A Two-Point Boundary Value Problem for a Second Order Differential Equation with Parameter

by

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In this paper we study the problem of finding the value of the parameter λ for which the differential equation.

$$\frac{d^2x}{dt^2} = f(t,x, \frac{dx}{dt}, \lambda) \tag{1}$$

has a solution satisfying the boundary conditions

$$x(o) = x'(o) = o \quad , \quad x(T) = x_T, \tag{2}$$

where $t \in [o,T]$,x,x',λ are elements of some Banach space E. Sufficient conditions for the existence of a unique solution are found as well as an iterative procedure for finding the solution $(x^*(t),\lambda^*)$.

Assume that the following conditions are satisfied:

1) The operator $f(t,x,y,\lambda)$ is continuous with the totality of its arguments in $Q = \{ [o,T] , \| x \| \leq R, \|y\| \leq R' , \|\lambda\| \leq \rho \}$ and satisfies the Lipschitz condition

$$\| f(t,x,y,\lambda) - f(t,\bar{x},\bar{y},\bar{\lambda}) \| \leq L \| x-\bar{x} \| + M \| y-\bar{y} \| + N \| \lambda - \bar{\lambda} \| \tag{3}$$

2) There exists a linear bounded and continuously invertible operator B, such that in Q the following inequality is satisfied:

$$\| \int_0^{\tau} (T-t) \{ f(t,x,y,\lambda) - f(t,x,y,\bar{\lambda}) \} dt - B(\lambda - \bar{\lambda}) \| \leq \in \| \lambda - \bar{\lambda} \| \tag{4}$$

3) The Lipschitz constants and $\in$ are such that

$$\frac{T^2}{2}(LR+MR'+N\rho+m) \leq R \tag{5}$$

$$T(LR+MR'+N\rho+m) \leq R', \tag{6}$$

$$\| B^{-1} \| \{ \in \rho + \frac{T^2}{2}(LR+MR') + m + \| x_T \| \} \leq \rho \tag{7}$$

where $m = \underset{o \leq t \leq T}{MAX} \| f(t,o,o,o) \|$,

For the solution of problem (1), (2) wee construct the following two sequences.

$$\lambda_{n+1} = \lambda_n - B^{-1} \{ X_n(T) - X_T \} \quad (8)$$

$$X_{n+1}(t) = \int_0^t (t-s) f(s, x_n(s), x_n'(s), \lambda_n) ds, \; n = o,1,2, \ldots, \quad (9)$$

where $x_o'(t)$ is continuous in $[o,T]$, $x_o'(o) = o$

$\| x_o'(t) \| \leq R'$, $\| x_o(t) \| \leq R$ and $\| \lambda_o \| \leq \rho$.

Theorem

Let the conditions (3) – (7) be satisfied and let the

$$\text{Matrix } A = \begin{bmatrix} \frac{LT^2}{2} & \frac{MT^2}{2} & \frac{NT^2}{2} \\ LT & MT & NT \\ \frac{LT^2}{2} \|B^{-1}\| & \frac{MT^2}{2} \|B^{-1}\| & \in \|B^{-1}\| \end{bmatrix}$$

be an a-matrix i.e. the determinants of all principal minors of the matrix $(I_3 - A)$ are positive, where I_3 is the unit 3×3 matrix (see [3]). The problem (1), (2) has a unique solution $(x^*(t), \lambda^*)$ which is the limit of the sequences (8), (9) and the speed of convergence is given by

$$\begin{bmatrix} \underset{O \leq t \leq T}{\max} \| x_n(t) - x^*(t) \| \\ \underset{o \leq t \leq T}{\max} \| x_n'(t) - x'^*(t) \| \\ \| \lambda_n - \lambda^* \| \end{bmatrix} \leq 2A^n (I_3 - A)^{-1} \begin{bmatrix} R \\ R' \\ \rho \end{bmatrix} \quad (10)$$

Proof

Let $\tilde{E}$ be a space of vector functions

$$X(T) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \lambda \end{bmatrix}$$

where x(t) is a continuously differentiable abstract function in $[o,T]$ with values in E, $\dot{x}(t)$ its derivative and $\lambda \in E$ - with the following norm.

$$\| X(t) \| \tilde{E} = \begin{bmatrix} \max\limits_{o \leqslant t \leqslant T} & \|x(t)\| \\ \max\limits_{o \leqslant t \leqslant T} & \|\dot{x}(t)\| \\ & \|\lambda\| \end{bmatrix}$$

Define in $\tilde{E}$ an operator

$$G(X(t)) \equiv \begin{bmatrix} \int_o^t (t-s)f(s,x(s),\ x'(s),\ \lambda)\ ds & \equiv U(X(t)) \\ \int_o^t f(s,x(s), x'(s),\lambda)\ ds & \equiv V(X(t)) \\ \lambda - B^{-1} \{ X(T) - X_T \} & \equiv W(X(t)). \end{bmatrix}$$

Let $X(t) \in \tilde{S}$, where

$$\tilde{S} = \left\{ X(t) \in \tilde{E}, \| X(t) \| \tilde{E} \leqslant \begin{bmatrix} R \\ \dot{R} \\ \rho \end{bmatrix} \right\}$$

Then from (4) — (8) we get:

$$\| U(X(t) \| \leqslant \| \int_0^t (t-s)f(s,x(s),\ x'(s),\ \lambda)\ ds - \int_0^t (t-s)f(s,o,o,o,)$$

$$ds \,\| + \frac{mT^2}{2}$$

$$\leqslant \frac{T^2}{2}(LR+M\grave{R}+N\rho+m) \leqslant R;$$

$$\| V\,(x(t))\,\| \leqslant \| \int_o^t f(s,x(s),x'(s),\lambda)\,ds - \int_0^t f(s,o,o\text{-}ds \,\| + mT,$$

$$T(LR+M\grave{R}+N\rho+m) \leqslant \grave{R}\,;$$

$$\| W(X(t))\| \leqslant \| B^{-1}\| \Big[\| B(\lambda) - \int_o^T (T-s)\Big\{f(s,x(s),\grave{x}(s),\lambda) - f(s,x(s),$$
$$\grave{x}(s),o)\Big\}\,ds\,\|$$

$$+ \| \int_o^T (T-s)\{f(s,x(s),\grave{x}(s),o) - f(s,o,o,o)\}\,.ds\,\| + \frac{mT^2}{2}$$

$$+ \| X_T \|\Big] \leqslant \| B^{-1}\| \{ \in\rho + \| x_T \| + \frac{T^2}{2}(LR+M\grave{R})\} \leqslant \rho\,.$$

Then it follows that

$$\| G(X(t))\|_{\widetilde{E}} \leqslant \begin{bmatrix} R \\ \grave{R} \\ \rho \end{bmatrix},$$

$$\text{I.E. } G(X(t)) \in \widetilde{S}. \tag{11}$$

Let $X(t)$ and $\overline{X}(t) \in \widetilde{S}$. then it follows from (3) that:

$$\| U(X(t)) - U(\overline{X}(t))\| \leqslant \frac{LT^2}{2}\max_{o\leqslant t\leqslant T}\| x(t)-\bar{x}(t)\| + \frac{MT^2}{2}\max_{o\leqslant t\leqslant T}\| x'(t)$$
$$-\bar{x}'(t)\| + \frac{NT^2}{2}\| \lambda - \bar{\lambda}\|\,;$$

$$\| V(X(t)) - V(\overline{X}(t))\| \leqslant LT,\max_{O\leqslant t\leqslant T}\| x(t\text{-}-\bar{x}(t)\| + MT\max_{o\leqslant t\leqslant T}\| x(t)-\bar{\grave{x}}(t)\|$$
$$+ NT\,\| \lambda - \bar{\lambda}\|\,;$$

$$\| W(X)(t))-W(\overline{X}(t)) \| \leqslant \| B^{-1} \| \left[\frac{LT^2}{2} \max_{o \leqslant t \leqslant T} \| x(t)-\bar{x}(t) \| + \frac{MT^2}{2} \max_{o \leqslant t \leqslant T} \| x(t)-\bar{x}(t) \| + \in \| \lambda - \bar{\lambda} \| \right].$$

These inequalities lead to:

$$\| G(X(t)) - G(\overline{X}(t)) \|_{\tilde{E}} \leqslant A \| X(t) - \overline{X}(t) \|_{\tilde{E}} \tag{12}$$

From (11), (12) using the generalized principle of contraction mapping [3], it follows that the operator G posseses in $\tilde{S}$ a unique stationary point $x^*(t) = G(X^*(t))$ and this point is the limit of the successive iterations.

$$X_{n+1}(t) = G(X_n(t)),\ n = o,1,\ ....,$$

starting with an arbitrary element $X_O(t) \in \tilde{S}$; the speed of convergence is determined by

$$\| X_{n+1}(t) - X^*(t) \|_{\tilde{E}} \leqslant A^n(I_3-A)^{-1} \| X_o(t)-G(X_o(t)) \|_{\tilde{E}}$$

from which (10) follows. This proves the theorem.

Remarks 1:

If the R. H. S. of equation (1) can be written in the form

$$f(t,x,y,\lambda) = b(t)\lambda \pm g(t,x,y,\lambda) \text{ and}$$

$$\| g(t,x,y,\lambda) - g(t,x,\bar{\lambda}) \| \leqslant \theta(t) \| \lambda - \bar{\lambda} \|,$$

then we can take

$$B = \int_o^t (T-t)b(t)\, dt$$

In this case

$$\in = \int_0^t (T-t)\, \theta(t)\, dt$$

Remarks 2:

The inequalities (5), (6) (7) can be replaced by the following:

$$\| f(t,x,y,\lambda) \| \leqslant r(t)$$

$$\int_0^t (T-t)r(t)dt \leqslant R, \qquad \int_0^t r(t)\,dt \leqslant \acute{R} \tag{13}$$

$$\| f(t,x,y,O) \| \leqslant p(t)$$

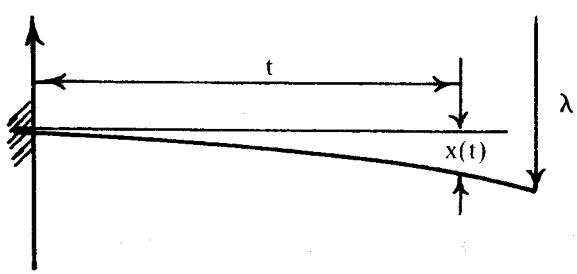
$$\| B^{-1} \| \{ \int_\sigma^t (T-t)\, p\,(t)\, dt + \| x_T \| \} \leqslant (1- \in \| B^{-1} \|) \rho \tag{14}$$

Also the condition that the matrix A is an a-matrix can be replaced by the assumption that $\| A \| = \propto < 1$ (15)
In the last case equation (10) will reduce to

$$\| X_{n+1}(t) - \overset{*}{X}(t) \|_{\tilde{E}} \leqslant \frac{2 \propto^n}{1 - \propto} \begin{bmatrix} R \\ \grave{R} \\ \rho \end{bmatrix}$$

Example:

To show that the conditions of the theorem are consistent, we introduce the following example.



Consider the scalar differential equation of the deflection of a contilever of unit length under the action of a weight λ at its end. The relevant equation is

$$\ddot{x}(t) = \frac{-1}{eI} (1+(\acute{x}(t)^2)^{3/2} \lambda(1-t)$$

where e is the modules of elasticity and I is the centroidal second moment of the cross-section of the lever. It is required to calculate the value of λ that will give a maximum deflection x_1 at the end point.

Taking $x_1 = 0.05$ and $eI = 10$ the problem becomes

$$\ddot{x}(t) = \frac{-1}{10}\left[1+(\dot{x}(t))^2\right]^{3/2} \lambda (1-t)$$

$x(o) = \dot{x}(o) = o$ and $x(1) = 0.05$

Here

$E = (-oo\ ,\ + oo),\ T = 1,\ x_T = 0.05$

$$f(t,x,y,\lambda) = \frac{-1}{10}(1+y^2)^{3/2}\lambda(1-t)$$

Choosing $R = 0.1$, $\dot{R} = 0.11$ and $\rho = 2$,

Then $r(t) = 0.21\ (1-t)$, $p(t) = 0$ and the inequalities (13) are satisfied.

Taking $b(t) = \frac{-1}{10}(1-t)$ then $B = \frac{-1}{30}$

$\theta\ (t) = 0.0037(1-t)$

$\therefore\ \epsilon = 0.002$

from which it follows that (14) is satisfied.

$L = O\ ,\ M = 0.08\ ,\ N = 0.105$

It can be easily seen that the matrix A has a norm less than unity and at the same time it is an a-matrix

i.e. All the assumptions of the theorem are satisfied.

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المسـألة الحدية ذات النقطتان للمعـادلات التفاضلية من الدرجـة الثانيـة ذات البارامتر

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ملخــــــــــص

في البحث ندرس ايجاد قيمة البارامتر التي تجعل حل المعادلة التفاضلية المتجهة .

$$\frac{d^2x}{dt^2} = f(t,\ x,\ \frac{dx}{dt},\ \lambda)$$

يحقق الشروط الحدية الآتيـــة :

$$X(o) = X'(o) = o, \qquad X(T) = X_T$$

حيث كل من λ ,x, متجهات في فراغ بناخ t, Banach space

متغير قياسى .

تعطى النظرية المثبته الشروط الكافية لأن يكون لهذه المسألة حل وحيد وكذلك تعطى طريقة الحل بتكوين متسلسلتين . كما تحدد النظرية أيضا سرعة تقارب المتسلسلتان من الحل المضبوط . وفي نهاية البحث تطبق النظرية على مثال للمعادلة التفاضلية القياسية الخاصة بانحناء كوبولى محمل في نهايته بثقل λ ويبين هذا المثال تطابق شروط النظرية على المسألة المعطاه .