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SOME RESULTS CONCERNING COBB – VOXMAN'S COJNECTURE

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بعض النتائج حول حدسية كوب-فوكسمان

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في هذا البحث سوف نعطي بعض النتائج الجزئية للحدسية التالية (٣) : إفرض أن X فضاء مترابط ويملك نقطة تبعثر p ، ولتكن $f: X \longrightarrow X$ دالة متصلة غير ثابتة، فإن $f(p) = p$.

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ABSTRACT

In this paper we shall give some partial results of the following conjecture raised in [3]: Suppose that X is a connected space with a dispersion point p and let f: X → X be a nonconstant continuous function. Then f(p) = p.

1. INTRODUCTION

In 1921 Knaster and Kuratowski [10] first introduced the notion of a space with a dispersion point and the example they described in this paper the "*Cantor teepee*" remains as perhaps the most famous (and of the most ingenious) examples of such a space.

The relative topology (subspace topology) on the set A inherited by τ will be denoted τ_A. The cardinality of the set A will be denoted by $|A|$. Throughout this paper all spaces having more than two points.

Definition 1.1

A point $\in$ X of a connected space X is called a *dispersion point* if $X \setminus \{p\}$ is totally disconnected.

As might be expected, spaces having dispersion points are in general both rare and unusual. Let us give some general properties of spaces having dispersion points. The most famous property of spaces having dispersion points is that the dispersion point in a connected space is unique whenever it exists [2]. Since connectedness and totally disconnectedness are topological properties, then having a dispersion point p, then X is a T_0-space [2]. Since connectedness is not hereditary, therefore having a dispersion point is not hereditary.

In 1946 Gustin [6] gave the first example of a countable connected Hausdorff space with a dispersion point,in 1966 Roy [13] gave an example of a countable connected Urysohn space having a dispersion point; subsequently, a number of such spaces with varying properties have appeared in the literature, [4], [5], [8], [11], [12], [14], and [15].

A. Al-Bsoul in [1] gave an exact characterization of finite spaces having dispersion points, in fact we have,

Theorem 1. 2 [1]. *Let X be a finite connected topological space with a dispersion point p and let $\setminus X \setminus = k$. Then X is homeomorphic to either one of the following spaces.*

(I) $N_k = \{0,1,2,\ldots,k\text{-}1\}$ $\tau = (\beta_k)$ where β_k is the base defined by

$\beta_k = \{N_k, \{j\} \setminus j = 1,2,\ldots,k\text{-}1\}$;

(II) $N_k = \{0,1,2,\ldots,k\text{-}1\}$ $\tau = \tau(\beta^*_k)$ is the base defined by

$\beta^*_k = \{\{0\}, \{n,0\} \setminus n = 1,2,\ldots,k\text{-}1\}$;

where 0 is the dispersion point of both spaces.

Question 1. 3 [1]. *Is it possible to characterize countable infinite spaces having dispersion points?*

Let us close this section by the following theorems which are used in Section 4.

Theorem 1. 4 [2]. *Let X be a connected space with a dispersion point p. Let U be any open set containing p. Then $\{x\}$ is closed for each $x \in X \setminus U$.*

Theorem 1. 5 [2]. *Let X be any connected space with a dispersion point p. If $|\tau_p| < \aleph_0$, then $|\tau_p| = 2n$ where $n = |X \setminus \cap \tau_p|$.*

2. FIXED POINTS AND DISPERSION POINTS

Left $f : X \to X$ be a function. A point $x_0 \in X$ is called a fixed point of f if $f(x_0) = x_0$. A topological space X is said to have the fixed point property if each continuous function $f : X \to X$ has a fixed point.

Since a space having the fixed point property must be a T_0 – space, and a space having dispersion point must be a T_0 – space, so, it is meaningful to ask the following question.

Question 2. 1 [9]. *If X isconnected space having a dispersion point p, does X have the fixed point property?*

The converse of the above question is, of course, false. These spaces with dispersion points are particularly interesting in this regard; in fact, in 1980 J. Cobb and W. Voxman [3], have shown that some particular spaces with dispersion points have the fixed point property. Evenmore, they have shown that some particular spaces with dispersion points have the fixed point property. Evenmore, they have shown that in each of these spaces, the dispersion point remains fixed under any nonconstant continuous function from the space into itself. They were optimistic in conjecturing the following.

Conjecture 2. 2 [3]. *Suppose that X is a connected space with a dispersion point p and let $f : X \to X$ be a nonconstantcontinuous function. Then $f(p) = p$.*

In the case of Cantor teepee space Conjecture 2. 2 is valid [3]. The above conjecture is not valid in this generality, in fact, H. Katsuura [9], and A. Gutek [7] proved that Conjecture 2.2 is not true in general. So, we have the following question.

Question 2. 3 [3]. *What conditions could be imposed on the* *space X such that Conjecture 2.2 would hold?*

In fact Conjecture 2.2 is valid in case of finite spaces with dispersion points and having cardinally $k \geq 3$ [1].

Theorem 2. 4 [1]. *Let X be a finite connected space with a dispersion point p, and let $f: X \to X$ be a nonconstant continuous function. Then f(p) = p.*

3. MORE CONJECTURES REGARDING SPACE WITH DISPERSION POINTS.

Several attempts were done to disprove Conjecture 2.2 by using the well known examples of spaces with dispersion points. Unfortunately these attempts fail to succeed. Indeed several authors (see for example [4], [7], [14]), proved that those examples served the affirmativity of the previous conjecture.

In [9] there is a counterexample shows that Question 2.1 and Conjecture 2.2 are false in general. Now we shall give other conjectures as modifications of Conjecture 2.2

Question 3. 1 [3]. *Is Conjecture 2.2 valid in general, and if not, what "conditions" could be imposed on X such that Conjecture 2.2 would hold?*

Now, it is reasonable to make the following conjecture which is the converse of Conjecture 2.2

Conjecture 3. 2 [3]. *Let X be a connected topological space that has a unique point p such that f(p) = p for every nonconstant continuous function $f : X \to X$, then p is a dispersion point of X.*

In fact Cobb and Voxman [3], gave an example showing that the above conjecture is not true in general. In [1] Al-Bsoul proved that this conjecture is true for finite spaces.

Question 3. 3 [3]. *Do there exist connected T_2 (countable, metric, etc.) spaces without dispersion points but having a unique oint that is left by all any nonconstant continuous functions mapping the space into itself?*

Question 3.3 remains open for spaces more like those having dispersion points, for example, countable T_2 – spaces, noncompact spaces,planar spaces.

4. Contributions Concerning Conjecture 2.2. As we said the well known examples of spaces with dispersion points fail to disprove Conjecturte 2.2, so we tried to prove Conjecture 2.2 by addint more assumptions, hence we managed to prove Conjecture 2.2 in some special cases, as in the following results:

Theorem 4.1. *Let X be a connected space with a dispersion point p, andleft $f : X \to X$ be a nonconstant continuous function. If the only open set containing p is X, then f(p) = p.*

Proof: Suppose that f(p) = q ≠ p. Since f(X) is connected, hence $p \in f(X)$, so there exists $r \in X$ such that f(r) = p. Choose an open set U such that $q \in U$ and $p \notin U$ (such open set U exists because if not then the only open set containing q is X and hence $X \setminus \{p\}$ is not totally disconnected). Since U sis an open set and f is continuous then f^{-1} (U) is open in X. Since $q \in U$, so $p \in f^{-1}(U) = X$. But since $p \notin U$, then $r \notin f^{-1}$ (U).

This contradiction completes the proof.■

Theorem 4. 2. *Let X be a connected space with a dispersion point p, and let $f : X \to X$ be a nonconstant continuous function. If $x \setminus \{p\}$ has the discrete topology, then f(p) = p.*

Proof : Since p is a dispersion point of X so we have two cases:

(I) If $\{p\}$ is open. In this case there is no $x \notin X \setminus \{p\}$ such that $\{x\}$ is open; because if $\{x\}$ is open for some $x \notin X \setminus \{p\}$ and since x does not belong to $\{p\}$, hence by Theorem 1.4, $\{x\}$ is closed, so $\{x\}$ is open which is absurd.

Suppose f(p) = q ≠ p. Then since f(X) is connected and f is a nonconstant function, there exists a point $r \notin X$ such that f(r) = p. Now, p f^{-1} ($\{p\}$), so f^{-1} ($\{p\}$) is not open, therefore, f is not consinuous.

(II) If $\{p\}$ is not open. In this case $\{x\}$ is open for all $x \notin X \setminus \{p\}$; because if $\{x_0\}$ is not open, then $\{x_0,p\}$ is open and there is no $y \notin X \setminus \{x_0,p\}$ such that $\{y\}$ is not open (if there is such y then $\{p\} = \{x_0, p\} \cap \{y, p\}$ is open), so $X \mid \{x_0, p\}$ is open but then X is disconnected, therefore, the only open set containing p is X. So by Theorem 4.1, f(p) = p.

Therefore, by (I) and (II), f(p) = p for every nonconstant continuous function f from X into X. ■

An obvious conclusion of the above result is that if X is a connected finite space with a dispersionpoint p then every nonconstant continuous function $f : X \to X$ fix p.

Corollary 4.3 *Let X be a connected space with a dispersion point p, and let $f : X \to X$ be a nonconstant continuous function. If X is a countable, P – space. Then f(p) = p.*

Proof. Consider the subspace $Y = X \setminus \{p\}$ of X. Since Y is totally disconnected, then it is a T1 – subspace, so for eacy y $\in$ Y, then $\{y\}$ is closed in Y, hence $Y \setminus \{y\}$ is open in Y. Now for each $y \in Y$ then $\{y\} = \cap \{Y \setminus \{x\} \mid x \in Y \setminus \{y\}\}$ is a countable intersection of open sets in Y, since X is a P – space, then $\{y\}$ is open in Y. Therefore Y has the discrete topology. By Theorem 4.2, f(p) = p for every nonconstant continuous function f from X into X. ■

Before proving the next result we need the following lemma.

Lemma 4.4. *Let X be a connected space with a dispersion point p, and let $f : X \to X$ be a nonconstant continuous function. If there are only finite number of open sets that contain p (i.e. $|\tau_p| < \aleph_0$) and f(p) = r, then $r \in U_P$, Where U_p is the "smallest" open set contains p.*

Proof : Assume that $|\tau_p| = t$, then by Theorem 1.5, t = 2n where $n = | X \setminus \cap \tau_p|$. Hence we may assume that $U_p = X \setminus \{q_1, q_1, \dots 1_n\}$. Without loss of generality, suppose that $f(p) = q_1$. First we shall show that f(X) must be infinite (in this case; $f(p) \notin U_p$). If f(X) is finite, and since $f(X) \setminus \{p\}$ there exists an open set V_x such that $q_1 \in Vx$ and $x \notin V_x$ and let V_p be any open set containing q_1 and does notcontain p (such V_p exists; if not,then every open set containing q_1 must contain p, but then $[f(X) \cap U_p] \subset V_p$ and hence $q_1 \notin U_p$), so $\{q_1\} = \cap \{V_x \mid x \neq q, x\ V \in f(X)\}$. Therefore, f(X) is disconnected; because $q_1 \notin U_p$ and hence $\{q_1\}$ is closed, so $\{q_1\}$ is clopen. Then f(X) must be infinite. Finally, let $\{s_1, \dots, s_n\} \subset f(X) \setminus \{p, q_1\}$ (such subject exists; because f(X) is infinite). Since $f(X) \setminus \{p\}$ is a T_1 – subspace, and by a similar way as above we select an open set G_{q_1} in f(X) such that $G_{q_1} \cap \{p, s_1, \dots, s_n\} = \phi$. Now, since $p \in f^1(G_{q_1})$ and $f^1(G_{q_1}) \cap f^1(p) = \phi$, so $X \setminus \{q_1, \dots, q_n\} = Up \subset f^1(G_{q_1})$. Therefore $\{p, s_1, \dots, s_n\} \subset f\{q_1, \dots, q_n\}$. Contradiction; because $n \geq |f(\{q_1, \dots, q_n\})| \geq |\{p, s_1, \dots, s_n\}| = n + 1$, and thiscontradiction completes the proof.■

Theorem 4. 5. *Let X be a connected space with a dispersion point p, and let $f : X \to X$ be a nonconstant continuous function. If τp is infinite and $f \circ f$ is nonconstant, then f(p) = p. Moreover $|f(X)| = 2$ provided $f(p) \neq p$ and $f \circ f$ is constant.*

Proof. If X is finite then f(p) = p. Hence we may assume that X is infinite. By Theorem 1.5, $|\tau_p| = t < \infty$, then $t = 2^n$ where $n = |X \setminus \cap \tau_p|$. We shall prove the theorem by induction on n.

(I) If n = 0, then t = 1. In this case the only open set that contains p is X, by heorem 4.1, f(p) = p.

(II) If n = 1, then t = 2. In this case there are only two open sets that contain p, X and $X \setminus \{q\}$ for some $q \in X$. By Lemma 4.4, $f(p) = \in X \setminus \{q\}$. If $f(p) = r \neq p$, then $|f(X)| = 2$; if $|f(p)| > 2$, then there exists $s \in f(X) \setminus \{p,r\}$ and hence there are r_0, $s_0 \in X$ such that $f(r_0) = p$ and $f(s_0) = s$. Since X is infinite then there exists an open set V_p and that $p \in V_p$ and since $X \setminus \{p\}$ is T_1 then there exists an open set V_s such that $s \in V_s$ and $r \in V_s$, take $U_r = V_P \cap f(X)$, so p, $s \notin U_r$, and $r \in U_r$. Hence, $p \in f^1(U_r)$, which is open in X, and since $s_0 \notin f^1(U_r)$, which is open in X, and since s0 $\notin f^1(U_r)$, so $f^1(U_r) = X \setminus \{q\}$. But since $\{p, s\} \cap U_r = \phi$, then $\{p, s\} \subset f(\{q\})$. Contradiction, therefore $f(X) = \{p, r\}$, so $\tau_{f(X)} = (\phi, \{r\}, f(X)\}$, hence $f^1(\{r\}) = X \setminus \{q\}$ and $f^1(\{P\}) = \{q\}$. Hence, f must be defined as follows:

$$f(x) = \begin{Bmatrix} r \text{ if } x \in X \setminus \{q\} \\ p \text{ if } \quad x = q \end{Bmatrix},$$

therefore

$$(f \cdot f)(x) = \begin{Bmatrix} f(r) \text{ if} \in X \setminus \{q\} \\ f(p) \text{ if } \quad x = q \end{Bmatrix}$$

$$= r \text{ for all } x \in X$$

but this mean that $f \circ f$ is constant, contradiction. Hence f(p) = p.

(III) Assume that $t = 2^n$. Hence, the "smallest" open set that contains p is $U_P = X \setminus \{q_1, \dots, q_n\}$. By Lemma 4. 4, f(p) U_P, if $f(p) = r \neq p$, then $|f(X)| = 2$ (first we show that $|f(X)| \leq n + 1$ and then $|f(X)| = 2$), suppose $|f(X)| > n + 1$ then there is a subset $\{s_1, \dots, s_n)$ $f(X) \setminus \{p, r\}$, $s_i \neq s_j$ for $i \neq j$. Select an open set U_r if f(X) sueh that $r \in U_r$ and $\{s_1, \dots, s_n, p\} \cap U_r = \phi$ (such U_r exists since $X \mid \{p\}$ is totally disconnected and use the same way as in (II). Now $f^1(U_r)$ contains U_P, hence $f(\{q_1, \dots, 1_n\})$ contains $\{p, s_1, \dots, s_n\})$ which is a contradiction, so $|f(X)| \leq n + 1$. If $|f(X)| > 2$, take $s \in f(X) \setminus \{p, r\}$, hence $\{s\}$ is open in f(X) (if not then every open set contains s must contain p so $X \setminus \{p\}$ is not totally disconnected hence $f^1(\{s\})$ $\{q_1, \dots, 1_n\}$ and f-1 ($\{s\}$) is open, but since each $q_i \notin U_P$, i = 1,…,n, $\subset$

then by Theorem 1.4, $\{q_i\}$ is closed, therefore f^1 ($\{s\}$) is clopen. Since $p \notin f^1$ ($\{s\}$) then X is disconnected. Contradiction, so there is no such s, hence $|f(X)| = 2$.

As in (II) fof is constant. Contradiction, hence $f(p) = p$ provided that fof is nonconstnat (in this case $n > 1$).

For any $k > 1$ (i.3. $t = 2^k$) we can apply case (III). And this completes the proof.

To complete this subject we give the following results.

Theorem 4. 6. *Let X and Y be two connected spaces with the dispersion points p and q respectively. Assume that for each mnonconstant continuous function $f : X \to X$ and $g : X \to Y$, then $f(p) = p$ and $g(p) = q$. If $H : X \to Y \times$ is nonconstant in each variable (i.e. H(c,y) is nonconstant and H(x,d) is nonconstant where c, d are constants). Then H(p,q) - (p,q) provided that H is continuous.*

Proof : Assume $H(p,q) = (x_0,y_0)$. Take the projection on the first variable π_1, and then the restriction on $X \times \{q\}$ (i.e. $(\pi_1 \text{ o } H)|_{X\times\{q\}}$. So, $(\pi_1 \text{ o } H)_{X\times\{q\}}: X \times \{q\} \to X$ is a nonconstant continuous function, hence $(\pi_1 \text{ o } H)|_{X\times\{q\}} (p,q) = p$, but $(\pi_1 \text{ o } H)|_{X\times\{q\}} (p,q) = \pi_1 (H(p,q)) = \pi_1 (x_0, y_0) = x_0$ hence $x_0 = p$. Similarly $y_0 = 1$. There $H(p,q) = (p,q)$. ■

As an easy corollary of the previous thorem is the following:

Corollary 4. 7. *Let X and Y be two connected spaces with the dispersion points p and q respectively. If $X \setminus \{p\}$ and $Y \setminus \{q\}$ have the discrete topology and $H : X \times Y \to X \times T$ is a continuous nonconstant in each variable function, then $H(p,q) = (p,1)$.*

Proof. By Theorem 4.2, every nonconstant continuous function $f : X \to X$ fix p (i.e. $f(p) = p$), and similarly $g(q) = q$ for every nonconstant continuous function $g : Y \to Y$. Hence by Theorem 4.6, $H(p, q) = (p,q)$. ■

If p, q are dispersion points and X and Y respectively, then it does not imply that (p,q) is a dispersion point of $X \times Y$. So we shall introduce the following result:

Theorem 4.8 *If p, q are dispersion points of the connected spaces X, Y respectively. Then (p,q) is not a dispersion point of $X \times Y$.*

Proof. Let $y \in Y \setminus \{q\}$, so $(p, y) \neq (p, q)$. Now $\{y\}$ and X are connected, so $X \times \{y\}$ is connected. Consider the component of (p, y) in $X \times Y \setminus \{(p, q)\}$, call it $C_{(p,y)}$. Since $C_{(p,y)}$ is the maximal connected set connected set containing (p, y), hence X ¥ {y} $C_{(p,y)}$. Since $|X| > 2$, so $| x \times \{y\} > 2$, hence $| C_{(p,y)}| > 2$, then $X \times Y \setminus \{(p, q)\}$ is not totally disconnected. Therefore, (p, q) is not a dispersion point of $X \times Y$. ■

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