DECOMPOSITION OF THE SET OF CONDITIONALLY EXPONENTIAL CONVEX FUNCTIONS

By

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INTRODUCTION

Berg, Christensen and Ressel(4) and independently Okb-El-Bab and El-Shazli(12) studied conditionally exponential convex functions on semigroups. For G compactly generated, a compact base for $E_0(G)$, the set of conditionally exponential convex functions defined on a locally compact group G, was constructed in(11). Also, the author in(11) obtained the extreme points of that base.

In this article we use the theory of C*-algebra to develop a connection between the local extreme points and the topology of the spectrum space $G$.

DEFINITIONS AND NOTATION

Let $G$ be a separable locally compact group equipped with left Haar measure $dx$ and modular function $\Delta$, where the identity element is denoted by $e$, and let $G$ be the set of irreducible representations of $G$. If $G$ is abelian, $G$ is its dual. By $C_c(G)$ ($C_0(G)$) we denote the set of compactly supported continuous functions on $G$ (with total left Haar integral zero)(5).

Let $C^*(G)$ be the enveloping C*-algebra of $L^1(G)$ equipped with the involution $\phi$ defined by $\phi(f)(x) = f(x^{-1})f^*(x)\Delta$ where $f^*(x) = f(x^{-1})$. The dual of $C^*(G)$ is $B(G)$ and its double dual is $W^*(G)$. For the universal representation $\omega$ we write $\omega(\mu)$ to indicate that $\mu$ belongs to $W^*(G)(7)$.

Now let $S$ be a separable compact convex set. A subset $F$ of $S$ is called a face if each line segment in $S$ whose interior intersects $F$ is contained in $F$. The complementary set $F'$ of $F$ is the union of all faces of $F$ disjoint from $F$. In this case, $S$ is the direct convex sum of $F$ and $F'$; i.e., every $x \in S$ can be written uniquely in the form $x = \lambda y + (1 - \lambda) z$, $0 \leq \lambda \leq 1, y \in F, z \in F'(2)$.

If $\chi$ is a subset of the set $\text{ext} S$, of extreme points of $S$, its facial closure is $F \cap \text{ext} S$, where $F$ is the smallest closed split face of $S$ containing $\chi$. The topology defined in this way is called the facial topology. This topology is coarser than the weak *-topology (2).

In the following we write $P(G)$ for the set of all exponentially convex functions defined on $G$; i.e., functions satisfying

$$\sum_{i,j=1}^{n} \psi(g_i, g_j) c_i c_j \geq 0,$$

where $g_1, ..., g_n \in G$ and $c_1, ..., c_n \in \mathbb{R}$. The set of elements from $P(G)$ with norm equals 1 is $P_1(G)$. Clearly $P_1(G)$ is a convex set(3). Also we write $E_0(G)$ for the set of all conditionally exponential convex functions defined on $G$ and vanishing at the group identity; i.e., functions satisfying

$$\sum_{i,j=1}^{n} [\psi(g_i) + \psi(g_j) - \psi(g_i g_j)] c_i c_j \geq 0,$$

where $g_1, ..., g_n \in G$ and $c_1, ..., c_n \in \mathbb{R}(4,12)$. 
Elements of $E_o(G)$ can be characterized geometrically as semi-tangents to $P_1(G)$ at the identity, and if $\varphi$, $-\varphi \in E_o(G)$ then $\varphi$ becomes a tangent vector to $P_1(G)$ at the identity.

By a Levy weight for $\varphi \in E_o(G)$ we mean the linear functional (also denoted by $\varphi$) defined by
$$\varphi(a) = \langle 1, \delta_{\varphi} a \rangle,$$
where $\delta_{\varphi}$ is a linear functional from $C^*(G)$ to $C^*(G)$ which is densely defined and
$$\varphi(1) = 0,$$
and $N_{\varphi} = \{ a \in ker 1 \mid \varphi(a) < \infty \}$, then the weight is called local if $sup \{ \rho_{\alpha}(a; \psi) \mid a \in ker 1 \}$ for all $a \in ker 1$.

Finally, for $\varphi$, $\psi \in E_o(G)$ we say that $\varphi$ dominates $\psi$ if $\varphi - \psi \in E_o(G)$. If $\varphi$ and $\psi$ dominate each other then they are equivalent, and they are weakly equivalent if one is equivalent to a weak multiple of the other.

A DECOMPOSITION THEOREM FOR $E_o(G)$

In this section we study the connection between the topology on $G$ and the structure of $E_o(G)$. Let $C^*_e(G)$ be the smallest $C^*$-algebra containing $C^*(G)$ and has an identity and let $S$ be its state space which is compact in the relative weak *-topology. If $C^*(G)$ has an identity then $S$ coincides with $P_1(G)$. Otherwise $P_1(G)$ is a split face in $S$, and $S$ is the direct convex sum of $P_1(G)$ and the state $f_0$, defined by $f_0\mid C^*(G) = 0$.

Let $\{O_n\}_{n \geq 1}$ be a collection of relatively weak *-open subsets of ext $S$ such that $O_n \subseteq O_{n+1} \cap \overline{ext \ S}$ and $\cap_{n \geq 1} O_n = \{1\}$. If $U$ is the irreducible representation of $S$ obtained from $p \in O_n \cap Ext S$ then we define the two sided ideal $In = \cap \{ ker U \cap \forall n \}$.

For each $n \geq 1$ there exists $\delta_n \in C_c(G)$ with $\delta_n \geq 0$, $\int \delta_n(x) \, dx = 1$, and $B_n \in W^*(G)$ such that $q_n = B_n \omega(\delta_n - \delta_n)$ where $\delta_n$ is the point mass at $e$.

Proof
Suppose that $V_\varphi$ is a neighborhood of the identity of $S$ contained in the face $F_{\varphi}$ annihilated by $I_\varphi$. For this neighborhood we construct $\delta_n \in C_c(G)$ and $\delta_n > 0$ as in Lemma 3.1. Then
$$\| q_n \ast \omega(\delta_n) \| = sup \{ \langle p, q_n \ast \omega(\delta_n) \rangle \mid p \in S \}$$
$$= sup \{ \langle p, \omega(\delta_n) \rangle \mid p \in S \}.$$
Lemma 3.3.

If $\psi$ is the Levy weight of an element of $E_0 (G)$ then there exists $p_n \in P (G)$ such that $\psi \upharpoonright I_n = p_n \upharpoonright I_n$ for all values of $n$.

Proof

Let $a \in I_n \subset \ker 1$. Then

$$0 \leq \psi (a \otimes a) = \psi (q_n (a \otimes a) q_n)$$

$$= \psi (\omega (\delta_c - \varepsilon_n) B_n a \otimes a B_n)$$

$$= \psi (\omega (\mu_n) B_n a \otimes a B_n) ,$$

where $\mu_n$ is the measure defined by $d\mu_n = \delta_c - \varepsilon_n \, dx$. It is clear that $\mu \in M_0^C (G)$, the set of compactly supported Borel measure on $G$ of total mass zero. Since

$$\sum_{k=0}^{\infty} a (\omega (\varepsilon_n) k \cdot a (\omega (\varepsilon_n) k) = a B_n$$

we have a $B_n \in I_n \subset C^* (G)$ Applying Lemma 2.1 in (11) we obtain

$$\psi (a \otimes a) = - \psi_{\mu_n} (B_n a \otimes a B_n) .$$

This shows that $\psi \upharpoonright I_n$ is a bounded weight. On the other hand, we define $p_n$ by the product $p_n (\cdot) = - \psi_{\mu_n} (B_n \cdot B_n)$.

Then $p_n \in P (G)$ and $\psi \upharpoonright I_n = p_n \upharpoonright I_n$.

Corollary 3.4

$\psi$ dominates $p_n (\cdot) - p_n$ in $E_0 (G)$.

Theorem 3.5

Suppose that $p_n \psi \in E_0 (G)$ has a local Levy weight and $F$ is the closed split face of $S$ given by $F = \cap \, F_n$. Then for each $\mu \in M_0^C (G)$ such that $- \psi_{\mu} (e) \neq 0$ we have $- \psi_{\mu - \psi_{\mu}} (e) \in F$.

Proof

If the Levy weight for $\psi$ is local then, by definition, $\psi$ does not dominate any semitangent of the form $p (\cdot) \cdot p \in P (G)$. This implies that $\psi \upharpoonright I_n = 0$ for all $n \geq 1$ and hence $\psi_{\mu} \upharpoonright I_n = 0$ for each $\mu \in M_0^C (G)$. This means that $- \psi_{\mu}$ is annihilated by the closed two-sided ideal $\cap I_n = F$. //

Theorem 3.6

Each $\psi \in E_0 (G)$ can be written uniquely in the form $\psi = \psi_1 + \psi_2$, $\psi_1, \psi_2 \in E_0 (G)$, where

i) for each $\mu \in M_0^C (G)$ such that $- \psi_{\mu_1} (e) \neq 0$, $- \psi_{\mu_1} - \psi_{\mu_1}$ ($e) \in F$.

ii) $\psi_2 = \lim \frac{1}{n} (p_n (e) - p_n)$, where $p_n \in P (G)$ and $p_n \upharpoonright I_n = \psi_{\mu} \upharpoonright I_n$.

Proof

We note that if $\psi_2 = \lim \frac{1}{n} (p_n (e) - p_n)$, then from Corollary 3.4, $\psi_2 , \psi_1 , = \psi - \psi_2$ belong to $E_0 (G)$. Moreover, $\psi_1$ vanishes on each $I_n$ and the proof of Theorem 3.5 applies. //

Now, let $U$ be a factor representation of $G$. By Proposition 5.2.7 of(7) and Corollary 2 of(1) ker $U$ is a primitive ideal of $C^* (G)$.

Definition 3.7(9)

A factor representation $U$ of $G$ is said to be separated from the trivial representation if there exist disjoint open sets $V_1$ and $V_2$ in Prim (G), the primitive ideal space of $C^* (G)$, such that $ker I_1 \in V_1$ and ker $U \in V_2$.

A group $G$ has a property (P) if each factor representation of $G$ on a separable Hilbert space which is separated from the trivial representation has a trivial first cohomology group $H^1 (U)$.

In fact, every locally compact group has this property. As an application of Theorem 3.6 we have:

Theorem 3.8

Let $G$ be a separable locally compact group.

i) If $F = \cap \{ \varepsilon \mid \varepsilon$ is an open neighborhood of the trivial representation in $G \}$ and if $U \in G \cdot F$, then $H^1 (U) = 0$.

ii) $G$ has property (P).

Proof

i) We need the following Lemma for proving this part.

Lemma 3.9

If $U$ is an irreducible representation of $G$ and if $c \in V^* (U)$ then $\psi (c (x)) = | | c (x) \psi (x) | | ^2$ generates an extreme ray in $E_0 (G)$; i.e., each of its dominated elements is either a tangent vector or weakly equivalent to $\psi$.

Proof

Let $\phi \in E_0 (G)$ be dominated by $\psi$ and let $\tilde{G} \times \mathbb{R}$ be the multiplier extension of $G$ by $R \times \mathbb{R}$ the trivial action of $G$ on $\mathbb{R}$, defined by the multiplier

$$m (g, h) = (c (h), c (y)) \psi (g, s) \psi (g, s) - \psi (g) + s \in E_0 (G) ,$$

where $c : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in G \times \mathbb{R}$.

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ii) $\psi_2 = \lim \frac{1}{n} (p_n (e) - p_n)$, where $p_n \in P (G)$ and $p_n \upharpoonright I_n = \psi_{\mu} \upharpoonright I_n$.

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where $c : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in G \times \mathbb{R}$. //
Construct the corresponding representation \((U_{W'}, H_{W'})\) of \(\hat{G}\); \(U_{W'}\) is the trivial extension of \(U\) to \(G\), therefore irreducible. It is to be noted that \(\psi'\) is extreme in \(E_0(G)\). Now extend \(\phi\) to \(G\) by \(\phi'(g, s) = \phi(g)\). Clearly, \(\psi'\) dominates \(\phi\) in \(E_0(G)\) and since \(\psi'\) is extreme there exists \(\lambda > 0\) such that \(\lambda \psi = \phi\); i.e., \(\lambda \psi = \phi\).

**Lemma 3.10**

B' (U) is precisely the set of bounded 1-cocycles of U. The proof follows directly from 3.7 of \((10)\).

**Lemma 3.11**

Let \(c\) be a cocycle for the representation \(U\) of \(G\) and let \(\psi(x) = \| c(x) \|^2_{L_2} \in E_0(G)\). Then for each \(\mu \in M^0_c\) (G) such that \(-\psi\mu(\epsilon) = 1\) we have \(-\psi\mu\) is a diagonal coefficient of \(U\).

**Proof**

The proof follows immediately because,

\[-\psi\mu(g) = -\int \psi(xg) \mu(x) \, d\mu(y) = -\int (U(g)c(x), c(y)) \mu(x) \, d\mu(y) = (U(g)c_\mu, c_\mu).\]

**Proof of i)**

Let \(U \in G-F, c \in Z'(U)\) and \(\psi(x) = \| c(x) \|^2_{L_2}\). By Lemma 3.9, \(\psi\) generates an extreme ray in \(E_0(G)\). In fact \(\psi\) is either local or bounded. If \(\psi\) is bounded then by Lemma 3.10 the result follows. If it is local, we choose \(\mu \in M^0_c(G)\) such that \(-\psi\mu(e) = 1\). By Theorem 3.5 we have \(-\psi\mu \in F\). In the same time, by Lemma 3.11, (see also Theorem 4.2 in \((11)\) ), there is a diagonal coefficient \(p\) of \(U\) such that \(-\psi\mu = (p+ \bar{p})/2\). Clearly, \(\bar{p}\) is also a diagonal coefficient of \(U\) in \(G-F\). Now \(p\) and \(\bar{p}\) belong to the set of extreme points of \(P_1(G)\), say \(\text{ext} P_1(G)\). So \(p\), \(\bar{p}\) \(\in F\), the complementary face of \(F\). This makes a contradiction with \(-\psi\mu \in F\) and hence \(H'(U) = 0\).

Before starting on part ii) we have to prove the following:

**Lemma 3.12**

Suppose that \(U\) is a representation of \(G\) on a separable Hilbert space and it has a direct integral decomposition \(U(.) = \int_S U(s, \cdot) \, d\mu(s)\) over some probability space \((S, \mu, \cdot)\). For \(H'(U) = 0\) it is necessarily that there exist open sets \(V_1\) and \(V_2\) in \(G\), \(V_1 \cap V_2 = \emptyset\) such that \(V_1\) contains the trivial representation and \(V_2\) contains almost every \(U(s, \cdot)\).

**Proof**

Let \(c \in Z'(U)\). By Theorem 13.2(13) \(c\) has a decomposition in the form \(c(.) = \int c(s, \cdot) \, d\mu(s)\) where \(s \in S\) and \(c(s, \cdot) \in Z'(U(s, \cdot))\).

By part i), for almost every \(s \in S\), there exists \(\lambda_s > 0\) and a unit vector \(\xi_s\) in the Hilbert space of \(U(s, \cdot)\) such that \(c(s, \cdot) = \lambda_s (U(s, \cdot) \xi_s, A_s \xi_s).\) Let \(p(s, x) = \xi_s, \xi_s).\) Then

\[
\psi(x) = \int_S \lambda_s^2 (1-p(s, x)) \, d\mu(s) = \int_S \lambda_s^2 d\mu(s) - \int_S \lambda_s^2 p(s, x) \, d\mu(s).
\]

Now, \(H_{\perp}(U) = 0\) if \(\psi(x)\) is bounded and this is true if \(\lambda_s \in L^2(S, u)\). In fact, there exists an open set \(O \in P_1(G)\) containing the identity and excluding almost every \(p(s, \cdot)\). Let \(g\) be the non-negative function in \(\mathcal{C}(G)\) of Lemma 3.1 which corresponds to \(O\) and let \(\delta > 0\) be such that for \(p \in P_1(G)\) \(\cdot (p, g) < 1 - \delta\). By Fubini's Theorem we have

\[
0 > \int \psi(x) (g(x) dx) = \int_S \lambda_s^2 (1-p(s, x)) g(x) \, d\mu(s) \geq \delta \int_S \lambda_s^2 d\mu(s),
\]

so that \(\lambda_s \in L^2(S, u)\).

**Proof of ii)**

Let the assumptions of Lemma 3.12 be given and suppose that there exist disjoint open sets \(V_1\) and \(V_2\) in \(\text{Prim}(G)\) such that \(\ker I \subset V_1\) and \(\ker U \subset V_2\). For \(a \in C^* (G)\), \(\| U(a) \| = \text{ess sup} \| U(s, a) \|\), so that if \(a \in \ker U\), then \(a \in \ker U(s, .)\) for almost every \(s \in S\). Let \(I = \ker U\) and \(I = \ker U(s, .)\). Excluding a \(\mu\)-null set, then \(I = \cap \{ I(s) \mid s \in S \}\). Since factor representations are homogeneous, then for each measurable subset \(E\) of positive \(\mu\)-measure there exists \(E_0\) \(\subset E\) such that \(I = \cap \{ I(s) \mid s \in E_0 \}\). Evidently, the map \(s \mapsto (s)\) is a measurable function from \(S\) into \(\text{Prim}(G)\), hence the set \(E = \{ s | I(s) \subset V_1 \}\) is measurable. Now, suppose that \(\mu, E > 0\). Then \(I = \cap \{ I(s) \mid s \in E_0 \}\) \(\subset \{ I(s) \mid s \in E_0 \}\) \(\subset V_1\). Since \(V_1\) is open we arrive to a contradiction, and the proof can be completed by applying Lemma 3.12.

**REFERENCES**


