SOME PROPERTIES OF CLASS $Z_w$

By

R.P. EISSA

Faculty of Science, Mansoura University, Egypt

Key words: Properties of class $Z_w$.

ABSTRACT

In paper [5] it was established the majorant $w$. But examples of a function $f \in Z_w$ were not constructed. In this paper, in the case of a smooth closed curve, an example of such a function is constructed.

Historical Approach and Main Results:

The Premeli Privalov theorem [8], [9], [10] is the classical result of the behaviour of a singular operator in the space of continuous functions.

By $H_x$ we denote the class of functions defined on a piecewise smooth closed curve $\mathcal{Y}$ and satisfying Hölder condition with index $\alpha$.

After that the Premeli Privalov theorem was proved for $k-$curves [3], [6]. A closed rectifiable Jordan curve is called $k-$curve if there exists a constant $k \geq 1$ such that for any $t_1, t_2 \in \mathcal{Y}$, $s(t_1, t_2) \leq k |t_1 - t_2|$.

On the other hand, at 1924, Zygmund A. [14] established the following relationship between continuity modules of the singular integrals with Hilbert Kernels and the continuity modulus of the density (in the case of a circle):

$$w_g(\delta) \leq c \left( \int_0^\delta \frac{w_f(\mathfrak{f})}{\mathfrak{f}} \, d\mathfrak{f} + \delta \int_0^\pi \frac{w_f(\mathfrak{f})}{\mathfrak{f}^2} \, d\mathfrak{f} \right),$$

where

$$g(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\mathfrak{f}) \, \text{ctg} \left( \frac{(\mathfrak{f} - t)}{2} \right) \, d\mathfrak{f},$$

and

$$w_f(\delta) = \sup_{|\mathfrak{T}_1 - \mathfrak{T}_2| \leq \delta} |f(\mathfrak{T}_1) - f(\mathfrak{T}_2)|.$$
Some properties of class $Z_w$

In particular from this inequality the Premeli Privalov theorem follows:

Later this estimation was proved [7] for the case of integrals with smooth curves Cauchy Kernel.

In [1] Zygmund type results were proved for the case of arbitrary closed rectifiable Jordan curves in terms of the characteristic metrics $\alpha(\delta), \beta(\delta)$ of the curve, where

$$
\alpha(\delta) = \inf \{ |t - \tau| : s(t, \tau) \geq \delta, \delta \in (0, l/2) \}
$$

in the form

$$
\beta(\delta) = \sup \{ s(t, \tau), \delta \in (0, d = \max |\tau - t|) : |t - \tau| \leq \delta, \tau, t \in \gamma \}
$$

where $l$ is the length of the curve $\gamma$.

From this inequality, in particular, the Premeli Privalov theorem for the case of a $k$-curve follows. In [13] new characteristics of curves were introduced:

$$
\Theta(\delta) = \sup \Theta_1(\delta)
$$

where $\Theta_1(\delta) = \text{mes} \{ \tau \in \gamma, |t - \tau| \leq \delta \}, \delta \in (0, d)$.
For any arbitrary closed rectifiable Jordan curve the following inequality was proved in [13]:

\[
 w_r^*(\delta) \leq c \int_0^\delta \frac{w_t(\delta)}{\delta} d\Theta(\delta) + \delta \int_\delta^t \frac{w_t(\delta)}{\delta^2} d\Theta(\delta),
\]

\[= cz(\delta, w) \]

Hence, the Premeli Privalov theorem follows for the curves satisfying \( \Theta(\delta) \sim \delta \) (i.e. there exists \( c_1, c_2 > 0 \) such that for any \( \delta \in (0, d) \), \( c_1 \delta \leq \Theta(\delta) \leq c_2 \delta \). Notice that for all curves \( \Theta(\delta) \geq \delta \). So we can take \( c_1 = 1 \). The class of these curves is larger than the class of piecewise smooth curves and the class of \( k \)-curves.

In [11] the following inequality of the Zygmund type, was obtained

\[
 w_t^*(\delta) \leq c \int_0^\delta \frac{\Theta(\delta)}{\delta^2} w_t\left( \frac{\delta^2}{\Theta(\delta)} \right) d\delta + \delta \int_\delta^2 \frac{\Theta(\delta)}{\delta^2} d\delta.
\]

Zygmund (or Zygmund type) estimations allow us to study the behaviour of singular integrals in generalized Hölder spaces:

\[
 H_w = \{ f \in C_\gamma \mid w_t(\delta) = \Theta[w(\delta)] \},
\]

where \( w(\delta) \) is a continuity modulus such that \( w(\delta) > 0 \),

\[
 \lim_{\delta \to 0} w(\delta) = 0, w(\delta) \uparrow, w(\delta_1 + \delta_2) \leq w(\delta_1) + w(\delta_2).
\]

Define a norm in \( H_w \) as follows:

\[
 ||f||_{H_w} = ||f||_{C_\gamma} + \sup \frac{w_t(\delta)}{w(\delta)}
\]

It is clear that \( H_w \) is a B-space.
Some properties of class $Z_w$

THEOREM 1. [13]: Let $\mathcal{Y}$ be a curve with $\Theta(\delta) \sim \delta$ and let

$$\int_0^\delta \frac{w(\mathfrak{f})}{\mathfrak{f}} \, d\mathfrak{f} < \infty.$$ 

Then the operator $A\mathfrak{f} = \mathcal{Y}$ maps $H_w$ into $H_{w_1}$, it is bounded, and

$$Z(\delta, w) = \mathcal{O}(w_1(\delta)), \text{ where } w_1(\delta) = \int_0^\delta \frac{w(\mathfrak{f})}{\mathfrak{f}} \, d\mathfrak{f}.$$ 

On the other hand in [2] for the case of a circle (and in [13] for the case of curves such that $\Theta(\delta) \sim \delta$ and at any point of which the tangent is continuous) and in [12] an inequality was obtained which is the inverse of Zygmund’s inequality in some sense. These results gave necessary and sufficient conditions for the existence of a singular operator from $H_w$ to $H_{w_1}$. Hence we have shown the following:

THEOREM 2. ([2], [13], [12])

Let $\mathcal{Y}$ be a smooth closed curve. Then the operator $A\mathfrak{f} = \mathcal{Y}$ maps $H_w$ into $H_{w_1}$ iff

$$\int_0^\delta \frac{w(\mathfrak{f})}{\mathfrak{f}} \, d\mathfrak{f} < \infty \quad \text{and} \quad Z(w) = \mathcal{O}(w(\delta)).$$

In [4,5] the invariance of the class $Z_w$ with respect to the characteristics of $\Omega$ was discussed where

$$\Omega_f(\delta) = \sup_{\mathcal{Y} \in \mathcal{Y}} \left| \frac{\int_{\mathcal{Y}} \frac{f(\mathfrak{f}) - f(t)}{\mathfrak{f} - t} \, d\mathfrak{f}}{\mathcal{Y}(t)} \right|, \delta \in (0,d).$$

and

$$Z_w = \{ f \in C_{\mathcal{Y}} \mid |w(\delta) = \Theta(w(\delta)) \}, \Omega_f(\delta) = \Theta(w(\delta)) \}.$$ 

We notice that $Z_w$ is a $B$–space with respect to the norm

$$\| f \|_{Z_w} = \| f \|_{H_w} + \sup_{\mathcal{Y}} \frac{\Omega_f(\delta)}{w(\delta)}.$$
It is easy to see that $Z_w \subset H_w$ and for any

$$f \in Z_w, \|f\|_{H_w} \leq \|f\|_{Z_w}$$

i.e. the embedding of $Z_w$ into $H_w$ is continuous. If

$$\int_{0}^{d} \frac{w(\mathcal{J})}{\gamma} d\mathcal{J} < \infty$$

then $Z_w = H_w$ their norms are equivalent.

**THEOREM 3.** ([4], [5])

Let $\mathcal{C}$ be a smooth closed curve, such that

$$\delta \int_{0}^{d} \frac{w(\mathcal{J})}{\gamma} d\mathcal{J} = \Theta (w(\delta)), \delta \in (0,d].$$

Then the operator $Af = \tilde{f}$ is a mapping from $Z_w$ to $Z_w$ and is bounded.

It can be easily seen that the condition of theorem 2 is not weaker than the condition of theorem 3.

In [5] a majorant $w$ was established for which

$$\int_{0}^{\delta} \frac{w(\mathcal{J})}{\gamma} d\mathcal{J} = \infty, \delta \int_{0}^{d} \frac{w(\mathcal{J})}{\gamma^2} d\mathcal{J} = O(w(\delta)),$$

however, an example of a function $f \in Z_w$ was not constructed.

In this paper, in the case of a smooth closed curve, an example of such a function is constructed.

Let $\mathcal{C}$ be a closed smooth curve and $t = t(s)$, $0 \leq s \leq \ell$,

where $\ell$ is the length of the curve $\mathcal{C}$, and the equation of the curve in the arc coordinate has the form $t(s) = x(s) + iy(s)$.

Put $t(o) = t_0$, and $t(-s) = t(\ell-s)$. Without losing generality we can take $\ell \geq 2$. 
Consider the function:

\[ w(\delta) = \begin{cases} 
\frac{2}{\ln 2} (1-\delta), & \delta \in [1; t/2] \\
\frac{1}{\ln \frac{1}{\delta}}, & \delta \in [\frac{1}{\delta}; t/2] \\
\frac{1}{\ln \frac{1}{6}}, & \delta \in [0; t/e] 
\end{cases} \]

First we prove that the function \( w(\delta) \) satisfies the following properties:

1) \( w(\delta) > 0; \)
2) \( w(\delta) \) is a non-decreasing function of \( \delta \).
3) \( \lim_{\delta \to 0} w(\delta) = 0; \)
4) \( \frac{w(\delta)}{\delta} \downarrow. \)

The proof of properties 1, 2, 3 is easy.

Now we prove the 4th property. Since \( w(\delta) \) is constant on \( \left[ \frac{1}{\delta}, \frac{t}{2} \right] \), then it is enough to prove that \( w(\delta) \downarrow \) on \( \left[ 0, \frac{1}{\delta} \right] \), by calculating the following derivative.

\[ \frac{(w\delta)'}{\delta} = \left( \frac{1}{\delta \ln \delta} \right)' = \frac{1 - \ln \left( \frac{1}{\delta} \right)}{\delta^2 \ln^2 \left( \frac{1}{\delta} \right)} \leq 0. \]

Now we shall prove that the function \( w(\delta) \) satisfies the conditions of theorem 3.

Consider the expression

\[ A(\delta) = \frac{\delta \int_{\delta}^{\frac{t}{2}} \frac{w(\delta)}{\bar{J}^2} d\bar{J}}{w(\delta)} \]

- 16 -
By using L Hospital's rule, we have.

\[
\lim_{\delta \to 0} A(\delta) = \lim_{\delta \to 0} \frac{\delta \int_{\delta}^{\delta/2} \frac{w(\frac{3}{\delta})}{3^2} d\frac{3}{\delta}}{\frac{w(\delta)}{\delta}} = \lim_{\delta \to 0} \frac{-\frac{w(\delta)}{\delta^2}}{\frac{1-Ln(1/\delta)}{\delta^2 Ln^2(1/\delta)}}
\]

\[
= \lim_{\delta \to 0} \frac{Ln(1/\delta)}{Ln(1/\delta) - 1} = 1
\]

Therefore

\[
\delta \int_{\delta}^{\delta/2} \frac{w(\frac{3}{\delta})}{3^2} d\frac{3}{\delta} = o(w(\delta)).
\]

Therefore by theorem 3, the class \(Z_w\) is invariant under the considered singular operator.

Fix the above curve \(\gamma\) and consider the following function:

\[
f(t(s)) = \begin{cases} 
0 & , \; s \in [1 ; \frac{t}{2} ], \\
\frac{2}{Ln 2} (1 - s) & , \; s \in [\frac{1}{e} ; \frac{t}{2} ], \\
\frac{1}{Ln} \frac{1}{|s|} & , \; s \in [- \frac{1}{e} ; \frac{1}{e}], \\
\frac{2}{Ln 2} (1 + s) & , \; s \in [-1 ; - \frac{1}{e}], \\
0 & , \; s \in [-1 ; - \frac{t}{2} ].
\end{cases}
\]

Now we prove that

\[
\sup_{|s_1 - s_2| \leq \delta} |f(t(s_1)) - f(t(s_2))| \sim w(\delta)
\]
Some properties of class $Z_v$

Actually, it is sufficient to show the last relation for any small $\delta$. On the other hand outside the segment $[-\frac{1}{e}, \frac{1}{e}]$ we consider a function which satisfies Lipschitz condition on the segment $[-\frac{1}{e}, \frac{1}{e}]$ and for

$0 < s_2 - s_1 < s_1 < s_2 \leq \frac{1}{e}$ we have:

$$f(t(s_2)) - f(t(s_1)) = \frac{s_2 - s_1}{3 \ln^2 \frac{1}{3}}$$

where $3 \in [s_1, s_2]$, i.e. $\frac{1}{\ln \frac{1}{3}} \uparrow$. Then $\frac{1}{\ln \frac{1}{3}} \leq 1$.

On the other hand, $\frac{1}{\ln \frac{1}{3}} \downarrow$ and therefore

$$\frac{1}{\ln \frac{1}{3}} \leq \frac{1}{(s_2 - s_1) \ln \left( \frac{1}{s_2 - s_1} \right)}.$$

Hence,

$$0 \leq f(t(s_2)) - f(t(s_1)) \leq \frac{s_2 - s_1}{3 \ln^2 \frac{1}{3}} \leq \frac{1}{\ln \frac{1}{3} \frac{1}{s_2 - s_1}}.$$

For $s_1 \leq s_2 - s_1$ we have:

$$0 \leq f(t(s_2)) - f(t(s_1)) \leq f(t(s_2)) = \frac{1}{\ln \frac{1}{s_2}} =$$

$$= \frac{1}{\ln \frac{1}{s_1 + (s_2 - s_1)}} \leq \frac{1}{\ln \frac{1}{2(s_2 - s_1)}} \leq \frac{c}{\ln \frac{1}{(s_2 - s_1)}}$$

$-18-$
Therefore, for \( s_1, s_2 \in \left[ 0, \frac{1}{e} \right] \) and \( 0 < s_2 - s_1 < \delta \), we have:

\[
|f(t(s_2)) - f(t(s_1))| < \frac{c}{\ln \frac{1}{\delta}}.
\]

For \( s_1, s_2 \in \left[ -\frac{1}{e}, \frac{1}{e} \right] \). In the same way, we obtain

\[
w_t(\delta) \leq \frac{c}{\ln \frac{1}{\delta}}.
\]

On the other hand,

\[
w_t(\delta) = \sup |f(t(s_2)) - f(t(s_1))| \geq f(t(\delta)) - f(t(0)) = \frac{1}{\ln \frac{1}{\delta}}.
\]

From this we obtain that

\[
w_t(\delta) \sim w(\delta).
\]

Now we show that \( f \in Z_w \).

Let \( t(s) \) be any fixed point on the curve \( \mathcal{Y} \). Consider the integral:

\[
\int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{f(t(\tau)) - f(t)}{\tau - t} d\tau = \int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{f(t(\tau)) - f(t)}{\tau - t} d\tau.
\]

Let \( \delta_0 \) be any positive number sufficiently small and \( 0 < s < \delta_0 < \frac{1}{2e} \).

Consider the following cases:

1) If \( \varepsilon < \frac{s}{2} \) then

\[
f(t(\tau)) - f(t(s)) = \frac{1}{\ln \frac{1}{\tau}} - \frac{1}{\ln \frac{1}{s}} = -\frac{1}{\tau \ln^2 \frac{1}{\varepsilon}} (\tau - s)
\]
Some properties of class $Z_w$

where $s + \varepsilon \leq \tau \leq s - \varepsilon$ \ i.e. $\frac{s}{2} < \tau < \frac{3}{2}s$.

Therefore

$$| f(\tau(t)) - f(t(s)) | \leq c \frac{| 3 - s |}{s \ln \frac{1}{s}}$$

For the integral we have,

$$\left| \frac{\int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{f(\tau(s)) - f(t(s))}{\tau(s) - t(s)} \, d\tau(s)}{t(s-\varepsilon) t(s+\varepsilon)} \right| \leq \frac{c}{s \ln \frac{1}{s}} \cdot \frac{c}{\varepsilon \ln \frac{1}{\varepsilon}} = \frac{c}{\ln \frac{1}{\varepsilon}} = c \, w(\varepsilon).$$

2) If $\frac{s}{2} \leq \varepsilon \leq s$ then

$$\int \frac{f(\tau) - f(t)}{\tau - t} \, d\tau = \left[ \int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{f(\tau) - f(t)}{\tau - t} \, d\tau \right]_{\tau = t(s-\varepsilon) t(s+\varepsilon)}$$

$$+ \left[ \frac{f(\tau) - f(t)}{\tau - t} \right]_{\tau = t(s-\varepsilon) t(s+\varepsilon)} t(s-\varepsilon) t(s+\varepsilon)$$

$$= A_1 + A_2 + A_3$$

$A_1$ is estimated as in case 1 with $\varepsilon = \frac{s}{2}$. Therefore we get

$$| A_1 | \leq c \frac{1}{\ln \frac{1}{\varepsilon}} = c \, w(\varepsilon).$$
Since on smooth curves, \(| \tau - t | \leq s(t, \tau) \leq k | t - \tau |\)

where \(k \geq 1\) is a constant, we have

\[
|A_2| = \left| \int_{t(s-\varepsilon)}^{t(s)} \frac{f(\tau) - f(t)}{\tau - t} \, d\tau \right| \leq \frac{1}{s} \int_{t(s-\varepsilon)}^{t(s)} \frac{|f(\tau) - f(t)|}{|\tau - t|} \, d\tau \leq k \frac{1}{s} \int_{t(s-\varepsilon)}^{t(s)} \frac{|f(\tau) - f(t)|}{s(t, \tau)} \, d\tau \leq \frac{2k}{s} \int_{t(s-\varepsilon)}^{t(s)} \frac{1}{\frac{1}{s} \ln \frac{1}{s}} + \frac{1}{\ln \frac{1}{s}} \, d\tau \leq \frac{4k}{s \ln \frac{1}{s}} \int_{t(s-\varepsilon)}^{t(s)} \frac{1}{\frac{1}{s} \ln \frac{1}{s}} \, d\tau = \frac{4k(s-\varepsilon)}{s \ln \frac{1}{s}} \leq \frac{4k}{\ln \frac{1}{s}} \leq \frac{c}{\ln \frac{1}{\varepsilon}} = cw(\varepsilon).
\]

For the 3rd integral \(A_3\) we have:

\[
|A_3| = \left| \int_{t(\frac{3}{2})}^{t(\frac{3}{2}+s)} \frac{f(\tau) - f(t)}{\tau - t} \, d\tau \right| \leq \frac{1}{s} \int_{t(\frac{3}{2})}^{t(\frac{3}{2}+s)} \frac{|f(\tau) - f(t)|}{|\tau - t|} \, d\tau \leq \frac{c}{\ln \frac{1}{\varepsilon}} = cw(\varepsilon).
\]
Some properties of class $Z_w$

Therefore

$$\left| \int \frac{f(\tau) - f(t)}{\tau - t} \, d\tau \right| \leq cw(\varepsilon).$$

3) If $\delta_0 > \varepsilon > s$ then

$$\int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{f(\tau) - f(t)}{\tau - t} \, d\tau = \left[ \int_{t(0)}^{t(2s)} \right] \frac{f(\tau) - f(t)}{\tau - t} \, d\tau +$$

$$= B_1 + B_2 + B_3.$$

$B_1$ is estimated as in case (2) with $\varepsilon = s$. Therefore, we get

$$|B_1| < c \frac{1}{\ln \frac{1}{s}} < c \frac{1}{\ln \frac{1}{j}} = cw(\varepsilon).$$

Consider now $B_2 + B_3 = $

$$\int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{f(\tau))}{\tau} \, d\tau(\tau) +$$

$$\int_{t(2s)}^{t(s+\varepsilon)} \frac{f(\tau))}{\tau} \, d\tau(\tau) -$$

$$f(t) \int_{t(s-\varepsilon)}^{t(s)} \frac{dt}{\tau} + \int_{t(2s)}^{t(s+\varepsilon)} \frac{dt}{\tau}$$

$$= B_1' - B_2'$$
where \( B'_2 = f(t)\), and

\[
\mathcal{J} = \ln \frac{(t(o) - t(s))}{t(s - \varepsilon) - t(s)} + \ln \frac{t(s + \varepsilon) - t(s)}{t(2s) - t(s)}
\]

\[
= \ln \frac{|t(o) - t(s)|}{wt(s - \varepsilon) - t(s)|} + i \arg \frac{t(o) - t(s)}{t(s - \varepsilon) - t(s)} +
\]

\[
+ \ln \frac{|t(s + \varepsilon) - t(s)|}{t(2s) - t(s)} + i \arg \frac{t(s + \varepsilon) - t(s)}{t(2s) - t(s)} =
\]

\[
= \ln \frac{|t(o) - t(s)|}{|t(s - \varepsilon) - t(s)|} \cdot \frac{|t(s + \varepsilon) - t(s)|}{|t(2s) - t(s)|} + i(\alpha + \beta)
\]

Since the curves is smooth, then

\[
\frac{|t(o) - t(s)|}{|t(2s) - t(s)|} \rightarrow 1
\]

as \( s \rightarrow 0 \) and

\[
\frac{|t(s + \varepsilon) - t(s)|}{|(s - \varepsilon) - t(s)|} \rightarrow 1 \text{ as } s \rightarrow 0.
\]

Therefore, the logarithmic part in the last equation is bounded when \( s \) is small.

In the case of smooth curve we have

\[
\alpha = \arg \frac{t(o) - t(s)}{t(s - \varepsilon) - t(s)} \rightarrow 0 \quad \text{when } s \rightarrow 0,
\]

\[
\beta = \arg \frac{t(s + \varepsilon) - t(s)}{t(2s) - t(s)} \rightarrow 0 \quad \text{when } s \rightarrow 0.
\]
Some properties of class $Z_w$

because $\propto$ and $\beta$ are bounded for small $s$.

Therefore, if we take $\delta_o$ sufficiently small, and $0 < s < \delta_o$ we find that $|J| \leq$ constant.

Hence,

$$|B'_2| = |f(t(s))| \cdot |J| \leq \text{const} |f(t)| = \frac{c}{\ln \frac{1}{s}} \leq \frac{c}{\ln \frac{1}{\varepsilon}} = c w(\varepsilon)$$

For

$$|B'_1| = \int_{s-\varepsilon}^{\infty} \frac{f(t(s))}{\tau(s) - t(s)} \dot{\tau}(s) \, ds + \int_{2s}^{\infty} \frac{f(t(s))}{\tau(s) - t(s)} \dot{\tau}(s) \, ds = I_1 + I_2$$

Now, in the integral $I_1$, let $y = s - \tau$ and in the integral $I_2$, $y = \tau - s$.

Then we have:

$$|B'_1| = \int_{s-\varepsilon}^{s} \frac{f(t(s-y))}{\tau(s-y) - t(s)} \dot{\tau}(s-y) \, dy + \int_{s}^{\epsilon} \frac{f(t(y+s))}{\tau(y+s) - t(s)} \dot{\tau}(y+s) \, dy =$$

$$= \int_{s}^{\epsilon} \left( \frac{f(t(y+s))}{\tau(y+s) - t(s)} \dot{\tau}(y+s) - \frac{f(t(s-y))}{\tau(s-y) - t(s)} \dot{\tau}(s-y) \right) \, dy +$$

$$= \int_{s}^{\epsilon} \left( \frac{f(t(y+s)) - f(t(s-y))}{\tau(y+s) - t(s)} \dot{\tau}(s-y) \right) \, dy +$$

$$+ \int_{s}^{\epsilon} \left( \frac{f(t(s-y)) - f(t(s))}{\tau(s-y) - t(s)} \dot{\tau}(y+s) \right) \, dy -$$
\[ - \int_s^\varepsilon \left( \frac{f(\tau(s-y) - f(t(s))}{\tau(s-y) - t(s)} \right) \tau(s-y) \, dy + \]

\[ + f(t) \int_s^\varepsilon \left( \frac{\tau(y+s)}{\tau(y+s) - t(s)} - \frac{\tau(y+s)}{\tau(y-s) - t(s)} \right) \, dy = \]

\[ = A_1 + A_2 + A_3 + A_4. \]

\[ |A_1| \leq \int_s^\varepsilon \frac{1}{\tau(y+s) - t(s)} \, dy \leq k \int_s^\varepsilon \frac{1}{\ln \frac{1}{y+s}} - \frac{1}{\ln \frac{1}{y-s}} \, dy \]

If \( \varepsilon \leq 2s \) then

\[ |A_1| \leq \int_s^{2s} \frac{1}{\ln \frac{1}{y+s}} - \frac{1}{\ln \frac{1}{y-s}} \, dy \leq \frac{1}{s} (\frac{1}{\ln \frac{1}{3s}} - \frac{1}{\ln \frac{1}{3s}}) s = \frac{1}{\ln \frac{1}{3s}} \]

\[ \leq c \frac{1}{\ln \frac{1}{\varepsilon}} \]

If \( \varepsilon > 2s \); then

\[ |A_1| \leq (\int_s^{2s} + \int_s^{\varepsilon}) \frac{1}{\ln \frac{1}{y+s}} - \frac{1}{\ln \frac{1}{y-s}} \, dy \leq c \frac{1}{\ln \frac{1}{\varepsilon}} \]

\[ + \int_s^{\varepsilon} \frac{1}{\ln \frac{1}{y+s}} - \frac{1}{\ln \frac{1}{y-s}} \, dy \]

\[ \leq \frac{1}{\ln \frac{1}{y+s}} - \frac{1}{\ln \frac{1}{y-s}} \leq \frac{\ln \frac{1}{y+s} - \ln \frac{1}{y-s}}{\ln \frac{1}{y+s} - \ln \frac{1}{y-s}} \]

\[ \leq \frac{\ln y+s}{y-s} \leq \frac{c}{\ln^2 \frac{1}{y}} \]

-25-
for \( y \in [2s, \varepsilon] \)

\[
\ln \frac{y+s}{y-s} = \ln \left( 1 + \frac{2s}{y-s} \right) \leq \frac{2s}{y-s} \leq 2.
\]

Therefore,

\[
|A_1| \leq C \frac{1}{\ln \frac{1}{\varepsilon}} + \int_{2s}^{\varepsilon} \frac{dy}{y \ln^2 \frac{1}{y}} = C \frac{1}{\ln \frac{1}{\varepsilon}} - \int_{2s}^{\varepsilon} \frac{dy}{\ln^2 \frac{1}{y}} \, d\ln \left( \frac{1}{y} \right)
\]

\[
= C \frac{1}{\ln \frac{1}{\varepsilon}} - C \int_{2s}^{\varepsilon} \frac{dt}{t^2} = C \frac{1}{\ln \frac{1}{\varepsilon}} + C \left| \frac{1}{t} \right|_{2s}^{\varepsilon}
\]

\[
= C \frac{1}{\ln \frac{1}{\varepsilon}} + C \left| \frac{1}{\ln \frac{1}{y}} \right|_{2s}^{\varepsilon} = C \frac{1}{\ln \frac{1}{\varepsilon}} + C \left( \frac{1}{\ln \frac{1}{2s}} \right)
\]

\[
\leq C \frac{1}{\ln \frac{1}{\varepsilon}} = CW(\varepsilon)
\]

Estimating the integrals \( A_2 \) and \( A_3 \) similarly, we find that \( |A_1| + A_2 + A_3 | \leq CW(\varepsilon) \).

For \( A_4 \):

\[
|A_4| \leq |f(t)| \left| \int_{s}^{\varepsilon} \left( \frac{1}{\tau(y+s) - \tau(t(s))} - \frac{1}{\tau(s-y) - \tau(t(s))} \right) \, dy \right|
\]

\[
= |f(t)| \left| \int_{s}^{\varepsilon} \frac{\tau(s-y) - \tau(s+y)}{(\tau(y+s) - \tau(t(s))) \cdot (\tau(s-y) - \tau(t(s)))} \, dy \right|
\]

\[
- 26 -
\]
This ends the proof.

REFERENCES


Some properties of class $Z_w$


بعض خواص دالة $z_w$

روزيف عيسى

في بحث سابق تم تعريف دالة جديدة من الدوال $z_w$ ويتم إيجاد قيمة
حدًا أعلى $w$ لأي دالة من هذا الفصل ولكن لا يعطى أي مثال على دوال هذا
الفصل.

وفي هذا البحث تم وضع مثال أي إيجاد دالة $f$ من دوال هذا الفصل والمعرفة
على المحدودات المغلقة المنسية. ثم أثبت أن هذه الدالة تنتمي إلى الفراغ $z_w$.