# *Relation between Quasi-normed Ideals of Entropy Numbers and of Approximation Numbers*

by

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## ABSTRACT

Let $e_n(T)$ and $a_n(T)$ be the entropy numbers and approximation numbers of operator T between Banach spaces, respectively. Let $\zeta_p$ be the operator ideal defined by

$$\zeta_p := \left\{ T \in L; \left\{ e_n(T) \right\} \in l_p \right\},$$

and $S_p$ be the operator ideal defined by

$$S_p := \left\{ T \in L; \Sigma a_n(T)^p < \infty \right\}.$$

Then if $0 < p < 1$, and $1 \leqslant u, v \leqslant \infty$ we have $S_p(l_u, l_v) \subsetneq \zeta_p(l_u, l_v)$.

# العلاقة بين
# مثاليات مؤثرات اعداد الانتروبي
# والاعداد التقريبية

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عنى هذا البحث بدراسة العلاقة بين مثالي كل من مؤثر اعداد الانتروبي $\mathfrak{S}_p$ ومؤثر الاعداد التقريبية $S_p$ في فراغات باناخ $l_v , l_u$ .

وفي حالة $1 \leqslant V \leqslant \infty,\ 1 \leqslant U \leqslant \infty,\ 0 < p < 1$ . فقد اثبتت العلاقة

$S_p\ (l_u , l_v) \subsetneqq \mathfrak{S}_p\ (l_u , l_v)$ .

## Introduction

In [1] the quasi-normed ideal $\zeta_p$ has been introduced. The relation between $\zeta_p$ and $S_p$ in Hilbert spaces has been studied and it has been found that [1] $S_p(l_2,l_2) = \zeta_p(l_2, l_2)$ for all exponent $0<p<\infty$. In particular, $\zeta_2(l_2, l_2)$ is the ideal of Hilbert-Schmidt operators.
In Banach spaces a little is known about the relation between $\zeta_p$ and $S_p$ [1]. The relation between $\zeta_p$ and the ideal of p-nuclear operators $N_p$, $0<p<1$ has been previously estimated [7].

This work investigates the relation between $\zeta_p(l_u,l_v)$ and $S_p(l_u,l_v)$ for all exponents $0<p<1$.

## Basic Notations

In the following L denotes the class of all bounded linear operators between arbitrary Banach spaces and L(E,F) the set of all such operators between specific Banach spaces E and F. The closed unit ball of E is denoted by U and of F is denoted by V. $L_n$ (E, F) denotes the subspace of L (E, F) of operators of rank $(T) < n$. Let $l_u$ denote the Banach space of all u-absolutely summable sequences provided with the norm

$$\| x \|_u = \left\{ \sum_i |\xi_i|^u \right\}^{1/u} \text{ if } 1 \leqslant u < \infty$$

and

$$\| x \| = \sup |\xi_i| \text{ if } u = \infty$$

respectively.

We mention that the ideals of entropy numbers, of p-nuclear and of approximation numbers are denoted by $[\zeta_p, E_p]$, $[N_p, \nu_p]$ and $[S_p, \sigma_p]$, respectively ([1, 4], [2, 3] and [5, 6]).

## Main Result

Our main result can be obtained through a series of theorems. We begin with a result of Pietsch [6].

*Theorem 1*
Each mapping $T \in S_p(E, F)$ with $0 < p \leqslant 1$ can be represented as

$$Tx = \sum_{r \in R} \lambda_r \langle x, a_r \rangle y_r$$

with linear forms $a_r \in U^\circ$ and elements $y_r \in V$, such that the inequality

$$\left\{ \sum_R |\lambda_r|^p \right\}^{1/p} \leqslant 2^{2+3/p} \sigma_p(T)$$

holds for the numbers $\lambda_r$

In the next theorem we need the well-known Auerbach's lemma.

*Lemma:* Let M be an n-dimensional normed linear space. Then there exists a basis $\{x_1, \dots, x_n\}$ for M and a subset $\{u_1', \dots, u_n'\}$ of M′ (the dual of M) such that

$$x = \sum_{i=1}^{n} \langle x, u_i' \rangle x_i \text{ for each } x \in M,$$

with $\|u_i'\| = \|x_i\| = 1$ and $\langle x_i, u_j' \rangle = \delta_{ij}$, $i, j = 1, 2, \dots, n$.

*Theorem 2*

Let E and F be Banach spaces, and $0 < p < 1$. Then we have

$$S_p(E, F) \subset N_p(E, F)$$

and

$$\nu_p(T) \leqslant 2^{2+3/p} \sigma_p(T) \quad \text{for each } T \in S_p(E,F).$$

*Proof:* By definition of approximation numbers, for $n = 1, 2, \dots$ there exists an $A_n \in L_{2^n-1}(E,F)$ such that

$$\| T - A_n \| \leqslant 2 a_{2^n-1}(T).$$

We now put

$$B_n = A_{n+1} - A_n,$$

$\dim R(B_n) = d_n$ ($R(B_n)$ denotes the range of $B_n$),

$$i_0 = 0, \; i_r = \sum_{n=1}^{r} d_n$$

and

$$I_r = \{\text{the integers in } [n_{r-1} + 1, n_r]\}, \quad r = 1, 2, \dots .$$

Then, since the sequence $\{a_j(T)\}$ is decreasing, we have

$$\|B_n\| \leqslant 4 a_{2^n-1}(T)$$

And since

$d_n < 2^{n+1} - 1 + 2^n - 1 < 2^{n+2}$, we have

$$i_r < 2^3 (2^r - 1) < 2^{3+r}, \quad r = 1, 2, \dots .$$

By Auerbach's lemma, there exist $\{u_n'\}_{n \in I_r} \subset F'$ and $\{y_n\}_{n \in I_r} \subset R(B_n)$ such that $\|u_n'\| = 1$, $\|y_n\| = 1$ and

$$B_r x = \sum_{n \in I_r} \langle B_r x, u_n' \rangle y_n, \quad r = 1, 2, \dots .$$

for each $x \in E$. Putting

$$x_n' = B_r' u_n' / \| B_r' u_n' \|,$$

$$\lambda_n = \| B_r' u_n' \| \leqslant \|B_r\| \quad \text{for } n \in I_r, \quad r = 1, 2, \dots,$$

we have

$$B_r x = \sum_{n \in I_r} \lambda_n \langle x, x_n' \rangle y_n, \quad r = 1, 2, \dots .$$

By making use of these $\{x_n'\}_{n \in I_r}, \{y_n\}_{n \in I_r}$, $r = 1, 2, \dots$, we can write

$$Tx = \lim_{r\to\infty} A_{r+1}x = \sum_{r=1}^{\infty} B_r x$$

$$= \sum_{r=1}^{\infty} \sum_{n\in I_r} \lambda_n \langle x, x_n' \rangle y_n \quad \text{for each } x\in E,$$

with $\|x_n'\| = 1, \|y_n\| = 1$, $n = 1, 2, \ldots$ . Therefore, for $0<p\leqslant 1$, we get

$$\{\nu_p(T)\}^p \leqslant \sum_{r=1}^{\infty} \sum_{n\in I_r} |\lambda_i|^p \leqslant \sum_{r=1}^{\infty} \sum_{n\in I_r} \|B_r\|^p$$

$$\leqslant \sum_{r=1}^{\infty} 2^{r+2} (4\, a_{2^r-1}(T))^p$$

$$\leqslant 2^{3+2p} \sum_{r=1}^{\infty} \sum_{n=2^{r-1}}^{2^r-1} \{a_n(T)\}^p$$

$$\leqslant 2^{3+2p} \{\sigma_p(T)\}^p.$$

Hence

$$\nu_p(T) \leqslant 2^{2+3/p}\, \sigma_p(T)$$

which finishes the proof.

*Theorem 3*

Let $0<p\leqslant 1$. Then there exist diagonal operators D from $1_u$ into $1_v$ such that $D\in N_p(1_u, 1_v)$ and $D\notin S_p(1_u, 1_v)$.

The following example shows that the above theorem is true

*Example:* Let $\lambda_k = 1/k - 1/(k+1) = 1/k(k+1)$ and define an operator $T \in L(l_\infty, l_1)$ by $T\{\xi_i\} := \{\lambda_i \xi_i\}$.

Since

$$\sum_{k=1}^{\infty} \lambda_k^p = \sum_{k=1}^{\infty} 1/(k^2+k)^p \leqslant \sum_{k=1}^{\infty} 1/k^{2p} < \infty \text{ , for } 1/2 < p \leqslant 1,$$

then T is p-nuclear, $1/2 < p \leqslant 1$.

But we have

$$\sigma_1(T) = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} (1/k - 1/(k+1)) = \sum_{n=0}^{\infty} 1/(n+1) = \infty,$$

which proves that $T \notin S_1(l_\infty, l_1)$. So $T \notin S_p(1_\infty, 1_1)$ for every $0<p\leqslant 1$, [5].

Theorems 2 and 3 prove the following

*Theorem 4*

Let $O < p \leqslant 1$. Then

$$Sp\,(l_u, l_v) \subsetneq N_p(l_u, l_v)$$

It is known that $N_p\,(l_2, l_2)$ is identical with $S_p\,(l_2, l_2)$ for $O < p \leqslant 1$, [3].

The relation between $N_p\,(l_u, l_v)$ and $\zeta_p\,(l_u, l_v)$ for $O < p < 1$ has been investigated in [7]. However, we state the main result of [7] as

*Theorem 5*

Let $O < p < 1$. Then

$$N_p\,(l_u, l_v) = \zeta_p\,(l_u, l_v).$$

As a consequence of theorems 4 and 5, we get

*Theorem 6*

Let $O < p < 1$. Then

$$S_p\,(l_u, l_v) \subsetneq \zeta_p(l_u, l_v)\ .$$

This result answers problem raised in [1] in case $O < p < 1$.

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