Continued Fractions Evaluation And Their Applications To Some Probability Distribution Functions

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ABSTRACT

This paper discusses some continued fractions in obtaining rational approximations to functions, emphasising those which are important in statistical applications. It will be shown that continued fractions are valuable tools in the evaluation of certain cumulative distribution functions. The probabilities of several convergents were evaluated over a wide range of functions, to determine the effectiveness of the expansion. Also forward and backward methods were used. However in conclusion, the continued fraction method as a mean of obtaining rational approximations to probability distributions is a very powerful and effective technique.

1. INTRODUCTION

This paper is primarily concerned with the use of continued fractions in obtaining rational approximations to functions.The rational functions can be written in fractional form, that is, as a polynomial divided by a polynomial, and except in a few degenerate cases, rational functions in fractional forms can be expressed as terminating continued fractions [1]. The number of divisions or "fractional lines," needed is n (the degree of the polynomial Q_n (x)).

There are certain advantages to be realized via this conversion process. First, evaluation is often faster if we express the approximation as a continued fraction. Furthermore, fewer arithmetic operations are required to evaluate a rational function if it is expressed as a continued faction than are required if it is expressed as a fraction, although divisions are a higher proportions. Therefore, rational function evaluation may be handled quite efficiently through the use of continued fraction expansions [2]. These ideas can be extended to rational approximations of functions. Snyder, states that "*one reason that rational approximations convenient for computer evaluation is that, they can be readily converted into continued fractions*" [3]. Thus evaluation of the use of continued fraction expansions is an economical procedure. Truncation of continued fraction is also considered one of the simplest rational approximations methods. Therefore, there are several advantages to the use of continued fractions in obtaining rational function approximations, two of such advantages are the rapid speed of convergence of continued fractions and the large regions of convergence common to these expansions. D. Tichroew uses these arguments to point out that continued fraction expansions may be more efficient to use than power series expansions in obtaining rational approximations to functions [4]. Also the region of convergence for a continued fraction expansion is often much larger that the region of convergence of a power series expansion of the function [5]. Finally, Kovanskii states that "*one can even transform a power series that has a convergence reddish of zero into a continued fraction which converges in the same domain*" [6]. Thus it is evident that continued fractions are an important and powerful tool in the study and evaluation of rational function approximations.

2. CONTINUED FRACTION FOR PROBABILITY DISTRIBUTION FUNCTIONS

This paper discusses some of the continued fractions emphasising those which are important in statistical applications. As a consequence of comparison of continued power series expansion, a derivation of continued fractions from a known power series will be considered in this paper. A discussion of the methods by which one may carry out such a derivation is appropriate. When no ready-made continued fraction expansion for a function is available, one can use the first few coefficients in an expansion from the MaClurian series for the function, if that series is known. Viskovativ's method is one way to convert a power series expansion into a continued fraction. A convenient notation for Viskovatov's method can be developed in this paper, which makes the actual computations quite easy as follows:

$$\text{Let } f(x) = \frac{a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4 + \dots\dots}{a_{00} + a_{01}x + a_{02}x^2 + a_{03}x^3 + a_{04}x^4 + \dots\dots} \qquad (1)$$

the above functions can be written as the following continued fraction

$$\frac{a_{10}}{a_{00}+}\ \frac{a_{20}x}{a_{10}+}\ \frac{a_{30}x}{a_{20}+}\ \frac{a_{40}x}{a_{30}+}\ \frac{a_{50}x}{a_{40}+}\ \frac{a_{60}x}{a_{50}+}\ \dots\dots\dots \qquad (2)$$

The calculation can be easily performed by using the following scheme:

a_{00}	a_{01}	a_{02}	a_{03}	a_{04}	a_{05} ....
a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15} ....
a_{20}	a_{21}	a_{22}	a_{23}	a_{24}	a_{25} ....
a_{30}	a_{41}	a_{42}	a_{43}	a_{44}	a_{45} ....

Where

$$a_{mn} = (a_{m-1,0}\ a_{m-2,n+1} - a_{m-2,0}\ a_{m-1,n+1})$$

Another method by which one may convert a power series of any other efficient polynomial approximation, into a continued fraction is given by Snyder [3]. He credits Euler with the basic identity, which follows;

$$\sum_{n=0} c_n x^n = \frac{a_0}{1-}\ \frac{a_1 x}{(1+a_1 x)-}\ \frac{a_2 x}{(1+a_2 x)-}\ \frac{a_3 x}{(1+a_3 x)-}\ \frac{a_4 x}{(1+a_4 x)-}\ \frac{a_5 x}{(1+a_5 x)-} \qquad (3)$$

where $a_0 = c_0$ and $a_n = \frac{c_n}{c_{n-1}}$ $\qquad (n>1)$

An equivalent representation of the transformation is given below:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + \frac{c_1 x}{1-}\ \frac{c_2 x}{(c_1+c_2 x)-}\ \frac{c_3 x}{(c_2+c_3 x)-} \cdots\cdots\ \frac{c_n x}{(c_{n-1}+c_n x)-} \qquad (4)$$

These algorithms provide a convenient methods by which one may obtain rational approximations to functions by using only the first few convergences of continued fraction. Khovanskii discusses various transformations of the form

$$b_0 + \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+}\ \frac{a_4}{b_4+}\ \frac{a_5}{b_5+} \cdots\cdots$$

The two more common transformations of this kind are given below.

One may obtain the "ordinary" continued fraction,

$$c_0 + \frac{1}{c_1+}\ \frac{1}{c_2+}\ \frac{1}{c_3+}\ \frac{1}{c_4+}\ \frac{1}{c_5+} \cdots\cdots$$

in which all of the partial numerators equal one, by means of the following transformation;

$$c_0 = b_0$$

$$c_{2k-1} = \frac{a_2 a_4 a_6 \cdots\cdots a_{2k-2}\, b_{2k-1}}{a_1 a_3 a_5 \cdots\cdots a_{2k-1}}$$

$$c_{2k} = \frac{a_1 a_3 a_5 \ldots\ldots a_{2k-1}\, b_{2k}}{a_2 a_4 a_6 \ldots\ldots a_{2k}}$$

where k= 1,2,3.....

Similarly, one can obtain a continued fraction of the form;

$$c_0 + \quad \frac{c_1}{1+} \quad \frac{c_2}{1+} \quad \frac{c_3}{1+} \quad \frac{c_4}{1+} \quad \frac{c_5}{1+} \quad \ldots\ldots$$

in which all the partial denominatiors are equal to one, by using the following transformation.

$$c_0 = b_0\,, \quad c_1 = \frac{a_1}{b_1} \text{ and } c_n = \frac{a_n}{b_{n-1}b_n} \qquad (n = 2,3,4 \ldots)$$

So far, this paper has been discussing the advantages of the use of the continued fraction expansion of a function as a mean of obtaining a rational approximation to that function. However, that does not mean to imply that a continued expansion for a function is unique. It may be that several such expansion are available for a single function. Furthermore, a continued fraction is unchanged in value if some partial numerator and partial denominator, along with the immediately succeeding numerators, are multiplied by the non-zero constant. Thus, one can always alter any continued fraction by means of this simple transformation.

Continued fractions may be used advantageously in the field of statistics. In this section of the paper, it will be shown that they are valuable tools in the evaluation of certain cumulative distribution functions. In particular, those continued fractions which are useable in the evaluation of the normal and incomplete beta distributions will be discussed in detail. A complete discussion on continued fraction evaluation of the incomplete gamma distribution can be found in [5] and [7]. First, however, a method by which continued fractions may be used to approximate binomial probabilities will be presented.

2.1 The Binomial Distribution

Suppose that it is desired to evaluate the probability that the number of successes, k, of the binomial distribution with n independent trials, and probability of success, p, on each trial, exceeds, k. Of course, one may evaluate by summing the individual probabilities of the well-known density function:

$$f(k) = \binom{n}{k} p^k q^{n-k}, \qquad k=0,1,2, \ldots, n, \qquad 0 < p < 1, \qquad q=1-p \tag{5}$$

However, if the number of trials, n, is large, hand calculation of the probabilities requires a substantial amount of tedious arithmetic. This is, of course, commonly avoided by approximation the probability determining the area under an appropriate region of the normal curve. Another approximation, given below, makes use of the continued fraction expansion. An advantage of the continued fraction method is that, it requires fewer mathematical assumptions.

To evaluate p(x>k), the probability that the number of successes, x, exceeds k. In this discussion, as usual, n denotes the number of trials, p denotes the probability of success and q = 1 - p. For the purposes of this development, it is assumed that k< np. This is not a restrictive assumption. If k < np using this approach evaluate:

$$P(x \leq k) = \sum_{x=0}^{k} f(k)$$

$$P(x > k) = \sum_{x=k+1}^{k} f(k)$$

$$= \frac{n!}{(k+1)!(n-k-1)!} p^{k+1}q^{n-k-1} + \frac{n!}{(k+2)!(n-k-2)} p^{k+2}q^{n-k-2} + \ldots\ldots + p^n$$

$$= \frac{n!}{(k+1)!(n-k-1)} p^{k+1}q^{n-k-1} + \{1 + \frac{(n-k-1)p}{(k+2)q} + \frac{(n-k-1)(n-k-2)p^2}{(k+2)(k+3)q^2} + \frac{(n-k-1)(n-k-2)(n-k-3)p^3}{(k+2)(k+3)(k+4)q^3} + \ldots\}$$

$$= p^{k+1} q^{n-k-1} \{\prod_{j=0}^{k} \frac{(n-j)}{(n-j-1)}\} \{1+ \frac{(n-k-1)p}{(k+2)q} + \frac{(n-k-1)(n-k-2)p^2}{(k+2)(k+3)q^2} + ...\} \quad (6)$$

The term outside the brackets may be evaluated quickly and easily by using logarithms. It is the evaluation of the finite sum inside the brackets will be denoted by S, that causes the difficulty. Note that S is a particular case of hypergeometric series,

$$F(a, b, g, z) = 1 + \frac{ab}{g} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{g(g+1)\ \ 2!} \frac{z^2}{2!} + ...$$

where

$$a = (-n+k+1), \quad b = 1, \quad g = (k+2) \quad \text{and} \quad z = \frac{-p}{q}$$

Define $x_{2n} \quad =F(a+n, b+n, g+2n, z)$

$x_{2n+1} \quad =F(a+n, b+n+1, g+2n+1, z)$

It can then be shown that

$$X_{2n} = X_{2n-1} + \frac{z(b+n)(g-a+n)}{(g+2n-1)(g+2n)} X_{2n+1}$$

$$X_{2n+1} = X_{2n} + \frac{z(b+n)(g-b+n)}{(g+2n)(g+2n+1)} X_{2n+2}$$

Let A_{2n} and A_{2n+1} denotes the coeffecients of x_{2n+1} and x_{2n+2} respectively. Then

$$X_{v-1} = X_v - A_v X_{v+1} z$$

Then a series of relationships between the X's is obtained. One of these relationships is given by

$$X_0 = X_1 + A_1 X_2 z \qquad \text{or}$$

$$\frac{X_1}{X_0} = \cfrac{1}{1 - A_1 \cfrac{z X_2}{X_1}}$$

Then a continued fraction series from these successive relations is:

$$\frac{X_1}{X_0} = \frac{1}{1-}\;\frac{A_1 z}{1-}\;\frac{A_2 z}{1-}\;\ldots\ldots\ldots\ldots\;\cfrac{A_v z}{\cfrac{X_v}{X_{v+1}}}$$

let

$$X_0 = 1,\ X_1 = F(a, b+1, g+1, z) = F\left(-n+k+1, 1, k+2, \frac{-p}{q}\right)$$

Thus

$$a = -n+k+1, \qquad b = 0, \quad g = k+1 \quad \text{and } z = \frac{-p}{q}$$

and let

$$d_i = -zA_{2i} \qquad \text{and } c_i = zA_{2i-1}$$

By direct substitution

$$d_i = \frac{i(n+i)}{(k+2i+1)(k+2i)} \tag{7}$$

$$c_i = \frac{(n-i-k)(n+i)p}{(k+2i-1)(k+2i)q}$$

Then

$$S = \frac{X_1}{X_0} = \frac{1}{1-}\;\frac{c_1}{1+}\;\frac{d_1}{1-}\;\frac{c_2}{1+}\;\frac{d_2}{1-}\;\ldots\ldots\ldots\ldots\;\cfrac{c_{n-k-1}}{\cfrac{1+d_{n-k-1}}{1+}} \tag{8}$$

and since

$c_1 = (n-k-1)p/(k+2)q$ and by assumption, $k > np$, it follows that $0 < c_1 < 1$. Therefore by the definition of c_1, $0 < c_1 < 1$ for all i. Thus, if we let

$$s_i = \cfrac{c_i}{1+\cfrac{d_1}{1-c_{i-1}}} \qquad \text{then } c_i > s_i > 0 \text{ and} \tag{9}$$

$$s_i = \cfrac{c_i}{1+\cfrac{d_1}{1-s_{i+1}}}$$

Then binomial probabalilities can be evaluated, using the above scheme. First, choose i to be any number desired. Then using the definition, calculate c_{i+1} to obtain the limit.

$$c_{i+1} > s_{i+1} > 0$$

Next calculate limits for s_1 where

$$s_i = \cfrac{c_i}{1+\cfrac{d_1}{1-s_{i+1}}}$$

(i.e. substitute the limits for s_{i+1}, into the above equation and solve to obtain the limits for s). Repeat the process to obtain limits for s_{i+1}, ... s_2, s_1 and finally for S where

$$S = \frac{1}{1-s_1}$$

This evaluation is efficiently done by recursive form as given in Table 1.

Table 1

Continued Fraction Algorithm for the Binomial Distribution Function.

STEP 1: Input values of p, k, n and I = the number of recursive steps to be used

STEP 2: Evaluate $s_i = c_i / (1+d_i)$ using (9) to evaluate c_w and d_w

STEP 3: For w = I-1, I-2, ..., 1 evaluate s_w where $s_w = c_w / (1+(d_w/(1-s_{w+1})))$

STEP 4: Evaluate $S = 1/(1-s_1)$ and LA=(k+1) Ln(p) + (n-k-1) Ln(1-p)

STEP 5: For j = 0,1, ..., k evaluate SUM = 0

LB = Ln(n-j) - Ln(k-j+1) and SUM= SUM + LB

STEP 6: Evaluate LF = LA + Ln(S) + SUM and F = EXP(LF)

In the above scheme, i could be chosen to be any number desired. Obviously, the approximation will be more accurate for larger value of i. However, this method was presented as a mean of obtaining good approximations to binomial probabilities while avoiding lengthy calculations. If the value chosen for i is too large, it needs a great deal of calculation.

A computer program, BINRC, evaluated on this research which evaluates exact probabilities using the following recursive relation,

$$f(x) = \frac{(n-x+1)p}{xq} f(x-1)$$

one would probably feel more justified and confident using this program rather than the approximation outlined above.

Using the method of continued fraction for n=80 and p=.4, the probability that the number of success exceeds 40 is approximately 0.027124. This result occurs if i=5. Calculating the exact probability using the computer program BINRC the correct result is 0.0271236. Thus, the result obtained using the continued fraction method, even for such a small value of i, is very good.

The real value of continued fractions is quite evident when one turns to continuous probability density functions. To demonstrate this, the normal and incomplete beta distributions will be discussed in detail. In these cases, unlike the binomial, no method is available by which one may determine the exact cumulative probabilities. Thus, effective approximations are required.

2.2 The Normal Distribution

Evaluation of probabilities for the normal distribution, with mean μ, and variance, σ^2, may always be reduced to the problem of determining the area under an appropriate region of the standard normal curve ($\mu = 0$, $\sigma^2 = 1$), the cumulative distribution function of the standard curve is denoted by $\Phi(x)$ where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t^2/2}\, dt \qquad \infty < x < \infty$$

The probability density function, denoted $\phi(x)$ is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{t^2/2} \qquad \infty < x < \infty$$

As will be demonstrated, continued fraction expansions are the basis of an effective method by which one may evaluate this cumulative distribution function. In this paper, it is found that the cumulative distribution function, Φ (x), is most effectively evaluated using continued fractions when two distinct expansions are utilized. The first of these is Laplace continued fraction, which is given by:

$$\Phi(x) = 1 - \phi(x) \left\{ \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \cdots\cdots\cdots \frac{n}{x+} \cdots \right\} \qquad (10)$$

The algorithm in recursive form is give in Table 2

Table 2

Laplace Continued Fraction Algorithm for the Normal Distribution Function

STEP 1: Input values of x and I = the number of recursive steps to be used
STEP 2: Evaluate $T(I) = I/x$ to evaluate $T(k)$
STEP 3: For $k=I-1, I-2, \ldots, 1$ evaluate $T(k) = x + k/(x+T(I+1))$
STEP 4: Evaluate $CF = 1/(x+T(1))$ and
$\phi = V2\pi \, EXP(-x^2/2)$
$\Phi = 1 - \phi . CF$

The second of these continued fractions is the following continued fraction, which was given by Shenton:

$$\Phi(x) = \frac{1}{2} - \phi(x) \left\{ \frac{x}{1-} \frac{x^2}{3+} \frac{2x^2}{5-} \frac{3x^2}{7+} \frac{4x^2}{9-} \cdots\cdots \frac{nx^2}{2n+1} \cdots\cdots \right\} \quad (11)$$

The above continued fraction can be evaluated recursively as on the algorithm given in Table 3.

Table 3

Shenton Continued Fraction Algorithm for Normal Distribution Function

STEP 1: Input values of α, ß, x and n = number of recursive steps to used, (effective number of divisions used in (11) is an odd number).
STEP 2: Evaluate
$w(I) = Ix^2/((2I + 1) + ((I+1)x^2/(2(I+1) + w(I+1))))$ to evaluate $w(k)$
STEP 3: For $k=I-2, I-4, I-6, \ldots, 1$ evaluate
$w(k) = kx^2/\{(2k + 1) + [(k+1)x^2/\{(2(k+1) + w(k+1))\}]\}$
STEP 4: Evaluate
$CF = x/(1-w(1))$ and
$\phi = V2\pi \, EXP(-x^2/2)$
$\Phi = .5 + \phi . CF$

These two continued fractions were examined in this research. The first expansion is the best utilized for large values of x, usually x greater than 3. Also it is found that this continued fraction, using 25 iterants only, guarantees 10 place accuracy for x = 2.8, but it is even better for larger values of x.

For these smaller values of x, (x<3), the second expansion converges much more rapidly than the Laplace continued fraction. But, as indicated below, these two expansions are powerful tools by which one may evaluate the cumulative distribution function of the standard normal curve.

A computer program was written, using these continued fractions to evaluate the cumulative distribution function to 16 decimal places in the most economical manner using algorithm in Tables 2 and 3. In the program, the Shenton continued fraction was used for x < 4.35, and the Laplace continued fraction was used for x < 11.3, values outside this range were set to zero and one respectively. It is interesting to note that, in writing a program of this sort, the cut-off point at which the Shenton continued fraction passes over the Laplace continued fraction is determined by the precision desired.

2.3 The Incomplete beta Distribution

Another problem of interest to statisticians is the evaluation of cumulative probabilities of the incomplete beta distribution, that denoted by $I_x(\alpha,\beta)$ and defined as follows:

$$I_x(\alpha,\beta) = \frac{\Gamma(\alpha,\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad 0<x<1,\ a>0,\ \beta>0 \qquad (12)$$

The mean of the distribution is $\alpha/(\alpha, \beta)$ and the variance is given by $\alpha/((\alpha, \beta)^2 (\alpha+ \beta-2))$. The mode, or anti-mode if the curve is U-shaped, occurs at $(\alpha-1)/(\alpha+\beta-2)$. The curve is bell-shaped if both α and β are greater than 1; it is U-shaped if both parameters are less than one; it is J-shaped decreasing if $0 < \alpha<1$ and $\beta >1$; and it is J-shaped increasing if $0 < \beta < 1$ and $\alpha > 1$. Finally, if $\alpha= \beta=1$, this is uniform distribution. Thus, it is clear that the incomplete beta distribution function takes on many shapes, depending on the value of the parameters. As a result, many curves closely fit this function.

Continued fraction provide an effective means of evaluation of the incomplete beta function [8]. Bouver and Barman, give a continued fraction, which they say it may be used to efficiently evaluate probabilities over entire range of the distribution. However, some errors in their report were discovered in this research [9]. An examination of the program they provide, reveals that the expansion should read as follows:

$$I_x(\alpha, \beta) = \frac{\Gamma(\alpha, \beta)\, x^{\alpha} (1-x)^{\beta}}{\Gamma(\alpha)\ \ (\beta)} \{ \frac{1}{1+} \frac{c_1}{1+} \frac{c_2}{1+} \frac{c_3}{1+} \frac{c_4}{1+} \frac{c_5}{1+} \ldots\ldots \} \qquad (13)$$

where

$$c_{2k-1} = \frac{-(\alpha+k-1)(\alpha+\beta+k-1)}{(\alpha+2k-2)(\alpha+2k-1)} \quad x$$

$$c_{2k} = \frac{k(\beta-x)}{(\alpha+2k-1)(\alpha+2k)} \quad x$$

where k=1,2,3,4, ...

The evaluation of the first continued fraction if beta distribution is efficiently done by a recursive form as given in Table 4.

Table (4)

Continued Fraction Algorithm for the first continued fraction of Beta Distribution Function

STEP 1: Input values of α, β, x and n = the number of recursive steps to be used (effective number of divisions used in the continued expansions (14) is 2n)

STEP 2: Evaluate $s_n = c_n/(1+d_n)$ using (9) to evaluate c_n and d_n.

STEP 3: For k = n-1, n-2, ..., 1 evaluate s_k where $s_k = c_k / (1+ (d_k- /(1-s_{k+1})))$

STEP 4: Evaluate $S = 1/(1- s_1)$ and LA=Ln ($\Gamma(\alpha, \beta)$) - Ln (α+1) - Ln (B)

STEP 5: Evaluate LF = LA+ αLn (x) - βLn (1-x) + Ln (CF)

$I_x(\alpha, \beta)$ = EXP (LF)

This expansion works most efficiently when x is less than the mean (i.e. $x < \alpha/(\alpha+\beta)$). However, if x is greater than the mean, one may still efficiently use the above scheme by simply taking complements, since

$$I_x(\alpha,\beta) = I_{1-x}(\beta,\alpha)$$

Several methods for the evaluation of the incomplete beta distribution were studied. Using the criteria outlined above, in order to find the regions over which each method provides the best approximation. In conclusion, the continued fraction is by far the most efficient method to evaluate the incomplete beta function. In fact, the continued fraction evaluation with variable number of terms, for 10 digits of precision, has proven to be vastly superior for all cases encountered in practice. The continued fraction works very efficiently even if the incomplete beta function extremely J-shaped or extremely U-shaped. Another expansion for the incomplete beta distribution function is

$$I_x(\alpha,\beta) = \binom{N}{\alpha} x^{\alpha}(1-x)^{\beta-1} S \qquad (14)$$

where

$$N = \alpha + \beta - 1$$

$$S = \{\frac{1}{1+}\,\frac{c_1}{1+}\,\frac{d_1}{1+}\,\frac{c_2}{1+}\,\frac{d_2}{1+}\,\frac{c_3}{1+}\,\frac{d_3}{1+}\,\ldots\}$$

$$c_{k-1} = \frac{(\beta-k)(\alpha-1+k)}{(\alpha+2k-2)(\alpha-1+2k)}\,\frac{x}{1-x}$$

and

$$d_k = \frac{k(\alpha+\beta-1+k)}{(\alpha+2k)(\alpha-1+2k)}\,\frac{x}{1-x}$$

This expansion has been shown to be an effective tool for the evaluation of the cumulative beta distribution and the evaluation of this second continued fraction expression of beta distribution is efficiently done by a recursive form similar to that recursive form given in Table 4.

It is evident from the above discussion that one could evaluate this distribution with a high degree of confidence by using one or both of the preceding fraction expansions. Two other expansions which may also prove to be useful in evaluating these probabilities are given by [10]. For completeness, they are listed below. The first is as follows:

Let the continued fraction W, be defined by

$$W = \frac{G_1}{1+G_1 -}\ \frac{G_2}{1+G_2 -}\ \frac{G_3}{1+G_3 -}\ \frac{G_4}{1+G_4 -} \quad \cdots\cdots\cdots$$

where

$$G_1 = \frac{\alpha+\beta}{\alpha+1}\ x$$

$$G_2 = \frac{\alpha+\beta+1}{\alpha+2}\ x$$

and in general,

$$G_r = \frac{\alpha+\beta+r-1}{\alpha+r}\ x$$

then

$$I_x(\alpha+\beta) = \frac{\Gamma(\alpha+\beta)\, x^a (1-x)^\beta}{\Gamma(\alpha)\ \Gamma(\beta)}\ \frac{1}{1-W} \tag{15}$$

The second continued fraction to be presented in connection with the incomplete beta function is given below

$$I_x(\alpha+\beta) = \frac{\Gamma(\alpha+\beta)\, x^a (1-x)^\beta}{\Gamma(\alpha)\Gamma(\beta)}\ \left\{1+ \frac{k_1 x}{1+I_1 x+}\ \frac{k_2 x^2}{1+I_2 x+}\ \frac{k_3 x^3}{1+I_3 x+}\ \cdots\cdots\cdots\right\} \tag{16}$$

where

$$k_1 = \frac{\alpha+\beta}{\alpha+1}, \qquad I_1 = \frac{\alpha+\beta+1}{\alpha+2}$$

$$k_{s+1} = \frac{s(s-\beta)(\alpha+\beta+s)}{(\alpha+2s-1)(\alpha+2s)^2(\alpha+2s+1)}$$

$$I_{s+1} = \frac{s(\beta-s)}{(\alpha+2s)(\alpha+2s+1)} - \frac{(\alpha+s+1)(\alpha+s+\beta+2)}{(\alpha+2s+2)(\alpha+2s+1)}$$

$s = 1,2,3, \ldots\ldots$

The unwieldy form of computation is clearly a disadvantage of the above expansion.

3. OTHER COMPUTER EVALUATION OF CONTINUED FRACTIONS

This final section of the paper deals with some other computer evaluation of a continued fraction of the form:

$$b_o + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \frac{a_4}{b_4 +} \ldots\ldots\ldots\ldots \frac{a_k}{b_k +}$$

Recall that the n^{th} convergent of this continued fraction is given by the following expression;

$$b_o + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \frac{a_4}{b_4 +} \ldots\ldots\ldots\ldots \frac{a_n}{b_n}$$

The first method of evaluation presented in the previous section is to compute a_n / b_n first, and then work up the continued fraction. Also the following recursive technique is suggested;

let $q_n = b_n$

$$q_{k-1} = b_{k-1} + \frac{a_k}{q_k} \qquad k= n, n-1, n-2 \ldots\ldots, 2, 1$$

Then q_0 is the value of the truncated expansion. If one uses this algorithm, continued fraction may be quite suitable for computer evaluation.

A second algorithm which may be used to evaluate a continued fraction is;

Define the function c_k as follows:

$c_{n+1} = 0$

$c_k = a_k/(b_k + c_{k+1})$, $\quad$ k= n, n-1, n-2 ....., 2, 1

Then the value of the above truncated continued fractions becomes $b_0 + c_1$. Actually the algorithms in Tables 1 to 4 are similar to these two algorithms and the algorithm used in [7], and they called "backward" algorithms, or from bottom to top.

One may also evaluate a continued fraction in a "forward" direction, by using other, less obvious, algorithm. In this manner, one may generate, the successive convergence,

$$\frac{A_1}{B_1}, \quad \frac{A_2}{B_2}, \ldots\ldots, \frac{A_n}{B_n}$$

where (A_n / B_n) denotes the n^{th} convergent, Blanch gives the "forward A - B" method [11]. She proceeds as follows:
define

$$A_{-1} = 1, \quad A_0 = b_0, \qquad B_{-1} = 0, \quad B_0 = 1$$

Then one can generate A_k and B_k by means of the following recursive formula:

$$A_k = b_k A_{k-1} + a_k A_{k-2}$$

$$B_k = b_k B_{k-1} + a_k B_{k-2} \qquad K= 1,2,3, \ldots$$

In this research, this algorithm for the evaluation of continued fractions was used and the following continued program was evaluated:

$$\frac{a_1}{b_1 +} \quad \frac{a_2}{b_2 +} \quad \frac{a_3}{b_3 +} \quad \frac{a_4}{b_4 +} \ldots\ldots\ldots\ldots \frac{a_k}{b_k +}$$

A computer program was written, so that two convergences would be evaluated per cycle, by proceeding as follows

Let AEV = a_{2k}, AOD = a_{2k+1}, BEV = B_{2k} and BOD = B_{2k+1}
where K= 1,2,3, ...

Then for each cycle, set

ALO = BEV* AHI + AEV * ALO (one could get A_2, A_4, A_6..)
BLO = BEV* BHI + AEV * BLO (one could get B_2, B_4, B_6 ..)
AHI = BOD* ALO + AOD * AHI (one could get A_3, A_5, A_7 ..)
BHI = BOD* BLO + AOD * BHI (one could get B_3, B_5, B_7 ..)

where initially

ALO = $A_0 = 0$ AHI = $A_1 = a_1$,
BLO = $B_0 = 1$ BHI = $B_1 = b_1$,
Then (A_n / B_n) = (AHI/BHI) is the value of the truncated continued fraction.

4. CONCLUSION

In order to determine the effectiveness of the expansions, it was necessary to evaluate probabilities for several convergences over a wide range of the functions. In earlier paper [7], the author studied the effectiveness of the continued fractions for the evaluation of the incomplete gamma distribution. A similar investigation for the other continued fractions for the binomial, normal and the incomplete beta distributions were carried out in the research to determine the effectiveness of these expansions, it was necessary to evaluate probabilities for several convergences over a wide range of the functions. In the initial runs, results were obtained for the 10^{th}, 20^{th}, 50^{th}, 100^{th}, and 200^{th}, convergences. Originally the backward method was used. In contrast, if a forward algorithm is implemented one only needs to print the result at appropriate intervals of one loop. Thus it was felt that the forward algorithm would be more efficient.

However, there was one drawback in the forward algorithm which had not been foreseen. An analysis of the method reveals that many intermediate multiplications are required. In this particular case, the values increased rapidly, and overflow problems were encountered. Consequently only up to the 50^{th} convergences could be evaluated. Fortunately, no such problems surfaced when the backward algorithm was utilized.

Before the final conclusions are stated, the reader should be warned not to make hasty judgements based on the above comments. One should not conclude that the backward approach is always better because more convergences may be obtained. On the contrary, the opposite problem may present itself. Since an algorithm which has been discarded in one instance may be the most efficient in another, no preferences for any method have been stated.

This paper avoided stating preferences for any of the algorithms. Each has its merits, which were alluded to in the above discussions. There is another use of forward algorithms, however, that has not yet been discussed. The presentation above assumed that one is interested in calculating the n^{th}, convergent of the expansion of loop. Another approach would be the following, choose a small tolerance level for the error bound of the calculations.

Then using a forward algorithm, evaluate successive convergences, and their difference. When that difference is less than the prescribed tolerance level, stop the calculations and give the result. This approach, of course, could not be used if one evaluated the expansion using a backward algorithm, since evaluation of such successive convergences would be impractical.

Having thus concluded the discussion of several of the methods available for evaluating a continued fraction, a logical sequence would next dictate a discussion of error estimates. However, error estimates for rational and continued fraction approximations are, in general, quite difficult to obtain. One could compare the evaluation of continued fractions to the evaluation of the fractional form of a function. Fike concludes that, it may be necessary to pay for speed with some sacrifice in accuracy of computed results [1]. To compare those continued fractions that evaluate the probability distribution functions in this research, extra precision was used to provide a satisfactory remedy for the difficulty, and mathematical tables were used for comparison [12]. However, in conclusion, the continued fractions as a means of obtaining rational approximations to probability distribution is a very powerful and effective technique.

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