

THEORY AND COMPUTATIONS FOR SYSTEMS MODELED BY FIRST ORDER DIFFERENTIAL EQUATIONS

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نظريات وحسابات مُنمذجة بمعادلات تفاضلية من الرتبة الأولى

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باستخدام دوال «سك» تم بناء تقريب لحل مجموعة معادلتين تفاضليتين من المرتبة الأولى، كما تم برهان أن الخطأ في الحل التقريبي متقارب أُسيًا.

Key Words: First Order Differential Equations, Sinc functions, Collocation.

ABSTRACT

A numerical scheme using Sinc functions is developed to approximate the solution of a 2 x 2 system of first order differential equations. The error in the approximate solution is shown to converge exponentially.

1. INTRODUCTION

Systems of first order differential equations of the form

$$\begin{aligned} \frac{du_1}{dt} &= F_1(u_1, u_2, t) \\ \frac{du_2}{dt} &= F_2(u_1, u_2, t) \end{aligned} \quad (1.1)$$

are often encountered in many mathematical models, as in Neutron flow, electrical networks, residential segregation, and more (see, [2]). It is well known that the system (1.1) has a unique solution satisfying some boundary conditions $u_1(a) = \alpha_1$, and $u_2(a) = \alpha_2$.

Consider a system of two linear first order differential equation:

$$\begin{aligned} \frac{du_1}{dt} &= b_{11}(t)u_1 + b_{12}u_2 + f_1(t) \\ \frac{du_2}{dt} &= b_{21}(t)u_1 + b_{22}u_2 + f_2(t) \end{aligned} \quad (1.2)$$

on the interval $[a, T]$ with boundary values

$$\begin{aligned} u_1(a) &= u_1^0 \\ u_2(a) &= u_2^0 \end{aligned} \quad (1.3)$$

We write (1.2) in the vector form:

$$\frac{d\vec{u}}{dt} = B(t)\vec{u}(t) + \vec{f}(t) \quad (1.4)$$

where B is the matrix $[b_{ij}(t)]_{2 \times 2, i, j = 1, 2}$ and

$$\vec{u} = (u_1, u_2)^T, \vec{f} = (f_1, f_2)^T.$$

Generally we solve system (1.4) subject to (1.3) in terms of matrices and eigenvalues, or, using Laplace transformation. To solve this system exactly, sometimes, we are concountered to put some assumption on the entries of the matrix B . In this paper we shall instead look at a new method for solving (1.4) via the use of Sinc functions to find an approximate solution of system (1.4) and (1.3). The main idea is to replace integral equation by its discrete Sinc approximations. There are several reasons to approximate by Sinc functions. First they are easily implemented and give good accuracy for problems with singularities. Further, approximations by Sinc functions are typified by errors of the form $O(\exp(-c/h))$ where $c, h > 0$. The paper is organized as follows: in section 2 we give the relevant properties of Sinc functions, assumptions, and basic techniques. In section 3 we present and verify our approximation.

2. THE SINC FUNCTION

A general review of Sinc functions and their uses has recently been given by Stenger in [6]. We therefore only outline properties important to our present goals, and refer to [6] for further references.

The term $Sinc(x)$ which is defined by

$$Sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

is an entire function over \mathbb{R} . We shall adopt a more powerful notation for the Sinc function. Namely, if $h > 0$ and k is an integer, we define the k th Sinc function $S(k, h)$ by

$$S(k, h)(x) = \frac{\sin[\pi(x - kh)/h]}{\pi(x - kh)/h} \quad (2.1)$$

Approximations on $[a, T]$ are obtained from corresponding approximations on $\mathbb{R} = (-\infty, \infty)$ via a conformal map. To be approximable on \mathbb{R} , $u \in C^\infty(\mathbb{R})$ must obey certain analyticity and boundedness conditions in a strip in the complex plane \mathbb{C} . With this in mind we first list properties over \mathbb{R} . To approximate functions via Sinc method we make an assumption on the growth of the function in some domain.

Definition 2.1 for $d > 0$ define the domain $D_d = \{z : z = x + iy, |y| < d\}$.

Let us introduce the following definition which are fundamentals to the development of Sinc method.

Definition 2.2 Let D be a simply connected domain and $a, b \in \partial D$ be such that $a \neq b$, and let Γ be defined by $\Gamma = \{z \in X : z = \sigma^{-1}(u), u \in \mathbb{R}\}$. Then there exists a conformal map $\sigma : D \rightarrow D_d$ satisfying $\sigma^{-1}(\mathbb{R}) = \Gamma$ and such that for $z \in \Gamma$, $\lim_{z \rightarrow a^+} \sigma(z) = -\infty$, $\lim_{z \rightarrow b^-} \sigma(z) = \infty$.

Given σ and its inverse σ^{-1} and a constant number h , let us set $z_k = z_k(h) = \sigma^{-1}(kh)$, $k = 0, \pm 1, \pm 2, \dots$. Of particular interest for the present study is the class of functions $L_\alpha(D)$ that characterized in the definition

Definition 2.3 Let α and β denote positive numbers, and let $L_{\alpha, \beta}(D)$ denote the family of functions f analytic in D , and such that for some constant $c > 0$, and all $z \in D$, we have

$$|f(z)| \leq c \frac{|\exp(\alpha z)|}{(1 + |\exp(z)|)^{\alpha + \beta}}$$

We simply write $L_\alpha(D_s)$ for $L_{\alpha\alpha}(D_s)$. For the same of simplicity of representation of the material, we shall present our approximating formulas for the space $L_\alpha(D_s)$ instead of for the more complicated space $L_{\alpha b}(D_s)$.

Let us next describe Sinc definite integration formula over an interval, i.e., give general formula for approximating: $\int_a^v F(u)du$, $v \in \Gamma$. At the outset, we define the numbers σ_k and

$$\delta_k^{(-1)} \text{ by } \sigma_k = \int_0^k \text{Sinc}(x) dx, k \in Z \text{ and } \delta_k^{(-1)} = \sigma_k + 1/2.$$

We define a Toeplitz matrix $I^{(-1)}$ of order m by

$$I_m^{(-1)} = [\delta_{i-j}^{(-1)}] \quad (2.2)$$

with δ_{i-j} denoting the (i, j) the element of $I_m^{(-1)}$. We can thus state the following theorem, the result of which enables us to collocate (linear or nonlinear) initial value problems over an interval.

Theorem 2.1 (see, [6, p. 219]) *Let $F/\phi' \in L_\alpha(D_s)$ with $\alpha > 0$, $d > 0$, let $\delta_{i,j}$ be defined as above, and $h = \sqrt{\pi d / (\alpha N)}$. Then there exists a constant K which is independent of N , such that*

$$\left| \int_a^z F(t) dt - h \sum_{j=-N}^N \delta_{i-j}^{(-1)} \frac{F(z_j)}{\phi'(z_j)} \right| \leq K \exp(-\sqrt{\pi d \alpha N}) \quad (2.3)$$

3. DESCRIPTION OF SINC APPROXIMATION

It is convenient for deriving an approximate solution for the system (1.4) by Sinc-Galerkin method to start with the scalar first order differential equation

$$\frac{du}{dt} = B(t)u(t) + f(t), t \in (a, T) \quad (3.1)$$

with

$$u(a) = u^0 \quad (3.2)$$

Integrating with respecting with respect to t , we get

$$u(t) = \int_a^t [B(\tau)u(\tau) + f(\tau)] d\tau + u^0 \quad (3.3)$$

To obtain approximations over (a, T) , we make the use of the conformal map: $\phi(z) = \log((z-a)/(T-z))$. For any d such that $0 < d \leq \pi/2$, ϕ maps the region $D_s = \{z : \arg((z-a)/(T-z)) < d\}$ onto D_s where D_s is as defined in section 2. The mesh size h represent the mesh size in the infinite strip D_s for the uniform grid $\{ih\}$, $-\infty < i < \infty$. The Sinc grid points $t_i \in (a, T)$ in D_s are the inverse images of the equispaced grid points; that is $t_i = \phi^{-1}(ih) = (\exp(ih) + a) / (1 + \exp(ih))$. We shall assume that both B/ϕ' and f belongs to the class of

functions $L_\alpha(D_s)$. If B is matrix this shall imply that all the components of B/ϕ' (or, f as a vector) are in the class $L_\alpha(D_s)$. Now, in equation (3.3) we collocate via the use of the indefinite integration formula (2.3). Using the notation $D(1/\phi'(t)) = \text{diag}[1/\phi'(t_{-N}), \dots, 1/\phi'(t_N)]$, then equation (3.3) can be written as a system of $m = 2N + 1$ linear equations

$$U = H I_m^{(-1)} D(B/\phi')U + H I_m^{(-1)} D(1/\phi')F + U^0 \quad (3.4)$$

where $U = [u_{-N}, \dots, u_N]T$, $F = [f(z_{-N}), \dots, f(z_N)]T$ with the nodes $t_i = \phi^{-1}(ih)$ for $i = -N, \dots, N$ where $h = \sqrt{\pi d / (\alpha N)}$, and $I_m^{(-1)}$ as defined in section 2. And where U^0 denotes the vector of $2N + 1$ constants values $U^0 = [u^0(z_{-N}), \dots, u^0(z_N)]T$. Define the matrices A and E by $A = h I_m^{(-1)} D(B/\phi')$, $E = h I_m^{(-1)} D(1/\phi')$. Equation (3.4) can be written in the form

$$U = AU + EF + U^0 \quad (3.5)$$

To prove convergence of the method. Evaluate the integral in (3.3) at the notes t where $i = -N, \dots, N$, to get

$$u(t_i) = \int_a^{t_i} [B(\tau)u(\tau) + f(\tau)] d\tau + u^0$$

with the same matrices A , E and U^0 as mentioned above, and using the approximation in (2.3) we get, in matrix form, the approximation

$$U = AU + EF + U^0 + \tilde{K} \exp(-\sqrt{\pi d \alpha N})$$

where the constant \tilde{K} is a vector such that each entry is bounded by the constant K in equation (2.3). So, the error ERR can be bounded as

$$\|ERR\| \leq \|U - (AU + EF + U^0)\| \leq \tilde{K} \exp(-\sqrt{\pi d \alpha N})$$

i.e., the discretization error that arises when a differential equation is replaced by a discrete system of algebraic equations is exponentially small. With the notation as above, we just proved the following Theorem.

Theorem 3.1 *Let B/ϕ' , $f \in L_\alpha(D_s)$, let the function $u(t)$ be defined as in (3.3), and let the matrix U be defined as in (3.5). The for $h = \sqrt{\pi d / (\alpha N)}$ there exists a constant K independent of N such that*

$$\sup \| [u(t_i)] - U \| \leq K \exp(-\sqrt{\pi d \alpha N})$$

Now, we may attempt to solve the linear systems of equation s(3.5) by successive approximations, that is, by means of the iterative scheme.

$$U^{(n+1)} = AU^{(n)} + EF + U^0 \quad (3.6)$$

The idea is to produce a sequence of iteration that converges to the solution of the problem, as noted in the following theorem

Theorem 3.2 *The sequence $U^{(n)}$ defined in (3.6) converges for all N sufficient large, to the exact solution provided that $(T - a) < 11/(10 \|B\|_\infty)$*

Proof: Recall the definition of $\delta^{(-1)}$ as defined in section 2 satisfies the inequality (see [6, p. 477]) $|\delta^{(-1)}| \leq 11/10$, we have

$$\begin{aligned} \|A\|_\infty &= \|hI^{(-1)} D(B/\phi')\|_\infty \\ &= \max_i \sum_{j=-N}^N |\delta_{i,j}^{(-1)} B(z_j) / \phi'(z_j)| \\ &\leq \frac{11}{10} h \sum_{j=-N}^N |B(z_j) / \phi'(z_j)| \\ &\leq \frac{11}{10} \int_a^T |B(t)| dt \\ &\leq \frac{11}{10} (T - a) \sup_{t \in (a,T)} |B(t)| \\ &\leq \frac{11}{10} (T - a) \|B\|_\infty \end{aligned}$$

where in the third inequality we used Theorem 2.1, with the fact that $B / \phi' \in L^\alpha(D)$. For the iteration scheme to converge we require that $\|A\|_\infty < 1$. Therefore we can always achieve

convergence of the scheme (3.6) by choosing $(T - a) < \frac{10}{11\|B\|_\infty}$

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