THE ONE DIMENSIONAL INVERSE SPECTRAL PROBLEM OF A
GENERALIZED STATIONARY SCHRÖDINGER EQUATION

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ABSTRACT

In this paper we study the inverse problem on the half line for a generalized stationary one dimensional Schrödinger equation. We consider the solutions of the differential equation and study its properties. Given the spectral distribution function, we solve the inverse problem.

Key Words: Inverse problem, stationary Schrödinger equation, spectral function, Parseval's equality, discontinuous coefficient.

INTRODUCTION

Consider, in the space $L_2 (0, \infty; \rho(x))$, the generalized stationary Schrödinger equation

\[
\frac{d^2}{dx^2} u(x) - q(x) + \lambda \rho(x) u(x) = 0
\]

and

\[
\rho(x) = \begin{cases} \sum a_n x^{\alpha_n-1} & \text{if } x < 0, \\ a_n x^{\alpha_n-1} & \text{if } 0 \leq x < 1, \\ \sum a_n x^{\alpha_n-1} & \text{if } x \geq 0, \end{cases}
\]

where $a_n > 0$, $a_0 \neq 0$, $\alpha_n \neq \alpha_{n+1}$, $\alpha_n \neq 1$ and $\lambda$ is a constant. Let $W_n$ be the set of functions $\rho(x)$. Also denote with $L_{1,1}$ the class of potentials.
The inverse problem can be stated as follows: knowing the spectral distribution function of (1)-(2), can we reconstruct equation (1), i.e., can we determine the potential \( q(x) \) and the density function \( p(x) \)? The boundary value problem (1), (2) was discussed in [1, 2, 5, 8] for the case \( p=1 \) and \( u(0) = 0 \), and the coefficients was investigated in [4, 9, 11]. In [12] the inverse problem of (1), (2) was investigated using the scattering data by employing the technique of the spectral distribution function.

**Preliminaries**

In this section, we give some results for the case \( p(x) \in W_n \) which will be used in the subsequent sections. Denote with \( \phi(x,\lambda), x \in [0, a_1] \) the solution of equation (1) which satisfy

\[
\phi(0,\lambda) = 1, \quad \phi'(0,\lambda) = 0
\]  

and with \( \psi(x,\lambda) \) the solution of (1) which satisfy

\[
\psi(0,\lambda) = 0, \quad \psi'(0,\lambda) = 1
\]  

To obtain these solutions, we shall use the results of [7, 8]. Let \( p(x) \in W_1; \, a_1 = \alpha, a_0 = 0, \) and \( q \in L_{\alpha,1} \) then we have

\[
\phi(x,\lambda) = \cos(\sqrt{\lambda} x) + \int_0^x A(x, t) \cos(\sqrt{\lambda} t) dt; 0 \leq x \leq a_1,
\]

(7)

where the kernel \( A(x, t) \) has summable derivatives \( A_n, A_n \), and

\[
\frac{dA(x, t)}{dx} = \frac{1}{2} q(x) \text{ and } \frac{\partial A(x, t)}{\partial x} \bigg|_{x=0} = 0
\]

This solution \( \phi(x,\lambda) \) is an entire function of \( \sqrt{\lambda} \) for any fixed \( \alpha \). Moreover

\[
\phi(x,\lambda) = \cos(\sqrt{\lambda} x) \left[ 1 + O\left( \frac{1}{\sqrt{\lambda}} \right) \right] \text{ as } \text{Im} \sqrt{\lambda} \geq 0, \sqrt{\lambda} \to \infty
\]

Uniformly with respect to \( \sqrt{\lambda} \) on \([0, a_1]\)

The second solution \( \psi(x,\lambda) \) of equation (1) is given by

\[
\psi(x,\lambda) = \frac{\sin(\sqrt{\lambda} x)}{\sqrt{\lambda}} \left[ 1 + O\left( \frac{1}{\sqrt{\lambda}} \right) \right] + \int_0^x B(x, t) \frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} dt; 0 \leq x \leq a_1,
\]

(8)

where the kernel \( B(x, t) \) has summable derivatives \( B_n, B_n \), and

\[
\frac{dB(x, t)}{dx} = \frac{1}{2} q(x) \text{ and } B(x,0) = 0
\]

This solution \( \psi(x,\lambda) \) has also the property that it is an entire function of \( \sqrt{\lambda} \) for \( \text{Im} \sqrt{\lambda} \geq 0, \sqrt{\lambda} \to \infty, \)

Moreover

\[
\psi(x,\lambda) = \frac{\sin(\sqrt{\lambda} x)}{\sqrt{\lambda}} \left[ 1 + O\left( \frac{1}{\sqrt{\lambda}} \right) \right]
\]

uniformly with respect to \( \sqrt{\lambda} \) on \([0, a_1]\).

The proof of the following lemma has been done in [7].

**Lemma 1**

If condition (4) is satisfied for any \( \lambda \) from the upper half plane then equation (1) has a solution \( F(x,\lambda) \) which can be written in the form

\[
F(x,\lambda) = \exp(i\sqrt{\lambda} x) + \int_0^x K(x, t) \exp(i\sqrt{\lambda} t) dt; a_1 < x < \infty,
\]

(9)

The kernel \( K(x, t) \) is twice differentiable and satisfies the equation

\[
\frac{\partial^2 K(x, t)}{\partial x^2} - \frac{\partial^2 K(x, t)}{\partial t^2} = q(x) K(x, t),
\]

(10)

and the conditions

\[
\frac{dK(x, t)}{dx} = -\frac{1}{2} q(x), \quad \lim_{x \to \infty} \frac{K(x, t)}{t} = 0
\]

(11)

The solutions \( F(x,\lambda) \) is analytic in the upper half plane \( \text{Im} \sqrt{\lambda} > 0 \) and continuous on the real line. The asymptotic behaviour of the solution is

\[
F(x,\lambda) = \exp(i\sqrt{\lambda} x) \left[ 1 + O(1) \right];
\]

\[
F_x(x,\lambda) = i\sqrt{\lambda} \exp(i\sqrt{\lambda} x) \left[ \sqrt{\lambda} + o(1) \right],
\]

as \( x \to \infty \) for all \( \text{Im} \sqrt{\lambda} \geq 0, \sqrt{\lambda} \neq 0. \)

The proof of the following lemma could be obtained from [12].
Lemma 2

The boundary value problem (1)-(2) has a finite number of negative eigenvalues and they are all simple.

The main results

Lemma 3

Let \( f(x) \) be a finite function which has a continuous derivative in \( (0, \infty, \rho(x)) \) and satisfies the boundary condition (2). Then

\[
\int_0^\infty f^2(x) \rho(x) dx = \frac{1}{\pi} \int_0^\infty \frac{|F(\sqrt{\lambda})|^2 \sqrt{\lambda}}{W(\sqrt{\lambda}) W(-\sqrt{\lambda})} d\lambda + \sum_{n=1}^\infty c_n F^2(\sqrt{\lambda} n),
\]

where

\[
F(\sqrt{\lambda}) = \int_0^\infty f(x) \phi(x, \sqrt{\lambda}) \rho(x) dx, \quad c_n = -\frac{2\sqrt{\lambda}}{\int_0^\infty f(x) \phi(x, \sqrt{\lambda}) dx}.
\]

and \( w(\sqrt{\lambda}) = f'(0, \sqrt{\lambda}) \).

Proof:

The resolvent \( R_\lambda \) of (1)-(2) can be written in the form

\[
R_\lambda (t) = \frac{-1}{F'(0, \sqrt{\lambda})} \begin{cases} \int_0^\infty \phi(t, \sqrt{\lambda}) F(\sqrt{\lambda}) \phi(x, \sqrt{\lambda}) \rho(x) dx, & t < x \\ \int_0^\infty \phi(t, \sqrt{\lambda}) F(-\sqrt{\lambda}) \phi(x, -\sqrt{\lambda}) \rho(x) dx, & t > x \end{cases}
\]

(12)

Thus we have

\[
R_{\lambda,+0} - R_{\lambda,-0} = w(\sqrt{\lambda}) w(-\sqrt{\lambda}) u(t, \sqrt{\lambda})
\]

and

\[
w(\sqrt{\lambda}) = f'(0, \sqrt{\lambda}).
\]

Upon implementing the method of Titchmarsh [10], hence we obtain

\[
f(x) = \frac{1}{2\pi i} \int_0^\infty d\lambda \left\{ R_{\lambda,+0} (x, t) - R_{\lambda,-0} (x, t) \right\} \rho(t) f(t) dt + \sum_{n=1}^\infty \text{Res} \left( \frac{2\sqrt{\lambda}}{\pi} \frac{R(x, t, \lambda) \rho(t) f(t) dt}{w(\sqrt{\lambda}) w(-\sqrt{\lambda})} \right) \]

\[
= \frac{1}{\pi} \int_0^\infty \frac{u(x, \sqrt{\lambda}) F(-\sqrt{\lambda}) \sqrt{\lambda}}{w(\sqrt{\lambda}) w(-\sqrt{\lambda})} d\lambda + \sum_{n=1}^\infty c_n F(\sqrt{\lambda} n) \phi(x, \lambda_n) F(\lambda_n)
\]

Multiplying both sides by \( f(x) \rho(x) \) and then integrating from 0 to \( \infty \) with respect to \( x \), we obtain

\[
\int_0^\infty f^2(x) \rho(x) dx = \frac{1}{\pi} \int_0^\infty \frac{F(\sqrt{\lambda}) F(-\sqrt{\lambda}) \sqrt{\lambda}}{w(\sqrt{\lambda}) w(-\sqrt{\lambda})} d\lambda + \sum_{n=1}^\infty c_n F(\sqrt{\lambda} n) \phi(x, \lambda_n) F(\lambda_n)
\]

\[
= \frac{1}{\pi} \int_0^\infty \frac{F(\sqrt{\lambda}) \sqrt{\lambda}}{w(\sqrt{\lambda}) w(-\sqrt{\lambda})} d\lambda + \sum_{n=1}^\infty c_n F^2(\sqrt{\lambda} n),
\]

which completes the proof of the lemma.

The following lemma could be proved using the previous one.

Lemma 4

The following Parseval's equation holds

\[
\frac{1}{\pi} \int_0^\infty u(x, \sqrt{\lambda}) \sqrt{\lambda} u(t, \sqrt{\lambda}) d\lambda + \sum_{n=1}^\infty c_n u(x, \sqrt{\lambda_n}) u(t, \sqrt{\lambda_n})
\]

\[
= \int_{-\infty}^\infty u(x, \sqrt{\lambda}) u(t, \sqrt{\lambda}) d\sigma(\lambda)
\]

where

\[
\sigma(\lambda) = \begin{cases} \frac{1}{\pi} \int_0^\lambda \frac{d\lambda}{w(\sqrt{\lambda}) w(-\sqrt{\lambda})}, & \lambda \geq 0 \\ -\sum_{n=0}^\infty c_n \frac{4\lambda}{w(\sqrt{\lambda_n})^2}, & \lambda < 0 \end{cases}
\]

and

\[
c_n = \frac{4\lambda_n}{w(\sqrt{\lambda_n})^2}
\]

Corollary 1:

As \( q(x) = 0 \), and using formula (12), we get

\[
\sigma_0(\lambda) =
\]

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\[ \rho\left(\sqrt{a_i^2 - \sqrt{a_i a_1}}\right) + \cos^2\left(\sqrt{a_i a_1}\right) = 1, \lambda \geq 0 \]
\[ 0, \quad \lambda < 0. \]

**Lemma 5**

(i) In formula (7), the kernel \( A(x,t) \) satisfies the fundamental equation

\[ F(x,t) + A(x,t) + \int_0^x \text{d} F(x,s) F(s,t) \text{d} s = 0, 0 \leq t \leq x < a_1, \]

\[ \text{(13)} \]

where

\[ F(x,t) = \frac{\partial^2}{\partial x \partial t} \int_{-\infty}^\infty \frac{\sin(\alpha t \sqrt{\lambda}) \sin(\alpha t \sqrt{\lambda})}{\lambda} d\tau(\lambda), \]

\[ \tau(\lambda) = \begin{cases} \sigma(\lambda) - \sigma_0(\lambda), & \lambda \geq 0 \\ \sigma(\lambda), & \lambda < 0 \end{cases} \]

and

\[ \sigma_0(\lambda) = \begin{cases} \frac{1}{\pi} \int_0^\infty [\alpha^2 \sin^2(\alpha t \sqrt{\lambda}) + \cos^2(\alpha t \sqrt{\lambda})]^{-1} d\lambda, & \lambda \geq 0 \\ 0, & \lambda < 0. \end{cases} \]

(ii) The integral equation (13) has one and only one solution \( A(x,t) \) defined on \( 0 \leq t \leq x < a_1 \).

**Lemma 6**

(i) The kernel \( B(x,t) \) of formula (8) satisfies Gelfand-Levitan equation.

\[ H(x,t) + B(x,t) + \int_a^x B(x,s) H(s,t) \text{d}s = 0, a_1 \leq t \leq x < \infty, \]

\[ \text{(14)} \]

where

\[ H(x,t) = \frac{\partial^2}{\partial x \partial t} \int_{-\infty}^\infty \frac{\sin(\sqrt{\lambda} x a_1) \sin(\sqrt{\lambda} t a_1)}{\lambda} d\tau_\lambda(\lambda), \]

and

\[ \tau_\lambda(\lambda) = \begin{cases} \sigma_\lambda(\lambda) - \frac{2\sqrt{\lambda}}{\pi}, & \lambda \geq 0 \\ \sigma_\lambda(\lambda), & \lambda < 0. \end{cases} \]

Equation (14) has a unique solution \( B(x,t) \) as \( a_1 < t < x < \infty \)

**Proof**

Since \( \phi(x,\lambda) \) and \( \psi(x,\lambda) \) are solutions of equation (1) together with the initial conditions (2), hence we take \( \phi_\alpha(x,\lambda) \) and \( \psi_\alpha(x,\lambda) \) as the solutions at \( x = a_1 \). Also, denote by \( m_\alpha \) the Weyl’s function [5,7,8] of (1)-(2) and \( m_\alpha \) the Weyl’s function of (1) together with the initial condition \( y'(a_1) = 0 \). Thus, we have

\[ F(x,\lambda) = \phi(x,\lambda) + \psi(x,\lambda) m_\lambda(\lambda) \]

\[ f_\alpha(x,\lambda) = \phi_\alpha(x,\lambda) + \psi_\alpha(x,\lambda) m_\alpha(\lambda) \]

Since \( f(x,\lambda) \) and \( f_\alpha(x,\lambda) \) are independent solutions as \( x > a_1 \), hence we have \( f_\alpha(x,\lambda) = f(x,\lambda) \gamma(\lambda) \). Thus

\[ m_\alpha(\lambda) = \left[ \phi'(a_1,\lambda) + \psi'(a_1,\lambda)m_\lambda(\lambda) \right] \gamma(\lambda) \]

and

\[ 1 = \left[ \phi(a_1,\lambda) + \psi(a_1,\lambda)m_\lambda(\lambda) \right] \gamma(\lambda). \]

Hence

\[ m_\alpha(\lambda) = \left[ \phi'(a_1,\lambda) + \psi'(a_1,\lambda)m_\lambda(\lambda) \right]^{-1}. \]

This function is to be used to find the spectral function of equation (1) through the relation

\[ \sigma_\lambda(\lambda) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{Im}(m_\lambda(s + i\epsilon)) \text{d}s \]

Thus equation (14) can be obtained as in [5, 6, 7]. One can also prove the uniqueness of \( B(x,t) \) using [8].

**Theorem 1: (Uniqueness Theorem)**

If the condition (3) is satisfied and \( \rho(x) \in W_n \), then by using the spectral function \( \sigma(\lambda) \) of (1)-(2), the potential function \( q(x) \) and \( \rho(x) \) can be defined uniquely.
Proof

It is evident that if $a_1 \neq a_2$ and $\alpha_1 \neq \alpha_2$ then the function

$$\sigma_{0}(\lambda, a_1, \alpha_1) \sigma_{0}^{-1}(\lambda, a_2, \alpha_2)$$

has no limit as $\lambda \to \infty$.

Therefore the asymptotic behaviour of $\sigma_{0}(\lambda, a_1, \alpha_1)$ as $\lambda \to \infty$ determines $a$ and $\alpha$ uniquely. Hence the function $p(x)$ can be reconstructed uniquely. Here it should be mentioned that this case is true for $p(x) \leq w_n$. From lemma (3) we have already deduced the fundamental equation (13), $0 \leq x < a_1$, by using the spectral function $\sigma_{0}(\lambda)$. Moreover, and in view of lemma (3), equation (13) has the unique solution $A(x, t)$ as $0 \leq x \leq a_1$ on the form

$$q(x) = 2 \frac{dA(x, x)}{dx}$$

Thus, the function $q(x)$ is defined uniquely as $0 \leq x \leq a_1$. From lemma (4) we have

$$q(x) = 2 \frac{dB(x, x)}{dx} \text{ as } a_1 < x < \infty.$$

Hence, we conclude that the equation (1) can be reconstructed on the interval $(0, \infty)$ and the theorem is now completely proved.

REFERENCES


