Qatar Univ. Sci. J (2000), 20 : 25 - 30

AN EXTENSION OF THE SZASZ-MIRAKYAN OPERATOR ON THE WHOLE REAL AXIS

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توسيع مؤثر «شاش-ميراكيان» على طول محور الأعداد الحقيقية

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يهدف هذا البحث إلى عرض لتوسيع جديد لمؤثر «شاش-ميراكيان» على طول محور الأعداد الحقيقة، كما يبين نظرية التقريب التقاربي عليه.

Key Words: The Szasz-Mirakyan operator; extension; whole real axis; approximation.

ABSTRACT

The aim of this paper is to present a new extension of the Szasz-Mirakyan operator n the whole real axis, and to give its convergent approximation theorem.

1. INTRODUCTION

In 1950 O.ASzasz[1] introduced and investigated the following operator, the well known Szasz-Mirakyan operator,

$$S_n(f;x) = e^{-(nx)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}, \qquad x \in [0,\infty),$$

where f is defined on $[0, \infty)$.

Later on, J. Grof[2] gave the following extended operator of the Szasz-Mirakyan operator. Let f be defined in $(-\infty, \infty)$,

$$T_n(f;x) = \frac{1}{e^{nx} + e^{-nx}} \sum_{k=0}^{\infty} [f(\xi_{n,k}) + (-1)^k f(-\xi_{n,k})] \frac{(nx)^k}{k!}, \quad x \in (-\infty, \infty),$$

where $\frac{k}{n} \le \xi_{n,k} < \frac{k+1}{n}$, and investigated its approximation problem.

Recently, Rempulska lucyna and Skorupka Moriala [3] gave a modified operator as follows. Let f is defined on $[1, \infty)$,

$$A_n(f;x) = \frac{1}{1+Sinhnx} \left\{ f(o) + \sum_{k=0}^{\infty} f\left(\frac{2k+1}{n}\right) \frac{(nx)^{2k+1}}{2k+1)!} \right\},$$

$x \in [0,\infty)$,

and discussed corresponding approximation problem

In this paper we give a new extension of the Szasz-Mirakyan operator.

Let $f(x)$ be defined $(-\infty, \infty)$ and m be an natural number, the extended operator

$$R_n^m(f;x) = \frac{1}{2 + \operatorname{sgn} x(e^{(nx)^m} - e^{-(mx)^m})} \{2f(o)+$$

$$\operatorname{sgn} x \sum_{k=0}^{\infty} \left(f\left(\frac{\theta_k^{\frac{1}{m}}}{n}\right) - (-1)k\, f\left(-\frac{\theta_k^{\frac{1}{m}}}{n}\right) \right) \frac{(nx)^{km}}{k!} \Big\}, \; -\infty < x < \infty, \quad (1)$$

where $k \le \theta < k+1$ and sign x is the sign functioin.

Especially, when $\theta\, k=k, x \in [0,\infty)$ and f is an even function, we can see clearly that the operator $R_n^1(f;x)$ is just the operator $A_n(f;x)$, therefore $R_n^1(f;x)$ is also an extension of the operator $A_n(f;x)$.

2. RESULTS

By $\omega_{2A}(f; \delta)$ denote the modulus of continuity of f in the inteval [-2A,sA] (a>0), that is,

$$\omega_{2A}(f; \delta) = \sup_{|x-y| \le \delta} |f(x) - f(y)|,$$

$$\delta > 0, x, y \in [-2A, 2A], \qquad (2)$$

(in the following we use the shorter form $\omega_{2A}(\delta)$ it has the following properties[4]

(1) $\omega_{2A}(f;\delta) \ge 0$ is an increasing and $\omega_{2A}(\delta) \to 0, \delta \to 0$.

(2) $f(x) - f(y) | \le \omega_{2A}(|x - y/)$, $\quad x, y \in [-2A, 2A]$.

(3) $\omega_{2A}(\lambda \cdot \delta) \le (\lambda + 1)\, \omega_{2A}(\delta) \quad (\lambda > 0, \delta > 0)$

(4) $\frac{1}{n} = O\left(\omega_{2A}\left(\frac{1}{n}\right)\right)$.

In this paper we give the following theorem.

Theorem. Let $f(x)$ be a continuous function defined in $(-\infty, \infty)$. If for some $\alpha > 0$, $f(x) = O(e^{\alpha [x]^m})$ $(-\infty < x < \infty)$, then when $m \ge 1$ is an odd number, the inequality

$$Rn(f;x) - f(x) = O\left[\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\right]$$

holds in any inteval [-A,A] (A>0) for $n > \max\left\{8\alpha e^{2\alpha}, \frac{2}{A}\right\}$. Here and after the bounds of "O" are independent of n.

When m=1, $\theta_k = k$, From the above theorem, we immediately obtain the following.

Corollary 1. Let $f(x)$ be a cntinuous functin defined on $[1, \infty)$. If for some $\alpha > 0$, $f(x) = O(e^{\alpha x})$ $(0 \le x < \infty)$, then

$$A_n(f;x) - f(x) = O\left[\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\right] \quad (0 \le x \le A),$$

for $n > \max\left\{8\alpha e^{2\alpha}, \frac{2}{A}\right\}$.

Corollary 2. Under the conditions of Corollary 1 and $f(o)=0$. Let

$$B_n(f;x) = \frac{1}{1 + sinh\, nx} \sum_{k=1}^{\infty} f\left(\frac{2k}{n}\right) \frac{(nx)^{2k}}{(sk)!},$$

then in any interval [0,A[, we have

$$B_n(f;x) - f(x) = O\left(\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\right)$$

when $n > \max\left\{8\alpha e^{2\alpha}, \frac{2}{A}\right\}$.

3. LEMMAS

In order to prove Theorem, we need some lemmas.

Lemma 1 : For $x>0$ and m is an natural number, the following identity holds

$$\sum_{k=0}^{\infty} \left(k - (nx)^m\right)^2 \frac{(nx)^{km}}{k!} = (nx)^m e^{(nx)^m}.$$

Proof : Using the known formula $e^{(nx)^m} = \sum_{0}^{\infty} \frac{(nx)^{km}}{k!} (-\infty < x < \infty)$,

(4)

successively differentiating (4) and then multiplying by x, we can get two equalities as follows

$$\sum_{0}^{\infty} k \,;\, \frac{(nx)^{km}}{k!} = (nx)^m e^{(nx)^m}, \tag{5}$$

$$\sum_{0}^{\infty} k^2 \,;\, \frac{(nx)^{km}}{k!} = \left((nx)^m + (nx)^{2m}\right) e^{(nx)^m}. \tag{6}$$

Multiplying (4), (5) and (6) by the factors $(nx)^{2m}$, $-2(nx)^m$ and 1 respectively, and then adding three equalities up together, we immediately obtain the equality (3).

Lemma 2 : When $x>0$ and $m\geq 2$ and $k\geq 1$ are integers, then

$$\left|k^{\frac{1}{m}} - nx\right| \leq \frac{\left|k - (nx)^m\right|}{(nx)^{\frac{m}{2}}}$$

Proof : When x > 0, m ≥ 2and k ≥ 1, we have

$$|k - (nx)^m| = \left|(k^{\frac{1}{m}} - nx)\left((k^{\frac{1}{m}})^{m-1} + (k^{\frac{1}{m}})^{m-2}(nx) + \ldots + (nx)^{m-1}\right)\right|$$

$$\geq \left|k^{\frac{1}{m}} - nx\right| \left|1 + nx)^{m-1}\right|$$

but

$$1+(nx)^{m-1} \geq \begin{cases} 1 \geq (nx)^{\frac{m}{2}}, & 0<nx\leq 1, \\ (nx)^{m-1} \leq (nx)^{\frac{m}{2}}, & 1\leq nx<\infty,\ m\geq 2, \end{cases}$$

so

$$|k - (nx)^m| \geq (nx)^{\frac{m}{2}} \left|k^{\frac{1}{m}} - nx\right| \qquad (m\geq 2).$$

From this lemma 2 is obtained.

Lemma 3: When $x>1$ and $m\geq 2$, the following inequality holds

$$\sum_{k=0}^{\infty} (k^{\frac{1}{m}} - nx)^2 \frac{(nx)^{km}}{k!} \leq 2e^{nx^m}. \tag{7}$$

Proof : Using lemma 2 and lemma 1, we get

$$\sum_{k=1}^{\infty} (k^{\frac{1}{m}} - nx)^2 \frac{(nx)^{km}}{k!} \leq \frac{1}{(nx)^m} \sum_{k=1}^{\infty} (k - (nx)^m)^2 \frac{(nx)^{km}}{k!}$$

$$\leq \frac{1}{(nx)^m} \sum_{k=0}^{\infty} (k - (nx)^m)^2 \frac{(nx)^{km}}{k!} = e^{(nx)^m},\ x>0, m\geq 2,$$

so the left-hand side of (7) is

$$\sum_{k=}^{\infty} (k^{\frac{1}{m}} - nx)^2 \frac{(nx)^{km}}{k!} \leq (nx)^2 + \sum_{k=1}^{\infty} (k^{\frac{1}{m}} - nx^2) \frac{(nx)^{km}}{k!}$$

$$\leq (nx)^2 + e^{(nx)^m} = \left[\frac{(nx)^2}{e^{(nx)^m}} + 1\right] e^{(nx)^m}.$$

$$\text{But } \frac{(nx)^2}{e^{(nx)^m}} \leq \begin{cases} (nx)^2 \leq 1, & 0<nx\leq 1, \\ (nx)^{2-m} \leq 1, & 1\leq nx<\infty \quad (m\geq 2), \end{cases} \tag{8}$$

and hence we get the conclusion of lemma 3.

Lemma 4: I*f* m≥1, δ >0 and α>0, then in any inteval (0,A], the inequality

$$e^{-(nx)^m} \sum_{\left|\frac{k^{\frac{1}{m}}}{n} - x\right| \geq \delta} e^{\frac{\alpha k}{n^m}} \cdot \frac{(nx)^{km}}{k!} = O(\frac{1}{n}) \tag{9}$$

holds for $n > \dfrac{2A\alpha e^{\alpha}}{\delta}$.

Proof. The left-hand side of (9) is

$$e^{-(nx)^m} \sum_{\left|\frac{k^{\frac{1}{m}}}{n} - x\right| \geq \delta} e^{\alpha \frac{k}{n^m}} \cdot \frac{(nx)^{km}}{k!} = e^{(nx)^m (e^{\frac{\alpha}{n^m}} - 1)} I_n(x), \tag{10}$$

where

$$I_n(x) = e^{-(nx)^m e^{\frac{\alpha}{n^m}}} \sum_{\left|\frac{k^{\frac{1}{m}}}{n} - x\right| \geq \delta} \frac{\left[(nx)^m e^{\frac{\alpha}{n^m}}\right]^k}{k!}. \tag{11}$$

Say $t = xe^{\frac{\alpha}{mn^m}}$ $(m\geq 1)$. If $\left|\frac{k^{\frac{1}{m}}}{n} - x\right| \geq \delta$, then we know that

$$\left|\frac{k^{\frac{1}{m}}}{n} - t\right| = \left|\frac{k^{\frac{1}{m}}}{n} - x + x\left[1 - e^{\frac{\alpha}{mn^m}}\right]\right| \geq \delta - |x|\,|1 - e^{\frac{\alpha}{mn^m}}|. \tag{12}$$

Using the inequalities $e^x - 1 \leq xe^x$ and $e^x - 1 \geq x (x>0)$, we have

$$0 < e^{\frac{\alpha}{mn^m}} - 1 \leq \frac{\alpha}{mn^m} e^{\frac{\alpha}{mn^m}} \leq \frac{\alpha}{n^m} e^{\alpha} < \frac{\delta}{2A} \quad (n > \frac{2A\alpha e^{\alpha}}{\delta}).$$

From this and (12) when $0 < x \leq$ A, we have

$$\left|\frac{k^{\frac{1}{m}}}{n} - t\right| \geq \delta - A \left|1 - e^{\frac{\alpha}{mn^m}}\right| \geq \left|\delta - \frac{\delta}{2} = \frac{\delta}{2},\right.$$

and hence it is shown that if $m \geq 1, 0 < x \leq$ A $\left[n > \dfrac{2A\alpha e^{\alpha}}{\delta}\right]$,

(1) set $\left\{k;\left|\frac{k^{\frac{1}{m}}}{n}-x\right|\geq\delta\right\}$ set$\left\{k;\left|\frac{k^{\frac{1}{m}}}{n}-t\right|\geq\frac{\delta}{2}\right\}$;

(2) $\frac{4(k^{\frac{1}{m}}-nt)^2}{n^2\delta^2}\geq 1.$

So by (11) we obtain that when $0<x\leq A$,

$$I_n(x)\leq e^{-(nt)^m}\sum_{\left|\frac{k^{\frac{1}{m}}}{n}-t\right|\geq\frac{\delta}{2}}\frac{(nt)^{km}}{k!}\leq e^{-(nt)^m}\sum_{\left|\frac{k^{\frac{1}{m}}}{n}-t\right|\geq\frac{\delta}{2}}\frac{4(k^{\frac{1}{m}}-nt)^s}{n^2\delta^2}\cdot\frac{(nt)^{km}}{k!}$$

$$\leq\frac{4e^{-(m)^m}}{n^2\delta^2}\sum_{k=0}^{\infty}(k^{\frac{1}{m}}-nt)^2\frac{(nt)^{km}}{k!}. \tag{13}$$

When m=1, by lemn1 we get

$$I_n(x)\leq\frac{4e^{-(nt)}}{n^2\delta^2}\sum_{k=0}^{\infty}(k-nt)^2\frac{(nt)^k}{k!}=\frac{4t}{n\delta^2}=\frac{4xe^{\frac{\alpha}{n}}}{n\delta^2}=O\left(\frac{1}{n}\right),\quad 0<x\leq A \tag{14}$$

When $m\geq 2$, by Lemman 3 and (13) we get

$$I_n(x)\leq\frac{8}{n\delta^2}=O\left(\frac{1}{n^2}\right). \tag{15}$$

On the other hand, using the unequality $e^x-1\leq xe^x$ $(x>0)$, we obtain that in $0<x\leq A$,

$$e^{(nx)^m(e^{\frac{\alpha}{nm}}-)}\leq e^{(nx)^m\frac{\alpha}{n^m}e^{\frac{\alpha}{nm}}}\leq e^{A^m.\alpha e^{\alpha}}=O(1)$$

From this and (14), (15) and (10), we get Lemma 4.

Lemma 5[2,p305(4)] For x > 0, the following inequality holds.

$$\sum_{k=0}^{\infty}|k-nx|\frac{(nx)^k}{k!}\leq en^x\sqrt{nx}.$$

In this paper we yet need the following **Cauchy's inequality.**

If a_k b_k (k=1,2…) are real numbers, then

$$\sum_{k=0}^{\infty}a_kb_k\leq\left\{\sum_{k=0}^{\infty}a_k^2\cdot\sum_{k=0}^{\infty}b_k^2\right\}^{\frac{1}{2}}.$$

4. PROOF OF THEOREM

(i) the case $x=0$.

By (1): R $(f;0)=f(0)$, Theorem is valid clearly.

(ii) the case $0<x\leq A$.

By (4) we see that

$$\frac{1}{2+e^{(nx)^m}-e^{-(nx)^m}}\left\{2+\sum_{k=0}^{\infty}(1-(-1)\frac{(nx)^{km}}{k!}\right\}=1.$$

So by (1) and sgn x=1 $(x>0)$ we get

$$R_n^m(f;x)-f(x)=\frac{1}{2+e^{(nx)^m}-e^{-(nx)^m}}\{f(o)-f(x)\}+\frac{1}{2+e^{(nx)^m}-e^{-(nx)^m}}\bullet$$

$$\bullet\left\{\sum_{k=0}^{\infty}\left(\left(f\left(\frac{\theta_k^{\frac{1}{m}}}{n}\right)-f(x)\right)-(-1)^k\left(f\left(-\frac{\theta_k^{\frac{1}{m}}}{n}\right)-f(-x)\right)\right)\frac{(nx)^{km}}{k!}\right\}$$

$$+\frac{1}{2+e^{(nx)m}-e^{-(nx)^m}}\sum_{k=0}^{\infty}(f(x)-f(-x))(-1)^k\frac{(nx)^{km}}{k!}\stackrel{def}{=}I_1+I_2+I_3. \tag{16}$$

First estimate I_1 :

Using the properties (2) and (3) of the modulus of continuity and

$$\frac{1}{2+e^{(nx)^m}-e^{-(nx)^m}}\leq e^{-(nx)^m}\leq 1\qquad(x>0), \tag{17}$$

we get

$$|I_1|\leq e^{-(nx)^m}\omega_{2A}(x)=e^{-(nx)^m}\omega_{2A}(\sqrt{nx}\cdot\frac{1}{\sqrt{n}})\leq e^{-(nx)^m}(\sqrt{nx}+1)\omega_{2A}(\frac{1}{\sqrt{n}}). \tag{18}$$

Say $g(x)=xe^{-(nx)^m}$ $(x>0)$.

Applying the common method of finding extreme value, we can find that $g(x)\leq\frac{1}{n}(\frac{1}{m})^{\frac{1}{m}}e^{\frac{1}{m}}$. since $(\frac{1}{m})$ is an increasing sequance and lim $(\frac{1}{m})$ = 1,

we can see that $(\frac{1}{m})^{\frac{1}{m}}\leq 1$, so $g(x)=xe^{-(nx)^m}\leq\frac{1}{n}$ $(x>0)$. (19)

From this and (18) we get $I_1=O\left[\omega_{2A}(\frac{1}{\sqrt{n}})\right]$ (20)

Next estimate I_2:

By (16) and (17),

$$|I_2|\leq e^{-(nx)^m}\left\{\sum_{\theta_k^{\frac{1}{m}}\leq 2An}\left|\left[f\left[\frac{\theta_k^{\frac{1}{m}}}{n}\right]-f(x)\right]-(-1)^k\left[f\left[-\frac{\theta_k^{\frac{1}{m}}}{n}\right]-f(-x)\right]\right|\frac{(nx)^{km}}{k!}\right.$$

$$\left.+\sum_{\theta_k^{\frac{1}{m}}\leq 2An}\left|\left[f\left[\frac{\theta_k^{\frac{1}{m}}}{n}\right]-f(x)\right]-(-1)^k\left[f\left[-\frac{\theta_k^{\frac{1}{m}}}{n}\right]-f(-x)\right]\right|\frac{(nx)^{km}}{k!}\right\}\stackrel{def}{=}I_{21}+I_{22}. \tag{21}$$

Using the properties (2) and (3) of the nodules of continuity It is follows that

$$\left|\left(f\left(\frac{\theta_k^{\frac{1}{m}}}{n}\right)-f(x)\right)-1^k\left(f\left(\frac{\theta_k^{\frac{1}{m}}}{n}\right)-f(x)\right)\right|\leq 2\omega_{2A}\left(\left|\frac{\theta_k^{\frac{1}{m}}}{n}-x\right|\right)=2\omega_{2A}\left(\left|\frac{\theta_k^{\frac{1}{m}}}{n}-x\right|\sqrt{n}\cdot\frac{1}{\sqrt{n}}\right)$$

$$\leq 2\left\{\left|\frac{\theta_k^{\frac{1}{m}}}{n}-x\right|\sqrt{n}+1\right\}\omega_{2A}\left(\frac{1}{\sqrt{n}}\right) \tag{22}$$

But by $k \le \theta_k < k+1$ and $(k+1)^{\frac{1}{m}} - k^{\frac{1}{m}} \le 1$ $(k \ge 1)$, we have

$$\left|\frac{\theta_k^{\frac{1}{m}}}{n} - \frac{k^{\frac{1}{m}}}{n}\right| \le \begin{cases} \frac{\theta_0^{\frac{1}{m}}}{n} \le \frac{1}{n}, & k=0 \\ \frac{(k+1)^{\frac{1}{m}}}{n} - \frac{k^{\frac{1}{m}}}{n} \le \frac{1}{n}, & k \ge 1. \end{cases}$$

Further,

$$\left|\frac{q_k^{\frac{1}{m}}}{n} - x\right|\sqrt{n} \le \left|\frac{q_k^{\frac{1}{m}}}{n} - \frac{k^{\frac{1}{m}}}{n}\right|\sqrt{n} + \left|\frac{k^{\frac{1}{m}}}{n} - x\right|\sqrt{n} \le 1 + \left|\frac{k^{\frac{1}{m}}}{n} - x\right|\sqrt{n}.$$

From this and (22), (21) we get

$$I_{21} \le e^{-(nx)^m}\left\{4w_{2A}\left(\frac{1}{\sqrt{n}}\right)\sum_{k=0}^{\infty}\frac{(nx)^{km}}{k!} + \frac{2}{\sqrt{n}}w_{2A}\left(\frac{1}{\sqrt{n}}\right)\sum_{\theta_k^{\frac{1}{m}} \le 2An}^{?}\left|k^{\frac{1}{m}} - nx\right|\frac{(nx)^{km}}{k!}\right\}. \quad (23)$$

(1) When m=1. By (4) and Lemma 5, the above ineduality (23) becomes the following

$$I_{21} \le 4\omega_{2A}\left(\frac{1}{\sqrt{n}}\right) + e^{-(nx)^m}\cdot\frac{2}{\sqrt{n}}\,\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\sum_{k=0}^{\infty}\left|k^{\frac{1}{m}} - nx\right|\frac{(nx)^{km}}{k!} = O\left(\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\right) (0 < x \le A). \quad (24)$$

(2) When m≥2. Using Canchy's inequality and Lemma 3, we get by (4)

$$\sum_{k=0}^{\infty}\left|k^{\frac{1}{m}} - nx\right|\frac{(nx)^{km}}{k!} \le \left\{\sum_{0}^{\infty}\left(k^{\frac{1}{m}} - nx\right)^2\frac{(nx)^{km}}{k!}\cdot\sum_{0}^{\infty}\frac{(nx)^{km}}{k!}\right\}^{\frac{1}{2}} \le \sqrt{2}e^{(nx)^m}.$$

From this and (23), (4), we have

$$I_{21} \le 4\omega_{2A}\left(\frac{1}{\sqrt{n}}\right) + e^{-(nx)^m}\cdot\frac{1}{\sqrt{n}}\,\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\sum_{k=0}^{\infty}\left|k^{\frac{1}{m}} - nx\right|\frac{(nx)^{km}}{k!} = O\left(\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\right). \quad (25)$$

By the known condition that $f(x) = O(e^{\alpha|x|^m})$ and $k \le \theta_k < k+1$, we have

$$I_{22} \le 4e^{-(nx)^m}\sum_{\theta_k^{\frac{1}{m}} > 2.An} e^{\frac{2\alpha k}{m}}\frac{(nx)^{km}}{k!}. \quad (26)$$

If $\theta_k^{\frac{1}{m}} > 2An$, then $\frac{k^{\frac{1}{m}}}{n} - x \ge \frac{(\theta_k - 1)^{\frac{1}{m}}}{n} - x \ge \frac{\left[\left(\frac{3}{2}An\right)^m\right]^{\frac{1}{m}}}{n} - x \ge \frac{A}{2}$

$$\left(0 < x \le A,\ n > \frac{2}{A}\right)$$

It is shown that set $\left\{k; q_k^{\frac{1}{m}} > 2An\right\} \supset$ *set* $\left\{k; \frac{k^{\frac{1}{m}}}{n} - x \ge \frac{A}{2}\right\}$.

From this and (26) we get

$$I_{22} \le 4e^{-(nx)^m}\sum_{\frac{k^{\frac{1}{m}}}{n} - x \ge \frac{A}{2}} e^{\frac{2\alpha k}{nm}}\cdot\frac{(nx)^{km}}{k!},$$

again by Lemma 4 and the properties (4) and (1) of the modulus of continuity we get

$$I_{22} \le O\left(\omega_{2A}\left(\frac{1}{n}\right)\right) = O\left(\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\right),\ n > 3\alpha e^{2\alpha}.$$

Combining this with (25), (26), we get from (21) that

$$I_{22} \le O\left(\omega_{2A}\left(\frac{1}{n}\right)\right),\ n > max\left\{8\alpha e^{2\alpha}\frac{2}{A}\right\}\ (0 < x \le A). \quad (27)$$

Finally extimate I_3:

By the properties (2) and (3) of the modulus of continu ity, we have

$$|f(x) - f(-x)| \le \omega_{2A}(2x) \le (2x\sqrt{n} + 1)\omega_{2A}\left(\frac{1}{\sqrt{n}}\right),$$

again by (17), (19) and (4), we obtain from (16) that

$$|I_3| \le |-f(x) - f(-x)|\left|\sum_{k=0}^{\infty}(-1)^k\frac{(nx)^{km}}{k!}\right| \le e^{-(nx)^m}\left(2x\sqrt{n}+1\right)\, w_{2A}\left(\frac{1}{\sqrt{n}}\right)$$
$$\le \left(\frac{1}{\sqrt{n}} + 1\right)w_{2A}\left(\frac{1}{\sqrt{n}}\right) = O\left(w_{2A}\left(\frac{1}{\sqrt{n}}\right)\right).$$

From this and (27), (20), (16), we obtain finally that

$$R_n^m(f;x) - f(x) = O\left(w_{2A}\left(\frac{1}{\sqrt{n}}\right)\right),$$

Theorem is valid in 0<x≤A.

(iii) the case -A≤x<0.

Let $F(y)=f(-y)$ and $y = -x$. The above (ii) implies that

$$R_n^m(f;y) - F(y) = O\left(w_{2A}\left(F;\frac{1}{\sqrt{n}}\right)\right),\ 0 < y \le A,\ n > max\left\{8ae^{2a}, \frac{2}{A}\right\} \quad (28)$$

By the definition of the modulus of continuity, we know that

$$w_{2A}\left(F;\frac{1}{\sqrt{n}}\right) = w_{2A}\left(f;\frac{1}{\sqrt{n}}\right). \quad (29)$$

$$R_n^m(F;y) = \frac{1}{2 + e^{(ny)^m} - e^{-(ny)^m}}\left\{F(o) + \sum_{k=0}^{\infty}\left(F\left(\frac{\theta_k^{\frac{1}{m}}}{n}\right) - (-1)^k F\left(-\frac{\theta_k^{\frac{1}{m}}}{n}\right)\right)\frac{(ny)^{km}}{k!}\right\}$$
$$= \frac{1}{2 + \operatorname{sgn} x\,(e^{(nx)^m} - e^{-(ny)^m}}\left\{f(o) + \operatorname{sgn} x\sum_{k=0}^{\infty}\left(f\left(\frac{\theta_k^{\frac{1}{m}}}{n}\right) - (-1)^k f\left(-\frac{\theta_k^{\frac{1}{m}}}{n}\right)\right)\frac{(nx)^{km}}{k!}\right\}$$
$$= R_n^m(f;x) \quad (30)$$

and

$$F(y) = F(-x) = f(x) \quad (31)$$

Combining (38) with (29) (31) we obtain that

$$= R_n^m (f;x) - f(x) = O\left(\omega_{2A}\left(\frac{1}{\sqrt{n}}\right)\right).$$

The proof of Theorem is completed.

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