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Strong and Weak n-Homogeneous Spaces

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فضاءات n- متجانس القوية والضعيفة

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في هذا البحث سوف نستحدث أنواعاً جديدة من n- متجانس، والتي سوف ندعوها n-متجانس من النوع ٢، والنوع ٣، وسوف نرمز لها nH_2, nH_3 على الترتيب، سوف نبرهن بأن كل فضاء $3H_3$- هو ثنائي التجانس، وكذلك سوف نبرهن بأن كل فضاء nH_3- منته هو فضاء واضح (منفصل أو بسيط)، سوف نبرهن بأن كل فضاء $3H_3$- هو فضاء ٢- متجانس قوي. سوف ندرس بعض العلاقات الممكنة بين الفضاءات والفضاءات المعرفة سابقاً مثل n- متجانس بقوة و n- متجانس بضعف.

Keywords: *Homogeneous, bihomogeneous, n-homogeneous.*

ABSTRACT

In this paper we introduce new types of n-homogeneity, we call them n-homogeneous of type 2 and type 3, denoted by nH_2, nH_3 respectively. We show that every $3H_3$-space is bihomogeneous space. Also, every finite nH_3-space has the trivial topology (discrete or indiscrete). We show that every $3H_3$-space is strong 2-homogeneous. We study the implications between these spaces with the well known spaces, strongly n-homogeneous spaces and weakly n-homogeneous spaces.

1. Introduction

A space X is called *homogeneous* if for every x, y in X there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. Several authors studied the n-homogeneity spaces, for instance [1], [2] and [4]. In this section we shall give the definitions besides with the obvious implications between these definitions. For a set X, by $|X|$ we mean the cardinality of X. Let us start with the following definition, one may consult [1].

Definition 1.1. A space X is called *bihomogeneous* provided every two points in X can be interchanged by means of an autohomeomorphism on X.

Although every bihomogeneous space is homogeneous, the converse is not true. In fact the reals R with the left ray topology is homogeneous but not bihomogeneous. For the next definition one may see [2]. For a positive integer n we have the following definitions.

Definition 1.2. A space X is *n-homogeneous of type 1*, denoted by nH_1, if for every two subsets of X each having exactly n elements $A = \{x_1, ..., x_n\}$ and $B = \{y_1, ..., y_n\}$, there exists a homeomorphism h of X onto itself such that $h(A) = B$.

Definition 1.3. A space X is called *n-homogeneous of type 2*, denoted by nH_2, if for every two subsets of X each having exactly n elements $A = \{x_1, ..., x_n\}$ and $B = \{y_1, ..., y_n\}$, there exists a homeomorphism h of X onto itself such that $h(A) = B$ and, $h(z) = z$ for all $z \in A \cap B$.

Definition 1.4. A space X is called *n-homogeneous* of *type 3*, denoted by nH_3, if for every two subsets of X each having exactly n elements $A = \{x_1, ..., x_n\}$ and $B = \{y_1, ..., y_n\}$, there exists a homeomorphism h of X onto itself such that $h(A)=B$ and $h(x_1) = y_1$.

Definition 1.5 [4]. A space X is called *n-homogeneous of type 4*, denoted by nH_4, if for every two subsets of X each having exactly n elements $A = \{x_1, ..., x_n\}$ and $B = \{y_1, ..., y_n\}$, there exists a homeomorphism h of X onto itself such that $h(x_i) = y_i$ for all $i = 1, ..., n$.

Notice that n-homogeneous spaces of type 4 were introduced before, and in [4] such spaces are called *strongly n-homogeneous*. It is obvious that every nH_4-space is an $(n-1)H_4$-space, and hence every nH_4-space is homogeneous provided that $|X| > n$.

2. 1- and 2-Homogeneous Spaces of all Types

In this section we shall study all implications between 1-, and 2-homogeneous spaces of type 1, type 2, type 3, and type 4. Also, we shall give a characterization of n-homogeneous spaces of type 4. Besides we have the following results.

Theorem 2.1. *Every $2H_3$-space ($2H_2$-space) is homogeneous provided that* $|X| \geq 3$.

Proof. Let $x, y \in X$. Let $z \in X \setminus \{x, y\}$ and consider the sets $\{x, z\}$ and $\{y, z\}$. So, there exists a homeomorphism $h : X \to X$ with $h(x) = y$.

Remark. In the proof of Theorem 2.1 we assumed that $|X| \geq 3$. In fact, if $|X| = 2$ and X is $2H_3$ ($2H_2$), then the topology on X is trivial (discrete or indiscrete) and hence it is homogeneous. Also, one can notice that if $|X| \leq n$, then X is nH_m –space for $m = 1, 2, 3, 4$.

Theorem 2.2. *Let X be a finite space with $|X| > n$, then X is an nH_3-space iff X has the trivial topology (discrete or indiscrete).*

Proof. Suppose on the contrary, that is, X is finite and τ is not the trivial topology. Hence, there exists a proper open subset U of X. Let $x_0 \in U$, and let $k = |\cap\{V \in \tau : x_0 \in V\}|$, since τ is not trivial then $k > 1$. Since X is homogeneous, and $\cap\{V \in \tau : x_0 \in V\}$ is the smallest open set containing x_0, then for $x \in X$, there exists a unique open set U_x containing x of cardinality k, in fact such an open set U_x is the smallest open set containing x.

Let $x_1 \in X$, fix the unique open set U_{x_1} containing x_1 with $|U_{x_1}| = k$. Since τ is not trivial, then there exists $x_2 \in X \setminus U_{x_1}$. Consider the unique open set U_{x_2} containing x_2 with $|U_{x_2}| = k$. Hence $U_{x_1} \cap U_{x_2} = \emptyset$. If $X \setminus (U_{x_1} \cap U_{x_2})$ is not empty continue in this process. Finally, we get $X = \cup^n_{i=1} U_{x_i}$, each U_{x_i} is open with cardinality k. We have the following cases:

I. $n \leq k$. Choose disjoint open sets U_1, U_2 with $|U_1| = |U_2| = k$, and then take $\{x_1, ..., x_n\} \subseteq U_1$ and $y \in U_2$. Since X is nH_3, there exists a homeomorphism $h : X \to X$ such that $h(\{x_2, ..., x_n\}) = \{x_2, ..., x_n\}$ and $h(x_1) = y$. Hence $h(U_1) \cap U_2$ is an open set containing y of cardinality less than k, a contradiction.

II. $n > k$. Let $A = \{x_1, ..., x_n\}$, so there exist open sets U_1, U_2 of size k with $x_n \in U_1$ and $x_n \in U_2$. Hence, there exists a homeomorphism $h : X \to X$ such that $h(\{x_2, ..., x_n\}) = \{x_1, ..., x_{n-1}\}$ and $h(x_1) = x_n$. So, $h(U_1) \cap U_2$ is an open set containing x_n with $|h(U_1) \cap U_2| < k$ since $x_1 \in h(U_1)$ and $x_1 \notin U_2$. This contradiction completes the proof.

Theorem 2.3. *If $|X| > 2$ and X is a $2H_1$-space, then X is homogeneous.*

Proof. Let $x, y \in X$. Let $z \in X \setminus \{x, y\}$. Then there exists a homeomorphism $h : X \to X$ such that $h(\{x, z\}) = \{y, z\}$. If $h(x) = y$ we are done. If $h(x) = z$ then hoh is the required homeomorphism.

Theorem 2.4. *Every $2H_2$-space is $2H_4$-space provided that* $|X| \geq 5$.

Proof. Take $\{x, y\}$ and $\{u, v\}$ in X. We shall find a homeomorphism $h : X \to X$ such that $h(x) = u$ and

$h(y) = v$. Consider the sets $\{x, y\}$ and $\{x, z\}$ where $z \in X \setminus \{x, y, u, v\}$. Hence there exists a homeomorphism $h_1 : X \to X$ such that $h_1(y) = z$ and $h_1(x) = x$. Consider the sets $\{x, z\}$ and $\{u, z\}$, hence there exists a homeomorphism $h_2 : X \to X$ such that $h_2(x) = u$ and $h_2(z) = z$. Finally, consider the sets $\{u, z\}$ and $\{u, v\}$. So, there exists a homeomorphism $h_3 : X \to X$ such that $h_3(z) = v$ and $h_3(u) = u$. Therefore, $h_3 o h_2 o h_1$ is the required homeomorphism.

The condition that $|X| \geq 5$ is necessary, because; Sierpenski space is $2H_2$ which is neither $2H_3$ nor homogeneous. The next example is also $2H_2$-space which is not $2H_3$-space although it is homogeneous.

Example 2.5. Let $X = \{1, 2, 3, 4\}$ be topologized by the base $ß = \{\{1, 2\}, \{3, 4\}\}$. Hence $(X, \tau(ß))$ is a homogeneous space which is not $2H_3$-space.

It is easy to see that every $2H_2$-space with cardinality 3 has the trivial topology (discrete or indiscrete), and hence it is $2H_3$-space.

Since every $2H_4$-space is $2H_3$-space, hence every $2H_2$-space is $2H_3$-space. Also, the real R with the left ray topology is $2H_1$ which is not $2H_2$. Hence we have the following diagram:

diagram:

$$\begin{tikzcd}
2H_1 \arrow[dr, "|X|>2"'] & & 2H_2 \arrow[ll] \arrow[dl, "|X|>2"] \arrow[r, shift left, Rightarrow] & 2H_3 \arrow[l, shift left, "|X|\neq 2,4"] \arrow[r, leftrightarrow] & 2H_4 \arrow[r] & 1H_1 \\
& 1H_1 & & & &
\end{tikzcd}$$

3. 3H3-Spaces

In this section we shall study the implications of 3-homogeneous spaces of type 3. We shall see that such spaces are bihomogeneous and are 2-homogeneous spaces of type 4. Let us start with the following result.

Theorem 3.1. *Every $3H_3$-space is bihomogeneous provided that $|X| \geq 4$.*

Proof. Let $x, y \in X$ with $x \neq y$. We shall show that there exists a homeomorphism $h : X \to X$ such that $h(x) = y$ and $h(y) = x$. Since $|X| \geq 4$, there are two distinct points u, v in $X \setminus \{x, y\}$. Consider the triple

$C_1 = \{u, x, y\}$, so there exists a homeomorphism $h_1 : X \to X$ such that $h_1(C_1) = C_1$ with $h_1(x) = y$. If $h_1(y) = x$ we are done. If $h_1(y) = u$, consider the triples C_1 and $C_2 = \{v, x, y\}$. Hence there exists a homeomorphism $h_2 : X \to X$ such that $h_2(C_1) = C_2$ and $h_2(y) = y$. If $h_2(x) = v$, then h_2oh_1 is the required homeomorphism. If $h_2(x) = x$, consider the triples C_2 and $C_3 = \{u, v, y\}$, hence there exists a homeomorphism $h_3 : X \to X$ such that $h_3(C_3) = C_2$ and $h_3(y) = y$. So we have two cases:

Case 1: $h_3(u) = v$ and $h_3(v) = x$, in this case the homeomorphism $h = h_3oh_2oh_1$ is the required homeomorphism.

Case 2: If $h_3(u) = x$ and $h_3(v) = v$, consider the homeomorphism $h = h_3oh_1$.

Another consequence of $3H_3$-spaces is the following.

Theorem 3.2. *If X is an $3H_3$-space then for every $x, y, z \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$ and $h(z) = z$ provided that $|X| \geq 4$.*

Proof. Let x, y, z be any three points in X. Fix $u \in X \setminus \{x, y, z\}$, if $x \neq y$, consider the triples $C_1 = \{u, x, z\}$ and $C_2 = \{x, y, z\}$. Hence there exists a homeomorphism $h_1 : X \to X$ such that $h_1(C_1) = C_2$ with $h_1(z) = z$. If $h_1(x) = y$ we are done. If $h_1(x) = x$, consider the triples C_2 and $C_3 = \{y, u, z\}$. So, there exists a homeomorphism $h_2 : X \to X$ such that $h_2(C_2) = C_3$ with $h_2(z) = z$. If $h_2(x) = y$, then h_2 is the required homeomorphism. If $h_2(x) = u$, then $h = h_1oh_2$ is the required homeomorphism.

Now, we shall give the main result in this section.

Theorem 3.3. *Every $3H_3$-space is $2H_3$-space provided that $|X| \geq 6$.*

Proof. Let $A = \{x_1, x_2\}$ and $B = \{y_1, y_2\}$ be any two doubletons. So, we have the following cases:

I. $A \cap B = \varnothing$. Consider the triples $C_1 = \{y_1, x_1, x_2\}$ and $C_2 = \{x_1, y_1, y_2\}$. Hence, there exists a homeomorphism $h_1 : X \to X$ such that $h_1(C_1) = C_2$ and $h_1(x_2) = y_2$. If $h_1(x_1) = y_1$, we are done. If not, that is, $h_1(x_1) = x_1$, consider the triples C_2 and $C_3 = \{u, y_1, y_2\}$ where $u \in X \setminus \{x_1, x_2, y_1, y_2\}$. So, there exists a homeomorphism $h_2 : X \to X$ such that $h_2(C_2) = C_3$ and $h_2(y_2) = y_2$. If $h_2(x_1) = y_1$ then h_2oh_1 is the required homeomorphism. If $h_1(x_1) = u$, consider the triples $C_4 = \{u, v, y_2\}$ and $C_5 = \{v, y_1, y_2\}$ where $v \in X \setminus \{u, x_1, x_2, y_1, y_2\}$. Hence, there exists a homeomorphism $h_3 : X \to X$ such that $h_3(C_4) = C_5$ and $h_3(y_2) = y_2$. If $h_3(u) = y_1$, then $h_3oh_3oh_2oh_1$ is the required homeomorphism. If $h_3(u) = v$, then $h_3oh_3oh_2oh_1$ is the required homeomorphism.

II. $A = B$, in this case, assume $A = \{x, y\}$, this case follows from Theorem 3.1.

III. $|A \cap B| = 1$. If $A = \{x, z\}$ and $B = \{y, z\}$, then by Theorem 3.2, we are done. If $A = \{x, z\}$ and $B = \{z, y\}$, choose $u \in X \setminus \{x, y, z\}$. Consider the triple $C_6 = \{x, y, z\}$. Hence there exists a homeomorphism $h_4 : X \to X$ such that $h_4(C_6) = C_6$ and $h_4(z) = y$. If $h_4(x) = z$, then we are done. If $h_4(x) = x$, then by Theorem

3.2, there exists a homeomorphism $h_5 : X \to X$ such that $h_5(x) = z$ and $h_5(y) = y$, so $h_5 o h_4$ is the required homeomorphism.

Corollary 3.4. *Every $3H_3$-space is $2H_4$-space.*

4. The N-Homogeneity of Type I and their Consequences

In this section we give a characterization of nH_4-spaces. Also, we shall draw the main lines for possible implications between these kinds of n-homogeneity spaces. It is obvious that every nH_4-space is kH_i-space for all $1 \leq k \leq n$ and $i = 1, 2, 3, 4$. We also saw in Section 3 that, every $3H_3$-space is kH_i-space where $k = 1, 2$ and $i = 1, 2, 3, 4$. Before continuing this study let us give the following characterization.

Theorem 4.1. *Let n be a positive integer. A space X is $n\ H_3$ if and only if X satisfies the property that for any two subsets $A = \{x_1,...,x_{n-1}\}$ and $B = \{y_1,...,y_{n-1}\}$ of X both of $(n-1)$ elements, and for any $z \in X \setminus (A \cup B)$, there exists a homeomorphism $h : X \to X$ such that $h(x_i) = y_i$; $i = 1, ..., n\text{-}1$ and $h(z) = z$.*

Proof. ($\Leftarrow$) Let $A = \{x_1,...,x_n\}$ and $B = \{y_1,...,y_n\}$. We shall show that there exists a homeomorphism $h : X \to X$ such that $h(x_i) = y_i$ for all $i = 1, ..., n$. By assumption, there are homeomorphisms $h_1, h_2 : X \to X$ such that

$h_1(x_i) = y_i$ for $i = 1, ..., n\text{-}1$; and $h_1(x_n) = x_n$,

$h_2(y_j) = y_j$ for $j = 1, ..., n\text{-}1$; and $h_2(x_n) = y_n$.

Hence, $h = h_2 o h_1 : X \to X$ is the required homeomorphism. The converse is obvious.

Theorem 4.2. *If X is $2H_3$-space then $X \setminus \{u\}$ is homogeneous for every $u \in X$.*

For a finite space X of cardinality n we have the following result for homogeneous space of type 1.

Theorem 4.3. *Let X be a space of cardinality n. Then X is a kH_1-space if and only if X is $(n\text{-}k)H_1$, where $1 \leq k \leq n\text{-}1$.*

Question 4.4. *Is every nH_1-space an $(n\text{-}1)\ H_1$- space? ($n \geq 3$)*

The following typical example solves Question 4.4 partially.

Example 4.5. Let $X = \{x_1,...,x_n\}$ be topologized as follows

$$\tau = \{\varnothing, \{x_1\}, \{x_1, x_2\}, ..., \{x_1,...,x_{n-1}\}, X\}.$$

Then, X is nH_1, but it is not kH_1 for all $k = 1, 2, ..., n\text{-}1$.

Moreover, in Example 2.5, X is $3H_1$-space with $|X| = 4$ but it is not $2H_1$. We guess that if X is infinite then every nH_1-space is also a kH_1-space for all positive integers $k \leq n$.

It is easy to see that every nH_2-space is an nH_1-space, but the converse is not true. In fact R with the left ray topology is an nH_1 for all n positive integer, but is not nH_2-space for all $n \geq 2$.

For nH_1-space we have the following results.

Theorem 4.6. *If X is nH_1 and $|X| \geq n+1$, then X is homogeneous.*

Proof. Let $x, y \in X$, since $|X| \geq n+1$, there exists a set of distinct points $\{x_1,\ldots,x_{n-1}\} \subseteq X \setminus \{x, y\}$. Since X is an nH_1-space, there exists a homeomorphism $h : X \to X$ such that $h(\{x_1, \ldots, x_{n-1}, x\}) = \{x_1,\ldots,x_{n-1}, y\}$. If $h(x) = y$, we are done, if not, without loss of generality, assume that $h(x) = x_1$. If $h(x_1) = y$, then $h(h(x)) = h(x_1) = y$, and hence the composition hoh is a homeomorphism from X onto itself and takes x into y. If $h(x_1) \neq y$, then $h(x_1) = x_j$. Continuing this process, there exists $k \leq n$, such that $h^k(x) = y$, where $h^k = hoho\ldots oh$; k times.

Corollary 4.7. *If X is nH_2 and $|X| \geq n+1$, then X is homogeneous.*

Proof. Since every nH_2-space is nH_1-space and $|X| \geq n+1$, then by Theorem 4.6, X is homogeneous.

Theorem 4.8. *If X is nH_2 and $|X| \geq 2n-1$, then X is an $(n-1)\,H_2$-space, and hence kH_2 for all positive integers $k < n$.*

Proof. Let $A = \{x_1,\ldots,x_{n-1}\}$ and $B = \{y_1,\ldots,y_{n-1}\}$. Hence there exists $z \in X \setminus (A \cup B)$. Since X is an nH_2-space, there exists a homeomorphism $h : X \to X$ such that $h\ \{x_1,\ldots,x_{n-1}, z\}) = \{y_1,\ldots,y_{n-1}, z\}$ and, $h(z) = z$ and $h(u) = u$ for all $u \in A \cap B$. Hence $h(A) = B$ and $h(u) = u$ for all $u \in A \cap B$. Therefore, X is $(n-1)H_2$.

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