A CLASS OF FUNCTIONS STRONGER THAN M-PRECONTINUITY, PREIRRESOLUTE AND A-FUNCTIONS

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ABSTRACT

We propose a definition for a new class of functions that may be stronger than the previous types. Some of its characterizations, connections with known ones and the effect of them on some topological spaces are studied.

1- INTRODUCTION

In recent years, one type of functions between topological spaces has been defined by three independent authors. In 1982, Mashhour et al. (3) called it M-precontinuous, in 1985, Reilly and Vamanamurthy (6) named it a preirresolute function and in 1988, Rose (8) called it A-function depending on the concept of almost-open set (8).

Therefore, let A be a subset of a topological space X and its closure and interior are denoted by cl(A) and int(A), respectively. A is called preopen (2) or almost open (8) if A ⊆ int(cl(A)) and (X-A) is called preclosed or almost closed, respectively. The intersection (union) of all preclosed (preopen) sets which contain (contained in) A, is called preclosure (3) (preinterior (3)) and denoted by p.cl(A)(p.int(A)). The preboundary of A (p.b(A)) is the intersection of p.cl(A) and p-cl(X-A). The collection of all preopen sets in X will be denoted by PO(X). A function f:X→ Y is called M-precontinuous (3) (M.P.C) or preirresolute (6) (A-function(8)) if f'(V) is preopen (almost open) in X for each preopen (almost open) V in Y. f is called preopen (2) (preclosed (5)) if the image of each open (closed) is preopen (preclosed).

A space X is called strongly compact (4) (S-closed (9), strongly Lindelof (4)) if for each preopen (semi-open, preopen) cover of X, there exists a finite subcover (a finite subcover, countable subcover) covers (its closure covers) X. Also, X is called a
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resolvable space (1) if there is a subset D of X such that D and X-D are both dense in X. Spaces pre-T_i (i = 0,1,2) defined likewise the spaces T_i (i = 0,1,2,) except that open sets are replaced by preopen ones.

2 - DEFINITION AND CHARACTERIZATIONS

Definition 2.1: A function f:X→ Y is called strongly M-precontinuous (abbreviated as SMPC), if the inverse image of each preopen in Y is open in X.

Some characterizations of SMPC are established throughout the next result.

Theorem 2.1: Let f:X→ Y be a function, then the following statements are equivalent.

(i) f is SMPC.
(ii) For each x ∈ X and each V ∈ PO(Y) containing f(x), there exists an open set U of X containing x such that f(U)⊂ V.
(iii) the inverse image of each preclosed set in Y is closed in X.
(iv) cl (f^1(B))⊂ f^1 (p.cl(B)).
(v) f(cl A)⊂ p.cl (f(A))
(vi) f^1 (p. int (B))⊂ int (f^1 (B))
(vii) b (f^1(B))⊂ f^1(p.b (B))
(viii) f (b(A))⊂ p.b (f(A))

Proof: (i) → (ii) and (i) → (iii) are obvious.
(iii) → (iv) Since B⊂ p.cl (B), for each B⊂ Y. so., f^1 (B)⊂ f^1 (p.cl (B)), but f^1 (p.cl (B)) is closed, hence cl (f^1 (B))⊂ f^1 (p.cl(B)).
(iv) ⇒ (v). Replacing f (A) instead of B in (iv).
(v) ⇒ (iii). Let F⊂ Y be preclosed, then f^1(F)⊂ X and so f(cl f^1 (F))⊂ p-cl (ff^1 (F))⊂ p-cl(F) = F. This gives clf^1 (F)⊂ f^1 (F). Hence f^1 (F) is closed.
(i) ⇒ (vi). For any B⊂ Y, p.int (B) ∈ PO(Y) and so f^1 (p-int (B)) ∈ τ (X). Hence f^1 (p.int(B)) = int (f^1 (p.int(B)))⊂ int (f^1 (B)).
(vi) ⇒ (i). Follows directly by taking B∈ PO(Y).
(vi) ⇒ (vii). Let B⊂ Y, since b(f^1 (B)) = cl (f^1 (B)) - int (f^1 (B))⊂ f^1 (p.cl (B)) - f^1 (p.int (B)) = f^1 (p.b(B) ∪ (p.int(B))). f^1 (p.int(B)) = f^1 (p.b(B)).
(vii) ⇒ (viii). By putting f(A) instead of B, the result follows immediately.
(viii) ⇒ (ii). Let B⊂ Y be preclosed, then f^1 (B)⊂ X by (viii) f(b(f^1 (B)))⊂ p.b (ff^1 (B)) p.b (B)⊂ p.cl(B) = B and so b(f^1 (B))⊂ int f^1 (B))⊂ f^1 (B) i.e. cl f^1 (B)⊂ f^1 (B). Hence f^1 (B) is closed and therefore f is SMPC.

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3. SOME CONNECTIONS AND PROPERTIES OF SMPC

Next diagram shows the implications between SMPC and other known types of functions.

\[
\text{MPC} \quad \text{SMPC} \quad \preirresolute \quad \text{A-function} \quad \text{continuity} \quad \precontinuity
\]

These implications are not reversible as shown throughout the following examples.

**Example 3.1:** Let \( X = \{a, b, c, d\} \) having \( \tau_1 = \{X, \emptyset, \{a, b\}, \{a, b, d\}\} \) and \( \tau_2 = \{X, \emptyset, \{a, c\}, \{a, b, c\}\} \).

The function \( f: (X, \tau_1) \rightarrow (X, \tau_2) \) defined as \( f(a) = f(b) = a, f(c) = b \) and \( f(d) = c \) is continuous but not SMPC.

**Example 3.2:** Let \( X = \{a, b, c, d\} \) with \( \tau_1 \) be an indiscrete topology and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \). The identity function from \( (X, \tau_1) \) \( (X, \tau_2) \) is M-precontinuous but not SMPC.

**Theorem 3.1:** The following holds for any function \( f: X \rightarrow Y \)

(i) If \( X \) is submaximal, then \( f \) is SMPC iff it is M-precontinuous.

(ii) SMPC coincides on continuity if \( Y \) is submaximal.

(iii) If both \( X \) and \( Y \) are submaximal, then all types in previous diagram are equivalent.

**Proof:** Follows by using Theorem (4) (7).

**Theorem 3.2:** A function \( f: X \rightarrow Y \), where \( Y = \text{FUG} \) denote the Hewitt-representation, \( \text{cl}(G) \) is open and \( \{Y\} \in \text{PO}(Y) \) for each \( Y \in \text{int.}\ (F.) \) Then \( f \) is SMPC iff \( f : X \rightarrow \text{PO}(Y) \) is continuous.

**Proof:** Under these assumptions and by Theorem (5) of (1), we have \( f : X \rightarrow Y \) is SMPC iff \( f'(V) \) is open in \( X \), for each \( V \in \text{PO}(Y) \), that is iff \( f : X \rightarrow \text{PO}(Y) \) is continuous.

**Corollary 3.1:** Any function \( f : X \rightarrow Y \), where \( Y \) is resolvable and each open set in it is closed. Then \( f \) is SMPC iff \( f : X \rightarrow \text{PO}(Y) \) is continuous.

**Proof:** Since resolvability of any space in which each open set is closed is considered as one of the applications of the properties on \( Y \) in previous Theorem (1), Corollary (2). Then \( f : X \rightarrow Y \) is SMPC iff \( f : X \rightarrow \text{PO}(Y) \) is continuous.

The following two results concerning compositions and restriction can be shown, therefore their proofs are an immediate consequence of their definitions.
Theorem 3.3: The following holds for functions $f : X \to Y$ and $g : Y \to Z$.

(i) If both $f$ and $g$ gave were SMPC, then their composition is also SMPC.
(ii) If $gof$ is SMPC and $f$ is open (preopen), then $g$ is SMPC (preirresolute.)
(iii) If $g$ is SMPC and $gof$ is preopen (preclosed), then $f$ is open (closed).
(iv) If $f$ is SMPC and $g$ is precontinuous, then $gof$ is continuous.
(v) $gof$ is SMPC if one of the next is verified.
   (1) $f$ is SMPC and $g$ is preirresolute.
   (2) $f$ is continuous and $g$ is SMPC.

Theorem 3.4: Two statements are true for a function $f : X \to Y$

(i) For any $A \subseteq X$, $f/A$ is SMPC if $f$ is SMPC.
(ii) If \{U$_{A}$, $\alpha \in \nabla$\} is an open cover of $X$, then $f$ is SMPC, If $f/U_{\alpha}$ is SMPC, for each $\alpha \in \nabla$.

Lemma 3.1 (5): Let $\{X_{i} : i \in I\}$ be a family of topological spaces, $X = \prod_{i \in I} X_{i}$ the product space and $A_{i}$ be a nonempty subset of $X_{i}$ for $i \in I$
eq 1. For a positive integer $n$ if $A = \prod_{j=i}^{n} A_{i}$, then

\[ A \in PO(X) \text{iff} A_{i} \in PO(X_{i}), \quad i \in \{1, 2, 3, \ldots, n\} \]

Theorem 3.5: The function $f : X \to Y$ is SMPC if the graph function $g : X \to XxY$ of $f$ which defined by $g(x) = (x,f(x))$, for each $x \in X$ is SMPC.

Proof: Let $g$ be SMPC, $x \in X$ and $V \in PO(Y)$ containing $f(x)$. Then by Lemma (3.1) $XxV \in PO(XxY)$ containing $g(x)$, so there exists an open set $U$ of $X$ containing $x$ such that $g(U) \subseteq XxV$ and hence $f(U) \subseteq XxV$. Therefore $f$ is SMPC.

Theorem 3.6: Let $\{X_{\alpha} : \alpha \in \nabla\}$ be any family of spaces and $f : X \to \prod X_{\alpha}$ be SMPC. Then $P_{\alpha}$ of: $X \to X_{\alpha}$ is SMPC, for each $\alpha \in \nabla$, where $P$ is the projection of $\prod X_{\alpha}$ onto $X_{\alpha}$, for each $\alpha \in \nabla$.

Proof: We shall consider a fixed $j \in \nabla$ and let $V_{j} \in PO(X_{j})$, from projection properties, $P_{j}^{-1}(V) = V_{j} \times \prod_{\alpha \neq j} X_{\alpha} \subseteq PO(\prod_{\alpha \in \nabla} X_{\alpha})$ (see lemma 3.1)

and so $(P_{j}of)^{-1}(V_{j}) = f^{-1}(P_{j}^{-1}(V_{j})) = f^{-1}(V_{j} \times \prod_{\alpha \in \nabla} X_{\alpha}) \subseteq \tau(X)$.

Hence Pof is SMPC, for each $j \in \nabla$.

Theorem 3.7: Let $f_{\alpha} : X_{\alpha} \to Y_{\alpha}$, $\nabla Y_{\alpha}, \alpha \in \nabla$ be a family of functions and $f : \prod X_{\alpha} \to \prod Y_{\alpha}$ defined as $f(x_{\alpha}) = \{f_{\alpha}(x_{\alpha})\}$. Then $f$ is SMPC iff $f_{\alpha}$ is SMPC, for each $\alpha \in \nabla$.
Proof: let $f$ be SMPC, for a fixed $j \in \mathcal{V}$ and $V_j \in \text{PO}(Y_j)$.

Then $P_j^{-1}(V_j) = V_j \times \pi \ Y_\alpha \in \text{PO}(\pi \ Y_\alpha)$, Lemma (3.1) shows $\alpha \neq j$.

This fact and so $f^1 \big( P_j^{-1}(V_j) \big) = f_j^{-1}(V_j) \times \pi \ X_\alpha \in \mathcal{T}(\pi \ X_\alpha)$

Hence $f^1(V_j) \in \mathcal{T}(X_j)$. Therefore $f_j$ is SMPC, for each $j \in \mathcal{V}$.

Conversely, let $f_\alpha$ is SMPC, for each $\alpha \in \mathcal{V}$ and $V \in \text{PO}(\pi \ Y_\alpha)$. Then there exists finite $\mathcal{V}_0$ of $\mathcal{V}$ such that $V = \bigcap V_j \times \pi \ Y_\alpha$ by Lemma

$J \in \mathcal{V}_0 \ \alpha \neq j$

(3-1) $V_j \in \text{PO}(Y_j)$ for each $j \in \mathcal{V}_0$ and so $f_j^{-1}(V_j) \in \mathcal{T}(X_j)$ for each $j \in \mathcal{V}_0$ this leads to $f^1(V) = \bigcap f_j^{-1}(V_j) \times \pi \ X_\alpha \ \ (\pi \ X_\alpha)$. Then $f$ is SMPC.

Theorem 3.8: If $f_\alpha : X \to X_\alpha$ is defined $f_\alpha(x) = x_\alpha$ and $f : X \to \prod X_\alpha$ is given by $f(x) = \{ f_\alpha(x_\alpha) \}$ for each $x \in X$, $\alpha \in \mathcal{V}$. Then $f$ is SMPC iff $f_\alpha$ is SMPC for each $\alpha \in \mathcal{V}$.

Proof: Necessity, it is enough to show this fact at one fixed $j \in \mathcal{V}$. So, let $V_j \in \text{PO}(X_j)$, by Lemma (3.1) and projection function one can show that $V = V_j \times \pi \ X_\alpha \in \text{PO}(\pi \ X_\alpha)$ and by SMPC of $f$ we get $f^1(V) = f^1 V_j \times \pi \ X_\alpha \ \ \alpha \neq j$

$= f_j^{-1}(V_j) \times X \in \mathcal{T}(X)$ which gives $f_j^{-1}(V_j) \in \mathcal{T}(X_j)$. Therefore $f_j$ is SMPC for each $j \in \mathcal{V}$.

Sufficiently, let $f_\alpha$ b SMPC, for each $\alpha \in \mathcal{V}$ and $V \in \text{PO}(\prod X_\alpha)$. So, for a finite $\mathcal{V}_0$ of $\mathcal{V}$ we get $V = \bigcap V_j \times \prod X_\alpha$, hence by Lemma (3.1)

$J \in \mathcal{V}_0 \ \alpha \neq j$

$V_j \in \text{PO}(X_j)$ for each $j \in \mathcal{V}_0$ and therefore $f_j^{-1}(V_j) \in \mathcal{T}(X)$ for each $j \in \mathcal{V}_0$

and so, $f^{-1}(V) = \bigcap f_j^{-1}(V_j) \times X \in \mathcal{T}(X)$. Hence $f$ is MPC.

Theorem 4.1: The image of compact space under a surjective SMPC function is strongly compact.

Proof Let $f : X \to Y$ be surjective SMPC. Let also $X$ be compact and $\{ V_i : i \in I \}$ be a
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preopen cover of \( Y \). By SMPC of \( f, f^i(V_i) \in \tau(X) \) for each \( i \in I \). Hence \{ \( f^i(V_i) : i \in I \) \} is open cover of \( X \) which is compact, then there exists a finite subfamily \( I_0 \) of \( I \) such that \( X = \bigcup \{ f^i(V_i) : i \in I_0 \} \). So, \( Y = f(X) = \bigcup \{ V_i : i \in I_0 \} \). Hence \( Y \) is strongly compact.

**Corollary 4.1:** Strongly compactness is preserved under a surjective SMPC function.

**Corollary 4.2:** For any SMPC surjective function, the image of a compact space is nearly compact and almost compact.

**Theorem 4.2:** Let \( f: X \to Y \) be SMPC surjective function and \( X \) be S-closed, then \( Y \) is almost compact.

**Proof:** If \{ \( V_i : i \in I \) \} is preopen cover of \( Y \), then \{ \( f^i(V_i) : i \in I \) \} is open cover of \( X \). Since \( X \) is S-closed, then there exists a finite \( I_0 \subseteq I \) such that \( X = \bigcup \{ \text{cl} f^i(V_i) : i \in I_0 \} \). Thus, \( Y = \bigcup \{ f(\text{cl} f^i(V_i)) : i \in I_0 \} \subseteq \bigcup \{ \text{cl} (V_i) : i \in I_0 \} \) and so \( Y \) is almost compact.

**Theorem 4.3:** If \( f: X \to Y \) is SMPC surjective and \( X \) is Lindelof, then \( Y \) is strongly Lindelof.

**Proof:** Let \{ \( V_i : i \in I \) \} be preopen cover of \( Y \), then \{ \( f^i(V_i) : i \in I \) \} is open cover of \( X \). Since \( X \) is Lindelof, there exists a countable subcover with \( X = \bigcup \{ f^i(V_i) : i \in I_0 \} \) (countable). Hence \( Y = f(X) = \bigcup \{ V_i : i \in I_0 \} \subseteq \bigcup \{ \text{cl} (V_i) : i \in I_0 \} \), therefore \( Y \) is strongly Lindelof.

**Theorem 4.4:** The inverse image of pre-T\(_i\), \((i=0, 1, 2)\) under an injective SMPC function is T\(_i\), \((i = 0, 1, 2)\).

**Proof:** We prove this result in one case (say \( i=0 \). So, let \( f:X \to Y \) be SMPC injective, \( Y \) be pre-T\(_0\) and \( x_1, x_2 \) be two distinct points of \( X \). Then \( f(x_1) \neq f(x_2) \). Hence for each preopen set \( V \subseteq Y \) containing one of \( f(x_j), j \in \{1, 2\} \), there exists \( U_j \in \tau(X) \) containing a corresponding point \( x_j, (j = 1, 2) \) such that \( f(U_j) \subseteq V \). Then \( X \) is a T\(_0\)-space. The other parts of the Proof follows similarly.

**REFERENCES**


 השונים من الدوال الأقوى من
M - Precantinuity, Preirresalute and A-Functions

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