n+m)}{4(\beta+r_n-2s_0+s_m)^{\alpha+n+m}}.$$

Therefore, in this case, the Bayes estimator in this case is given by;

$$\hat{\theta}=\frac{(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{\Gamma(\alpha+n+m)}\left[\frac{\Gamma(\alpha+n+m)}{2(\beta+r_n-2s_0+s_m)^{\alpha+n+m}}+\frac{(r_0-s_0)\Gamma(\alpha+n+m+1)}{4(\beta+r_n-2s_0+s_m)^{\alpha+n+m+1}}\right]$$
$$=\frac{(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{2(\beta+r_n-2s_0+s_m)^{\alpha+n+m}}+\frac{(r_0-s_0)(\alpha+n+m)(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{4(\beta+r_n-2s_0+s_m)^{\alpha+n+m+1}}. \quad (53)$$

Combining (52) and (53), the Bayes estimator is given by;

$$\hat{\theta} = \begin{cases} \frac{(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{2(\beta+r_n-2s_0+s_m)^{\alpha+n+m}} + \frac{(r_0-s_0)(\alpha+n+m)(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{4(\beta+r_n-2s_0+s_m)^{\alpha+n+m+1}}, & s_0 < r_0 \\ 1 - \frac{(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{2(\beta+r_n-2r_0+s_m)^{\alpha+n+m}} - \frac{(\alpha+n+m)(s_0-r_0)(\beta+r_n-r_0+s_m-s_0)^{(\alpha+n+m)}}{4(\beta+r_n-2r_0+s_m)^{\alpha+n+m+1}}, & r_0 < s_0. \end{cases} \tag{54}$$

If we consider the non-informative prior distribution for the common scale parameter given by; $\pi(\sigma) = \frac{1}{\sigma}, \ \sigma > 0$, then using similar calculations we obtain the Bayes estimator of the stress–strength reliability as;

$$\hat{\theta} = \begin{cases} \frac{(r_n-r_0+s_m-s_0)^{(n+m)}}{2(r_n-2s_0+s_m)^{n+m}} + \frac{(r_0-s_0)(n+m)(r_n-r_0+s_m-s_0)^{(n+m)}}{4(r_n-2s_0+s_m)^{n+m+1}}, & s_0 < r_0 \\ 1 - \frac{(r_n-r_0+s_m-s_0)^{(n+m)}}{2(r_n-2r_0+s_m)^{n+m}} - \frac{(n+m)(s_0-r_0)(r_n-r_0+s_m-s_0)^{(n+m)}}{4(r_n-2r_0+s_m)^{n+m+1}}, & r_0 < s_0 \end{cases}. \tag{55}$$

This estimator is compared with maximum likelihood estimator obtained by inserting the MLEs of $(\mu_1, \mu_2, \sigma)$ in (44). These MLEs are given by; $\hat{\mu}_1 = r_0, \ \hat{\mu}_2 = s_0$ and $\hat{\sigma} = \frac{r_n-r_0+s_m-s_0}{n+m+2}$.

## 4. A simulation study and the results

We have designed a simulation study to investigate the performance of the Bayes estimators derived in this paper and to compare them with other estimators suggested in the literature. The main focus is on point estimation of the stress–strength reliability $\theta = \Pr(Y < X)$. We have considered four cases for the distributions of the stress and strength variables, that is, when both have;

**Table 1**
Biases, mean squared errors, and the relative efficiency of the Bayes estimator to the maximum likelihood estimator in the one parameter and the unrestricted two parameter cases.

| | | | The one parameter exponential case | | | | | The unrestricted two-parameter exponential case | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Bias | | MSE | | REff | Bias | | MSE | | REff |
| $n$ | $m$ | $\theta$ | MLE | Bayes | MLE | Bayes | | MLE | Bayes | MLE | Bayes | |
| 5 | 5 | 0.5 | 0.0002 | −0.0204 | 0.0234 | 0.0190 | 1.2300 | 0.0017 | 0.0031 | 0.1139 | 0.0553 | 2.0591 |
| 5 | 5 | 0.7 | −0.0129 | −0.0504 | 0.0177 | 0.0179 | 0.9898 | 0.0727 | −0.0637 | 0.0770 | 0.0493 | 1.5608 |
| 5 | 5 | 0.9 | −0.0137 | −0.0481 | 0.0049 | 0.0080 | 0.6154 | 0.0645 | −0.1439 | 0.0129 | 0.0472 | 0.2739 |
| 5 | 10 | 0.5 | −0.0115 | −0.0106 | 0.0173 | 0.0147 | 1.1724 | −0.0162 | 0.0209 | 0.1040 | 0.0526 | 1.9779 |
| 5 | 10 | 0.7 | −0.0216 | −0.0342 | 0.0148 | 0.0138 | 1.0693 | 0.0552 | −0.0534 | 0.0758 | 0.0481 | 1.5775 |
| 5 | 10 | 0.9 | −0.0158 | −0.0309 | 0.0040 | 0.0050 | 0.8060 | 0.0629 | −0.1409 | 0.0128 | 0.0454 | 0.2822 |
| 5 | 15 | 0.5 | −0.0134 | −0.0053 | 0.0157 | 0.0136 | 1.1526 | −0.0333 | 0.0213 | 0.1061 | 0.0557 | 1.9046 |
| 5 | 15 | 0.7 | −0.0224 | −0.0265 | 0.0132 | 0.0120 | 1.1021 | 0.0505 | −0.0472 | 0.0756 | 0.0478 | 1.5832 |
| 5 | 15 | 0.9 | −0.0172 | −0.0261 | 0.0038 | 0.0043 | 0.8944 | 0.0621 | −0.1401 | 0.0122 | 0.0454 | 0.2690 |
| 10 | 5 | 0.5 | 0.0102 | −0.0223 | 0.0175 | 0.0147 | 1.1902 | 0.0266 | −0.0140 | 0.1049 | 0.0543 | 1.9297 |
| 10 | 5 | 0.7 | −0.0016 | −0.0479 | 0.0123 | 0.0137 | 0.8925 | 0.0919 | −0.0809 | 0.0695 | 0.0524 | 1.3275 |
| 10 | 5 | 0.9 | −0.0064 | −0.0431 | 0.0029 | 0.0057 | 0.5064 | 0.0713 | −0.1661 | 0.0106 | 0.0546 | 0.1936 |
| 10 | 10 | 0.5 | −0.0013 | −0.0124 | 0.0120 | 0.0107 | 1.1194 | 0.0003 | 0.0009 | 0.0964 | 0.0528 | 1.8263 |
| 10 | 10 | 0.7 | −0.0064 | −0.0271 | 0.0087 | 0.0089 | 0.9760 | 0.0791 | −0.0680 | 0.0661 | 0.0491 | 1.3471 |
| 10 | 10 | 0.9 | −0.0071 | −0.0237 | 0.0021 | 0.0029 | 0.7084 | 0.0697 | −0.1644 | 0.0102 | 0.0535 | 0.1917 |
| 10 | 15 | 0.5 | −0.0034 | −0.0071 | 0.0100 | 0.0091 | 1.1031 | −0.0099 | 0.0038 | 0.0953 | 0.0529 | 1.7998 |
| 10 | 15 | 0.7 | −0.0109 | −0.0227 | 0.0077 | 0.0076 | 1.0057 | 0.0695 | −0.0693 | 0.0664 | 0.0499 | 1.3302 |
| 10 | 15 | 0.9 | −0.0083 | −0.0187 | 0.0018 | 0.0023 | 0.8016 | 0.0659 | −0.1665 | 0.0107 | 0.0549 | 0.1955 |
| 15 | 5 | 0.5 | 0.0131 | −0.0235 | 0.0163 | 0.0139 | 1.1747 | 0.0378 | −0.0172 | 0.1027 | 0.0531 | 1.9331 |
| 15 | 5 | 0.7 | 0.0022 | −0.0471 | 0.0106 | 0.0124 | 0.8560 | 0.0991 | −0.0872 | 0.0665 | 0.0531 | 1.2514 |
| 15 | 5 | 0.9 | −0.0033 | −0.0405 | 0.0023 | 0.0048 | 0.4712 | 0.0732 | −0.1725 | 0.0101 | 0.0573 | 0.1769 |
| 15 | 10 | 0.5 | 0.0059 | −0.0096 | 0.0101 | 0.0090 | 1.1163 | 0.0070 | −0.0050 | 0.0959 | 0.0538 | 1.7838 |
| 15 | 10 | 0.7 | −0.0025 | −0.0259 | 0.0071 | 0.0075 | 0.9436 | 0.0858 | −0.0760 | 0.0628 | 0.0500 | 1.2566 |
| 15 | 10 | 0.9 | −0.0056 | −0.0228 | 0.0016 | 0.0024 | 0.6706 | 0.0709 | −0.1686 | 0.0100 | 0.0554 | 0.1813 |
| 15 | 15 | 0.5 | 0.0012 | −0.0066 | 0.0081 | 0.0075 | 1.0871 | −0.0004 | −0.0010 | 0.0934 | 0.0529 | 1.7645 |
| 15 | 15 | 0.7 | −0.0062 | −0.0205 | 0.0058 | 0.0059 | 0.9709 | 0.0793 | −0.0745 | 0.0637 | 0.0501 | 1.2718 |
| 15 | 15 | 0.9 | −0.0050 | −0.0159 | 0.0012 | 0.0016 | 0.7635 | 0.0701 | −0.1675 | 0.0101 | 0.0549 | 0.1846 |

(a) The one parameter exponential distribution.
(b) The two-parameter exponential distribution with unrestricted location and scale parameters.
(c) The two-parameter exponential distribution with common location parameters.
(d) The two-parameter exponential distribution with common scale parameters

In each of these cases we used the sample sizes as all combinations of $n = 5, 10, 15$ and $m = 5, 10, 15$. The true values of the stress–strength reliability are chosen as $\theta = 0.5, 0.7, 0.9$. By the symmetry of $X$ and $Y$our results extends to the corresponding values of $\theta$ below 0.5. In each of the four cases above, the parameters of the stress and strength distributions are chosen to obtain the desired value of $\theta$. Our simulations are based on 5000 replications for each simulation run. In cases where Bayesian estimators are computed using Markov Chain Monte Carlo methods we used 2000 replications to estimate the desired posterior expectation. The Bayes estimators developed in this paper under non-informative priors are compared with the corresponding maximum likelihood estimators. The comparison is in terms of the bias and mean squared error of the estimators. The relative efficiency of the MLE to the Bayes estimator is calculated as the MSE of the MLE divided by the MSE of the Bayes estimator and it is calculated in the columns labeled REff in Tables 1 and 2.

The results are given in Tables 1 and 2. Concerning the performance of the estimators in terms of bias, it appears that in most cases the Bayes estimators are more biased for small sample sizes or for very high values of the stress–strength reliability, otherwise the bias performance is similar to that of the MLE or even better. An exception is for the case of exponential distributions with a common scale parameter where it appears that the Bayes estimator consistently have higher bias.

The MSE performance of the estimators is clear, the Bayes estimators have smaller MSE unless the true value of the stress–strength reliability is very high, in which case the MLE has better performance. This can be seen very clearly from the column representing the relative efficiency of the Bayes estimator to the MLE. A possible justification for this is that the Bayes estimator is more biased in such situations which contribute significantly to the MSE.

**Table 2**
Biases, mean squared errors, and the relative efficiency of the bayes estimator to the maximum likelihood estimator in the two parameter case, common location or common scale.

| | | | Two-parameter exponential with Common Location | | | | | Two-parameter exponential with Common Scale | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Bias | | MSE | | REff | Bias | | MSE | | REff |
| $n$ | $m$ | $\theta$ | MLE | Bayes | MLE | Bayes | | MLE | Bayes | MLE | Bayes | |
| 5 | 5 | 0.5 | −0.0031 | −0.0027 | 0.0274 | 0.0206 | 1.3269 | 0.0014 | 0.0012 | 0.1056 | 0.0563 | 1.8759 |
| 5 | 5 | 0.7 | −0.0039 | −0.0274 | 0.0210 | 0.0178 | 1.1808 | −0.0655 | −0.1088 | 0.0962 | 0.0603 | 1.5940 |
| 5 | 5 | 0.9 | 0.0015 | −0.0255 | 0.0042 | 0.0056 | 0.7541 | −0.0696 | −0.1592 | 0.0518 | 0.0549 | 0.9437 |
| 5 | 10 | 0.5 | −0.0240 | −0.0011 | 0.0212 | 0.0166 | 1.2817 | 0.0015 | 0.0014 | 0.0956 | 0.0540 | 1.7723 |
| 5 | 10 | 0.7 | −0.0206 | −0.0200 | 0.0164 | 0.0132 | 1.2408 | −0.0684 | −0.1061 | 0.0923 | 0.0616 | 1.4994 |
| 5 | 10 | 0.9 | −0.0097 | −0.0211 | 0.0037 | 0.0040 | 0.9138 | −0.0760 | −0.1507 | 0.0558 | 0.0561 | 0.9939 |
| 5 | 15 | 0.5 | −0.0361 | −0.0048 | 0.0190 | 0.0146 | 1.3027 | −0.0044 | −0.0040 | 0.0924 | 0.0538 | 1.7186 |
| 5 | 15 | 0.7 | −0.0335 | −0.0237 | 0.0162 | 0.0126 | 1.2843 | −0.0767 | −0.1101 | 0.0923 | 0.0631 | 1.4624 |
| 5 | 15 | 0.9 | −0.0133 | −0.0191 | 0.0036 | 0.0036 | 1.0038 | −0.0781 | −0.1468 | 0.0554 | 0.0563 | 0.9838 |
| 10 | 5 | 0.5 | 0.0289 | 0.0054 | 0.0207 | 0.0159 | 1.2984 | 0.0046 | 0.0039 | 0.0966 | 0.0545 | 1.7730 |
| 10 | 5 | 0.7 | 0.0231 | −0.0160 | 0.0143 | 0.0130 | 1.1007 | −0.0666 | −0.1066 | 0.0921 | 0.0618 | 1.4891 |
| 10 | 5 | 0.9 | 0.0110 | −0.0187 | 0.0026 | 0.0038 | 0.6950 | −0.0774 | −0.1518 | 0.0562 | 0.0568 | 0.9894 |
| 10 | 10 | 0.5 | 0.0016 | 0.0015 | 0.0130 | 0.0110 | 1.1813 | −0.0036 | −0.0027 | 0.0915 | 0.0531 | 1.7248 |
| 10 | 10 | 0.7 | −0.0003 | −0.0144 | 0.0094 | 0.0086 | 1.0864 | −0.0757 | −0.1112 | 0.0911 | 0.0629 | 1.4490 |
| 10 | 10 | 0.9 | 0.0003 | −0.0139 | 0.0019 | 0.0024 | 0.8098 | −0.0773 | −0.1456 | 0.0555 | 0.0554 | 1.0033 |
| 10 | 15 | 0.5 | −0.0067 | 0.0013 | 0.0106 | 0.0091 | 1.1591 | −0.0030 | −0.0033 | 0.0907 | 0.0535 | 1.6960 |
| 10 | 15 | 0.7 | −0.0086 | −0.0137 | 0.0081 | 0.0074 | 1.1010 | −0.0657 | −0.1002 | 0.0885 | 0.0609 | 1.4537 |
| 10 | 15 | 0.9 | −0.0032 | −0.0120 | 0.0016 | 0.0019 | 0.8753 | −0.0780 | −0.1438 | 0.0534 | 0.0539 | 0.9907 |
| 15 | 5 | 0.5 | 0.0363 | 0.0050 | 0.0200 | 0.0154 | 1.2964 | 0.0065 | 0.0059 | 0.0926 | 0.0537 | 1.7255 |
| 15 | 5 | 0.7 | 0.0301 | −0.0140 | 0.0124 | 0.0114 | 1.0952 | −0.0683 | −0.1049 | 0.0871 | 0.0592 | 1.4704 |
| 15 | 5 | 0.9 | 0.0138 | −0.0166 | 0.0022 | 0.0032 | 0.6868 | −0.0722 | −0.1430 | 0.0511 | 0.0523 | 0.9775 |
| 15 | 10 | 0.5 | 0.0090 | 0.0010 | 0.0111 | 0.0096 | 1.1611 | 0.0023 | 0.0021 | 0.0911 | 0.0539 | 1.6913 |
| 15 | 10 | 0.7 | 0.0070 | −0.0119 | 0.0073 | 0.0070 | 1.0528 | −0.0713 | −0.1064 | 0.0884 | 0.0611 | 1.4460 |
| 15 | 10 | 0.9 | 0.0037 | −0.0112 | 0.0014 | 0.0017 | 0.7906 | −0.0739 | −0.1399 | 0.0506 | 0.0513 | 0.9860 |
| 15 | 15 | 0.5 | 0.0004 | 0.0004 | 0.0085 | 0.0076 | 1.1254 | −0.0045 | −0.0035 | 0.0895 | 0.0536 | 1.6689 |
| 15 | 15 | 0.7 | 0.0024 | −0.0076 | 0.0060 | 0.0056 | 1.0635 | −0.0739 | −0.1061 | 0.0892 | 0.0627 | 1.4233 |
| 15 | 15 | 0.9 | 0.0003 | −0.0092 | 0.0012 | 0.0014 | 0.8551 | −0.0797 | −0.1432 | 0.0552 | 0.0556 | 0.9916 |

Both the MLE and the Bayes estimator have larger values of the MSE for true values of the stress–strength reliability around 0.5 and it decreases as the true value approaches the extremes. As for the effect of sample size, it is anticipated that the bias and MSE of both estimators are less for larger sample sizes and this supported by the simulation results.

## 5. An illustrative example

We simulated the records data from one parameter exponential distributions with $n = 4,\ m = 4, \sigma_1 = 1$ and $\sigma_2 = 0.1111$ with the true value of the stress–strength reliability being 0.9. The simulated records are, $r_4 = 2.4101$, and $s_4 = 0.4843$. Using this data we found that $\hat{\sigma}_1 = \frac{r_n}{n+1} = 0.482,\ \hat{\sigma}_2 = \frac{s_m}{m+1} = 0.0969$ and therefore the MLE of the stress–strength reliability is $\hat{\theta} = \frac{\hat{\sigma}_1}{\hat{\sigma}_1+\hat{\sigma}_2} = 0.8327$. The corresponding Bayes estimator is found using Monte Carlo methods and it is given by 0.7882.

Now assume that the same records used above were generated from two-parameter exponential distributions with unrestricted location and scale parameters. The records data given earlier in addition to the first record in each sample are $r_0 = 0.6065,\ s_0 = 0.3179,\ r_4 = 2.4101,$ and $s_4 = 0.4843$. Therefore, $\hat{\mu}_1 = r_0 = 0.6065,\ \hat{\mu}_2 = s_0 = 0.3179,\ \hat{\sigma}_1 = \frac{r_n - r_0}{n+1} = 0.3607,\ \hat{\sigma}_2 = \frac{s_m - s_0}{m+1} = 0.0333$ and therefore the MLE of the stress–strength reliability is $\hat{\theta} = \frac{\hat{\sigma}_1}{\hat{\sigma}_1+\hat{\sigma}_2} e^{-(\hat{\mu}_2-\hat{\mu}_1)/\hat{\sigma}_1} I_{\hat{\mu}_2 \geqslant \hat{\mu}_1} + \left(1 - \frac{\hat{\sigma}_2}{\hat{\sigma}_1+\hat{\sigma}_2} e^{-(\hat{\mu}_1-\hat{\mu}_2)/\hat{\sigma}_2}\right) I_{\hat{\mu}_2 < \hat{\mu}_1} = 0.9999$. The corresponding Bayes estimator is obtained using the Monte Carlo procedure described after Eq. (28) and it is found to be 0.7760.

If we consider the same records as being generated from two-parameter exponential distributions with common location, the records data are $r_0 = 0.6065,\ s_0 = 0.3179,\ r_4 = 2.4101$ and $s_4 = 0.4843$. The parameter estimates are given by $\hat{\mu} = \min(r_0, s_0) = 0.3179,\ \hat{\sigma}_1 = \frac{r_n - \hat{\mu}}{n+1} = 0.4278,\ \hat{\sigma}_2 = \frac{s_m - \hat{\mu}}{m+1} = 0.0427$ and therefore the MLE of the stress–strength reliability is $\hat{R} = \frac{\hat{\sigma}_1}{\hat{\sigma}_1+\hat{\sigma}_2} = 0.9093$. The corresponding Bayes estimator found using the simulation algorithm described after Eq. (38) and is found to be 0.8823.

We note here that this example mainly shows how the calculations are carried out. For comparison between the performance of the different estimator, it is dependent on the true value of the parameter as explained in the simulation study.

## 6. Conclusions

In this paper we have derived Bayes estimators based on record values for the stress–strength reliability when the stress and the strength variables follow the one parameter as well as the two-parameter exponential distributions. In the two parameter case we have considered the cases when the parameters unrestricted, when the location parameter is common or when the scale parameter is common. Except for the common scale parameter case, the Bayes estimators are not in closed form and numerical integration or Monte Carlo methods are needed. Under non-informative prior distributions, we have investigated their performance and compared them with the MLE. We found that the Bayes estimators generally perform better when the true value of the stress–strength reliability is not close to the extremes (0 or 1), while near the extremes the MLE is better. We found that the reason for this is the bias performance of the Bayes estimators which tend to be relatively highly biased near the extremes. Depending on the nature of the specific application, one can determine to use the Bayes estimator or the MLE. This is because in some applications like those in the medical sciences, values near 0.5 are most important while in other fields, like reliability engineering, very high values of the stress–strength reliability are most important.

This paper is concerned mainly with point estimation. Another very important problem is the construction of interval estimators for the stress–strength reliability based on record values in the one and two-parameter exponential distribution. While some of these intervals are already developed in this paper and in [2], other competing approaches are possible and currently under investigation by the author.

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