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# CONDITIONALLY EXPONENTIAL CONVEX FUNCTIONS ON LOCALLY COMPACT GROUPS

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# الدوال الأسية المحدبة المشروطة على الزمرات محليـــة التضـــاغط

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يهدف هذا البحث إلى دراسة فئة جميع الدوال الأسية المحدبة المشروطة وحيث أن هذه الفئة هي مخروط محدب فاننا في هذا البحث نبحث نقطتين أساسيتين هما :

١ ـ كيف يمكن تكوين أساس مضغوط لهذا المخروط ؟

٢ ـ ما هي الأجزاء المتطرفة لهذا المخروط ؟

*Key Words:* Conditionally Exponential Convex Functions

ABSTRACT

The main objectives of this study are:

1) The construction of a compact base for the convex cone of all conditionally exponential convex functions.

2) The determination of the extreme parts of this cone.

## INTRODUCTION

Conditionally exponential convex functions have been introduced and studied in references [2,8]. Berg [2] named it "negative-definite". The set of these functions, denoted $E_0$ (G), is a convex cone, hence is amenable to an analysis by Choquet theory. For the real line, such a study was done by Johansen[7]. We follow an idea of Johansen to construct a compact base for $E_0$ (G). This leaves us with the task of finding the extreme points of the base. To do so, we first study a weight, called the Levy weight, in its abstract form.

Now, let C* (G) be the enveloping C* -algebra of $L^1$ (G) equipped with # involution defined by: $f^{\#}(x) = \Delta(x^{-1}) f^{+}(x)$, where $\Delta$ is a modular function and $f^{+}(x) = f(x^{-1})$. The dual Banach space of C* (G) is B (G) [4]. The set of positive linear functionals in B [G] is P [G] and it is identified with the set of exponentially convex functions on G; i.e., the set of functions satisfying

$$\sum_{i,j=1}^{n} \psi(g_i g_j) c_i c_j \geq 0,$$

where $g_1, ..., g_n \in G$ and $c_1, ..., c_n \in R$. The set of elements form $P_1$ (G) with norm equals 1 is $P_1$ (G); this is a convex set whose set of extreme points is denoted by ext $P_1$ (G)[7]. Also, we write $E_0$ (G) for the set pf all conditionally exponential convex functions defined on G and vanishing at the group identity; i.e., functions satisfying

$$\sum_{i,j=1}^{n} [\psi(g_i) + \psi(g_j) - \psi(g_i g_j) c_i c_j \geq 0,$$

where $g_1, ..., g_n \in G$ and $c_1, ..., c_n \in R$[2,8].

Now, if $a \in C^*(G)$ and $p \in P(G)$, define the translation of a by p to be the unique element $T_p$ a of C* (G) such that for all $b \in B(G)$, (bp, a) = (b, $T_p$ a). The translation operator $T_p$ is a completely positive linear map on C* (G) and is norm decreasing if $p \in P_1(G)$. If p, q $\in$ P (G) and $\lambda \geq o$, then $T_{pq} = T_p T_q$ and $T_{p + \lambda q} = T_p + \lambda T_q$, so that $T_{(.)}$ is a homomorphism from P (G) into the set of completely positive linear maps on C* (G)[3].

Given the notion of translation, we can define differentiation as a limit of difference quotients. In this way, we obtain a tangent space at each point of $P_1$ (G).

A semitangent vector to $P_1$ (G) at the identity is any continuous real valued function $\psi$ on G satisfying

$$\psi = \lim_{k} \lambda_k (1-p_k),$$

where $\{\lambda_k\}_{k>1}$ is a diverging increasing sequence of nonnegative numbers and $\{p_k\}_{k>1}$ is a unformly convergent sequence belongs to $P_1$ (G). It can be seen that the collection of such semitangent vectors is identified with the set $E_0$ (G), then $\psi$ is called a tangent vector to $P_1$ (G) at the identity [1].

## LEVY WEIGHTS

Let $\psi \in E_0$ (G) and let $\partial : C^*(G) \to$ be $C^*(G)$ be a linear functional such that $\partial(a^\# a) \geq (\partial a^\#) a + a^\# (\delta a)$ if $a^\# a$ belong to D ($\partial$), the domain of $\partial$. Define a linear functional on C* (G), denoted by $\psi$, by

$$\psi(a) = (1, \partial_\psi a),\ a \in D(\partial).$$

Clearly, $\psi$ is densely defined and $\psi \mid (\ker 1)^+ \geq o$; i.e., $\psi$ is a weight for the elements of $E_0$ (G) and it is called a Levy weight.

Let W* (G) be the double dual of C* (G). We may consider G and C * (G) to be contained in W* (G); when it is necessary to emphasize that a measure $\mu$, say, belongs to W* (G)., we will write $\omega(\mu)$, where $\omega$ is the universal representation [6].

***Lemma 2.1***

Let $M^o{}_c$ (G) be the set of compactly supported Borel measures on G of total mass zero. For $\mu \in M^o{}_c$ (G) and $a \in C^*$ (G) we have

$$\psi(\omega(\mu^\#)\, a\omega(\mu)) = (-\psi^\mu, a), \text{ where } \psi^\mu = \mu * \psi * \mu^+.$$

***Proof:***

First we suppose that a takes the form $\omega$ (f), f belongs to the set $C_c$ (G) of compactly supported continuous functions on G. then

$$\psi(\omega(\mu^\# * f * \mu)) = (1, \partial_\psi \omega(\mu^\# * f * \mu)) = -\int \psi(x)\, \mu^\# * f * \mu(x)\, dx = -\int \psi^\mu(x)\, f(x)\, dx.$$

Now, we prove the lemma in the general case. Let $a \in C^*$ (G) be the strong limit of the sequence $a_n = \omega(f_n)$, $f_n \in C_c$ (G). Then

$$\psi(\omega(\mu^\#)\, a\omega(\mu)) = \psi(\omega(\mu^\#)\, a_n\, \omega(\mu)) + \psi(\omega(\mu^\#)(a - a_n)\, \omega(\mu)).$$

Applying Cauchy-Schwarz inequality to the second term of the right side and then taking the limit we get

$$\psi(\omega(\mu^\#)\, a\omega(\mu)) = \lim_n \psi(\omega(\mu^\#)\, a_n\, \omega(\mu))$$
$$= \lim_n (-\psi^\mu, a_n) = (-\psi^\mu, a).$$

Now let $Z_1$ be the central support of the weak closure of ker 1 in W* (G) and let $(\ker 1)_1$ be the unit ball of ker 1. The proof of the following theorem, which is similar to that of proposition 1.11 of[3] is omitted.

***Theorem 2.2***

The necessary and sufficient condition for the function to $\psi \in E_0$ (G) to be lower semi-continuous is that there exist positive linear functional $\{f_k\}_{k>1}$ on ker 1 such that $\psi(a) = \sum_{k=1}^{\infty} (f_k, a)$ for $a \in (\ker 1)^+$.

Now we reformulate this theorem in a more concrete form. choose $a \in (\ker 1)^+$ to be of the form $\mu^\# * h^\# * h * \mu$ where $h \in C_c$ (G), and let $p_k \in P$ (G) be the extension of $f_k$ with the same norm. Finally, put $\mu = \delta_e - \delta_y$ where $\delta.$ denotes the point mass at. Then

$$\psi(a) = (-\psi^\mu, h^\# * h) = \sum_{k=1}^{\infty} (\mu * p_k * \mu^+ * h^\# * h),$$

and we get that $-\psi^\mu$ is the monotonic limit of the exponentially convex functions $\sum_{k=1}^{\infty} \mu * p_k * \mu^+$.

## A COMPACT BASE FOR $E_0$ (G)

In this section we construct a base for $E_0$ (G). First we begin with the following definition.

***Definition 3.1 [3]***

If $\psi, \varphi \in E_0$ (G), we say that $\psi$ dominates $\varphi$ if $\psi - \varphi \in E_0$ (G). If $\psi$ and $\varphi$dominate each other, they they are said to be equivalent. They are weakly equivalent if one is equivalent to a positive multiplier of the other.

Now we construct a base for $E_0$ (G) which is compact in some suitable topology. This can be done by the selection of an element on each weak equivalence class in $E_0$ (G). Let G be compactly generated, N be a compact symmetric neighbourhood of the identity in G which generates the group and $K = \{\psi \in E_0(G) \mid \int_{N^3} \psi(x)\, dx = 1\}$. If $\psi \in E_0$ (G) satisfies $\int_{N^3} \psi(x)\, dx = 0$, then $\psi \mid N^3 = 0$,. Since N is a generating set, we have $\psi = 0$ everywhere. So, for $\psi \in E_0$ (G) we can devide it by $\int_{N^3} \psi(x)\, dx$ to obtain an element of K, which implies that K is a base for $E_0$ (G).

***Lemma 3.2***

Suppose that $X_{A_n}$ is the indicator function of the set $A_n$, where $A_1 = N$, $A_n = N^n - N^{n-1}$ for $n \geq 2$ and $c = \alpha^{-1} \sup \{1 + \Delta(y) \mid y \in N\}$ where $\alpha$ is the left Haar measure of N. Then for $\psi \in K$ we have $\psi(x) \leq f(x)$ where

$$f = c. \sum_{n=1}^{\infty} n^2 X_{A_n}.$$

***Proof:***

Let $y \in G$ and $\mu = \delta_e + \delta_y$. then for each K we have

$\psi^\mu(x) - \psi^\mu(e) = \psi(x) + \psi(yxy) + \psi(yx) + \psi(xy) - 2\psi(y) \in E_0(G)$ and by integration we get

$$2\psi(y) \le \alpha^{-1} \int_U (\psi(x) + \psi(yxy) + \psi(yx) + \psi(xy)\, dx.$$

If $y \in N$, then

$$2\psi(y) \le \alpha^{-1} \{ \int_N \psi(x)\, dx + \int_{yNy} \psi(x)\, \Delta(y)\, dx + \int_{yNy} \psi(x)\, dx + \int_{Ny} \psi(x)\, \Delta(y)\, dx \}$$

$$\le 2\alpha^{-1}(1+\Delta(y)) \int_{N3} \psi(x)\, dx = 2\alpha^{-1}(1+\Delta(y)),$$

by symmetry of N and sup, $\{\psi(y) \mid y \in N, \psi \in K\} \le C$.

Now, if $x \in G$, then $x \in A_n$ for fixed n and hence it belongs to $N^n$. So, there exist $Y_1, ..., Y_n \in N$ such that $x = Y_1 Y_2 ... Y_n$. Then

$$\psi^{1/2}(x) = \psi^{1/2}(Y_1 ..., Y_n) \le \sum_{k=1}^{n} \psi^{1/2}(yk) \le nc,$$

and we get

$$\psi(x) \le f(x) = c \sum_{n=1}^{\infty} n^2 X_{A_n}(x).$$

Let S be a separable compact convex set. A subset F of S is called a face if each line segment in S whose interior intersects F is contained in F. The complementary set F' of F is the union of all faces of S disjoint from F. If F' is a face and F is a closed face, then F is called a closed split face. In the latter case, S is the direct convex sum of F and F' which means that every element $x \in S$ can be written uniquely in the form

$$x = \lambda y + (1 - \lambda)z, O \le \lambda \le 1, y \in F, z \in F'$$

Now we notice that K is compact only if G is discrete, but we can compactify K by adding a point at $\infty$. Let $\psi^\infty(x) = |N^3|^{-1}$. It is clear that $\int_{N3} \psi^\infty(x)\, dx = 1$, and $\psi^\infty$ belongs to $E_0(G)$. Let $\tilde{K}$ be the convex hull of K and $\psi^\infty$.

***Theorem 3.3***

Suppose that $L^\infty(G)_1$ is the unit ball of $L^\infty(G)$ equipped with the $\sigma(L^\infty, L^1)$ topology and suppose also that $p: \tilde{K} \to L^\infty(G)_1$ is given by $p(\psi) = \psi/f$, where f is defined as in Lemma 3.2 and K is compact in the topology induced by p. Then the set $\{\psi^\infty\}$ is a closed split face of $\tilde{K}$.

***Proof***

It is clear that $L^\infty(G)_1$ is compact in the $\sigma(L^\infty, L^1)$ topology, and we only need to prove that $p(\tilde{K})$ is closed. Suppose $\{\psi_n\}^\infty{}_{n=1} \in \tilde{K}$ and such that $p(\psi_n) \to \psi \in L^\infty(G)_1$ and suppose also that $\psi = \varphi f$. For $g \in L^1(G)$ and $h \in C_c{}^o(G)$, where $g = (h^\# * h) f$, we have

$$\int \psi(x)\, h^\# * h(x)\, dx = \int \varphi(x)\, g(x)\, dx = \lim_n \int p(\varphi_n)\, g(x)\, dx$$

$$= \lim_n \int \varphi_n(x)\, h^\# * h(x)\, dx \le o.$$

Let V be any measurable subset of G with finite measure. Then

$$\int_V \varphi(x)\, dx = \lim_n \int V p(\varphi_n)(x)\, dx = \lim_n \int V \varphi_n(x)/f(x)\, dx \ge o$$

and hence $\varphi(x) \ge o$ a.e. and the same is true for $\psi$. Hence $\psi$ is a.e. equals to a unique element of E (G) and it remains to prove only that for $\psi \in E_0(G)$ such that $\int_{N^3} \psi(x)\, dx = 1$, $\psi \in \tilde{K}$. In fact, since $\psi(e) \le |N^3|^{-1} = \psi^\infty$ we put $\lambda = \psi(e)/\psi^\infty$. If $\lambda = 0$ then $\psi \in E_0(G)$ and hence $\psi \in K \subset K$. If $\lambda = 1$. then $\int_{\tilde{N}^3} \psi(e)\, dx = 1$, from which we get $\psi(x) = \psi(e) = \psi^\infty$ and $\psi \in K \subset K$. Finally, we suppose that $0 < \lambda < 1$ and we put $\psi(x)$ in the form $\psi(x) = \lambda\psi^\infty + (1-\lambda)\varphi(x)$, where $\psi(x) = (\psi(x) - \psi(o))/(1-\lambda) \in E_0(G)$. Then

$$\int_{N^3} \varphi(x)\, dx = (1-\lambda)^{-1} \int_{N^3} (\psi(x) - \psi(e))\, dx = 1,$$

so that $\psi \in K \subset \tilde{K}$ The decomposition of $\psi$ into a convex sum of $\psi^\infty$ and an element of K is easily seen to be unique.

## EXTREME RAYS OF $E_0(G)$

Since $E_0(G)$ is a convex cone, one way to understand its structure is to characterize its extreme rays.

***Definition 4.1***

We say that $\psi \in E_0(G)$ generates an extreme ray in $E_0(G)$ if each of its dominated elements is either a tangent vector or weakly equivalent to $\psi$.

***Theorem 4.2***

Suppose that $\psi \in E_0(G)$ has a lower semi-continuous Levy Weight. The necessary and sufficient condition for $\psi$ to generate an extreme ray is that it takes the form

$\lambda(1 - p) + h$ with $\lambda > 0$, $p \in \text{ext}\, P_1(G) - \{1\}$ and $h \in \text{hom}(G, R)$.

***Proof:***

First we suppose that $\psi$ is weakly equivalent to $1 - p$ for $p \in P_1(G) - \{1\}$. If $\varphi \in E_0(G)$ is dominated by $(1 - p)_1$ then

$\varphi$ is bounded. Hence, there exists $P_1 \in \{1\}'$ $\lambda \geq 0$ and $h \in$ Hom (G, R) such that $\varphi = \lambda\ (1 - p_1) + h$. So $1 - p$ dominates $\lambda\ (1 - p_1)$ in $E_0$ (G). Now supposing that $\lambda \geq 1$ and since $(1 - p) - \lambda\ (1 - p_1) \in E_0$ (G) then we can find a constant $K \geq 0$ such that

$$K + \lambda (1 - p_1) - (1 - p) = K + \lambda - 1 - \lambda p_1 + p \in p_1 (G);$$
$$1 - \lambda \geq 0.$$

This implies that $K + p - \lambda p_1 \in p_1$ (G). Since $p \in \text{ext } p_1$ (G), $p_1$ is a convex combination of the orthogonal exponentially convex functions 1, p. But $p_1 \in \{1\}'$ then $p_1 = p$ and we have $\psi$ is extreme.

Conversely, let $\psi$ be lower semi-continuous. Using Theorem 2.2 in its reformulated form, we can see that $\psi$ is bounded if it generates an extreme ray in $E_0$ (G). Hence, $\psi$ is weakly equivalent to an element of $E_0$ (G) in the form $1 - p$, $p \in P_1$ (G). Noting that $\{1\}$ is a closed split face in $P_1$ (G), we have to prove that $p \in \{1\}'$ In fact, if $p = \lambda + (1 - \lambda)\ p_1$ is the representation of p in $\{1\}$ and $\{1\}'$ then $1 - p = (1 - \lambda)\ (1 - p_1)$ and we get $1 - p_1$ is weakly equivalent to $\psi$.

Now suppose that $p \notin \text{ext } P_1 (G) - \{1\}$. Then p can be written in the form $p = (p_1 + p_2)/2$ with $p_1 \neq p_2$ from $P_1$ (G). Hence $1 - p = (1-p_1)/2 + (1 - P_2)/2$ contradicting the extremity of $\psi$ and we have $p \in \text{ext } P_1 (G) - \{1\}$.

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