$\zeta_{p,q}$ *Ideal of Operators in Banach Spaces*

by

E. El-Shobaky

Department of Mathematics, Faculty of Science, Qatar University, Doha, Qatar.
Department of Pure Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt.

المثالي للمؤثرات في فراغات باناخ

انتصارات محمد حسن الشبكي
كلية العلوم ـ جامعة عين شمس

في هذا البحث نقدم لأول مرة المثالي $\zeta_{p,q}$ كالتالي

$$\zeta_{p,q} = \left\{T\epsilon L;\ \Sigma_n n^{q/p-1}\ e_n(T)^q < \infty,\ 0<p/q\leqslant\infty.\right\}$$

حيث $e_n(T)$ هي اعداد الانتروبي من درجة n للمؤثر T بين فراغات باناخ ثم ندرس الخواص المختلفة لهذا المثالي والعلاقة بينه وبين المثلي $S^{app}_{p,q}$ في حالة خاصة وهي فراغات هيلبرت ، حيث $S^{app}_{p,q}$ عرف كالآتي :

$$S^{app}_{p,q} = \left\{T\epsilon L;\ \Sigma n^{q/p-1}\ \alpha_n(T)^q < \infty,\ 0<p,q\leqslant\infty\right\}$$

$\alpha_n(T).$ هي الاعداد التقريبية من درجة n للمؤثر T

Introduction

For every operator T between Banach spaces a sequence of entropy numbers $e_n(T)$ with n = 1,2, . . . and the ideal ζ_p of operators T such that

$\sum_{n=1}^{\infty} e_n(T)^p<\infty$, $0<p<\infty$, have been defined and investigated in [1].

The ideal S_p^{app} of operators with $\sum_{n=1}^{\infty} \alpha_n(T)^p<\infty$, where $\alpha_n(T)$ is the n-th approximation number has been defined [7]. Furthermore, it has been found that [1]

$$\zeta_p(l_2,l_2) = S_p^{app}(l_2,l_2).$$

In this paper we introduce and investigate the ideal $\zeta_{p,q}$, $0<p, q\leqslant\infty$, of operators such that $\sum_{n=1}^{\infty} n^{q/p-1} e_n(T)^q<\infty$, which provides a generalization of ζ_p. Moreover, we give a relation between $\zeta_{p,q}$ and $S_{p,q}^{app}$ in Hilbert spaces, where $S_{p,q}^{app}$ has been defined in [4].

Preliminaries

In the following we mention the notion and some properties of the Lorentz sequence space $l_{p,q}$, [5], which will be used throughout this work.

Definition 1. For $0<p\leqslant\infty$, $0<q\leqslant\infty$ the Lorentz sequence space $l_{p,q}$ is defined as the collection of sequences $\lambda=\{\lambda_i\}_{1\leqslant i\leqslant\infty}\in c_0$, such that

$$\|\{\lambda_i\}\|_{l_{p,q}} := \begin{cases} \left(\sum_{i=1}^{\infty} i^{q/p-1} |\lambda_i|^{*q}\right)^{1/q} & \text{if } q<\infty \\ \sup_i i^{1/p} |\lambda_i|^* & \text{if } q=\infty \end{cases}$$

is finite, where $|\lambda_j|^*$ denotes the j-th term in non-increasing rearrangement of the sequence $\{|\lambda_i|\}$.

Lemma 1. [3]. For $0<p\leqslant\infty$, $0<q\leqslant\infty$, $l_{p,q}$ is a quasi-normed space with respect to $\|.\|_{l_{p,q}}$.

Lemma 2. (i) If $0<p\leqslant\infty$ and $0<q_1<q_2\leqslant\infty$, then

$$l_{p,q_1} \subset l_{p,q_2}$$

and

$$\|\lambda\|_{l_{p,q_2}} \leqslant c\|\lambda\|_{l_{p,q_1}} \quad \text{for each } \lambda\in l_{p,q_1}$$

(ii) If $0<p_1<p_2\leqslant\infty$ and $0<q_1, q_2\leqslant\infty$, then

$$l_{p_1,q_1} \subset l_{p_2,q_2}$$

and

$$\|\lambda\|_{l_{p_2,q_2}} \leqslant c\|\lambda\|_{l_{p_1,q_1}} \quad \text{for each } \lambda\in l_{p_1,q_1} .$$

Here c stands for a positive constant depending on parameters p_1, p_2, q_1 and q_2 and independent of λ.

This is a direct consequence of the result concerning Lorentz sequence spaces $l_{p,q}$ obtained by the interpolation theory of Banach spaces [3].

Lemma 3 [2]. Let $\{c_i^*\}$ and $\{{}^*c_i\}$ be the non-increasing and non-decreasing rearrangements of a finite sequence $\{c_i\}_{1\leqslant i\leqslant n}$ of positive numbers, respectively. Then for two sequences $\{a_i\}_{1\leqslant i\leqslant n}$ and $\{b_i\}_{1\leqslant i\leqslant n}$ of positive numbers we have

$$\sum_i a_i^* \, {}^*b_i \leqslant \sum_i a_i \, b_i \leqslant \sum_i a_i^* \, b_i^* .$$

We note that these inequalities hold in case of infinite sequences if the right hand side is convergent.

In the following E, F and G are real Banach spaces. The closed unit ball of E is denoted by U_E. Furthermore, L denotes the class of all operators between arbitrary Banach spaces and L(E,F) denotes the Banach space of all bounded operators from E into F. We denote by $L_n(E, F)$ the subspace of L(E, F) of operators T of rank (T) < n. The logarithm is to base 2.

Definition 2. For each operator $T \in L(E, F)$ and for n = 1, 2, . . ., the approximation numbers $\alpha_n(T)$ are defined by

$$\alpha_n(T) := \inf \{ \|T - A\| \ ; \ A \in L_n(E, F) \}$$

For the general properties of the approximation numbers we may refer to [6].

By making use of the approximation numbers the following class of operators is defined, generalizing that in [7].

Definition 3. [4]. For $0 < p, q \leqslant \infty$,

$$S_{p,q}^{app} := \{ T \in L \, ; \{\alpha_n(T)\} \in l_{p,q} \}$$

As shown in [4] this class is an operator ideal for which every component $S_{p,q}^{app}(E, F)$ becomes a complete metric linear space with respect to the quasi-norm

$$\sigma_{p,q}(T) := \left(\sum_n n^{q/p-1} \alpha_n(T)^q\right)^{1/q}$$

Definition 4. For every operator $T \in L(E,F)$ the n-th entropy number $e_n(T)$ is defined to be the infimum of all $\sigma \geqslant 0$ such that there are $y_1, \ldots, y_q \in F$ with $q \leqslant 2^{n-1}$ and

$$T(U_E) \subseteq \bigcup_1^q \{ y_i + \sigma U_F \}$$

We recall without proof the following properties of entropy numbers [1].

Proposition 1 [1]. If $T \in L(E,F)$, then

$$\|T\| = e_1(T) \geqslant e_2(T) \geqslant \ldots \geqslant O.$$

Proposition 2 [1]. If $T_1, T_2 \in L(E,F)$, then

$$e_{n_1+n_2-1}(T_1+T_2) \leqslant e_{n_1}(T_1) + e_{n_2}(T_2).$$

Quasi-normed operator ideal $\zeta_{p,q}$

In this section we define and investigate the properties of $\zeta_{p,q}$ as the ideal of operators [6]. We begin with

Definition 5. Given $O < p,q \leqslant \infty$, we define

$$\zeta_{p,q} := \{ T \in L \ ; \{e_n(T)\} \in l_{p,q} \},$$

and

$$E_{p,q} := \{ \sum_{n=1}^{\infty} n^{q/p-1} \ e_n(T)^q \}^{1/q}$$

Theorem 1. Let $T_1, T_2 \in \zeta_{p,q}(E,F)$. Then

$$T_1 + T_2 \in \zeta_{p,q}(E,F)$$

and

$$E_{p,q}(T_1+T_2) \leqslant c_{p,q} \ [E_{p,q}(T_1) + E_{p,q}(T_2)]$$

Proof. Since the sequence $\{e_n(T)\}$ is non-increasing and additive, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{q/p-1} \ e_n(T_1+T_2)^q \\
&\leqslant \max(2,2^{q/p}) \sum_{n=1}^{\infty} n^{q/p-1} \ e_{2n-1}(T_1+T_2)^q \\
&\leqslant \max(2,2^{q/p}) \sum_{n=1}^{\infty} n^{q/p-1} \ (e_n(T_1) + e_n(T_2))^q \\
&\leqslant \max(2,2^{q/p}) \max(2^{q-1},1) \sum_{n=1}^{\infty} n^{q/p-1} \ (e_n(T_1)^q + e_n(T_2)^q)
\end{aligned}$$

where we have used proposition 2 and the inequality

$$(\xi + \eta)^a \leqslant \{ \max(2^{a-1},1) \ (\xi^a + \eta^a) \},$$

for any $\xi \geqslant O, \eta \geqslant O$ and $a > O$.

Hence we obtain

$$E_{p,q}(T_1 + T_2) \leqslant$$

$$\leqslant \max(2^{1/q}, 2^{1/p}) \max(2^{1-1/q}, 2^{1/q-1}) \{ E_{p,q}(T_1) + E_{p,q}(T_2) \}$$

$$= c_{p,q} \{ E_{p,q}(T_1) + E_{p,q}(T_2) \}$$

with $c_{p,q} = \max(2, 2^{2/q-1}, 2^{1/p-1/q+1}, 2^{1/p+1/q-1})$.

Without proof we state

Theorem 2. Let E, F, G and H be Banach spaces and $X \in L(E,F)$, $T \in \zeta_{p,q}(F,G)$ and $Y \in L(G,H)$. Then we have $YTX \in \zeta_{p,q}(E,H)$ and

$$E_{p,q}(YTX) \leqslant \|Y\| \, E_{p,q}(T) \, \|X\|.$$

By definition 4 and lemma 2, the next proposition could be easily proved. So the proof is omitted.

Proposition 3. (i) If $0 < p \leqslant \infty$ and $0 < q_1 < q_2 \leqslant \infty$, then

$$\zeta_{p,q_1}(E,F) \subset \zeta_{p,q_2}(E,F)$$

and

$$E_{p,q_2}(T) \leqslant c \, E_{p,q_1}(T).$$

(ii) If $0 < p_1 < p_2 < \infty$ and $0 < q_1, q_2 \leqslant \infty$, then

$$\zeta_{p_1,q_1}(E,F) \subset \zeta_{p_2,q_2}(E,F)$$

and

$$E_{p_2,q_2}(T) \leqslant c \, E_{p_1,q_1}(T).$$

Relation between $\zeta_{p,q}$ and $S^{app}_{p,q}$ in Hilbert spaces

In this section we shall investigate the relationship between $\zeta_{p,q}(l_2,l_2)$ and $S^{app}_{p,q}(l_2,l_2)$ to obtain the inequality concerning $E_{p,q}$ and $\sigma_{p,q}$. These are the (p,q)-version extending theorems 4 and 5 in [1].

Lemma 3 [6]. Let $S \in L(l_p, l_q)$ such that $S\{\xi_n\} = \{\sigma_n \xi_n\}$ and $\{\sigma_n\} \in c_o$. Then

$$\alpha_n(S) = \sigma_n$$

Theorem 3. Let $0 < p, q \leqslant \infty$. Then we have

$$\zeta_{p,q}(l_2, l_2) \subset S^{app}_{p,q}(l_2, l_2)$$

and

$$\sigma_{p,q}(S) \leqslant c \; E_{p,q}(S) \quad \text{for each } S \in \zeta_{p,q}(l_2, l_2) .$$

Proof. Let $S \in L(l_2, l_2)$ such that $S\{\xi_n\} = \{\sigma_n \xi_n\}$ and $\{\sigma_n\} \in c_0$.
Without loss of generality we may suppose that $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant 0$. If $\sigma_n = 0$, then

$$\sigma_n \leqslant 2e_n \quad \text{for every n.}$$

So assume that $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_n > 0$.
Put

$$Q_n(\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots) := (\xi_1, \ldots, \xi_n)$$

and

$$J_n(\xi_1, \ldots, \xi_n) := (\xi_1, \ldots, \xi_n, 0, \ldots) .$$

Then, $S_n = Q_n S J_n$ is invertible. If I_n denotes the identity map of l_2, it will follow from $e_n(I_n) \geqslant ½$ and the properties of entropy numbers that

$$½ \leqslant e_n(I_n) \leqslant e_n(S_n) \, \|S_n^{-1}\| \leqslant \|Q_n\| \, e_n(S) \, \|J_n\| \, \sigma_n^{-1} \leqslant e_n(S) \, \sigma_n^{-1} .$$

Then

$$\sigma_n \leqslant 2 \, e_n(S) .$$

Using lemma 3, we get

$$(\sigma_{p,q}(S))^q = \sum_{n=1}^{\infty} n^{q/p-1} \alpha_n(S)^q \leqslant \sum_{n=1}^{\infty} n^{q/p-1} \, (2 \, e_n(S))^q$$

$$= 2^q \sum_{n=1}^{\infty} n^{q/p-1} \; e_n(S)^q .$$

Then

$$\sigma_{p,q}(S) \leqslant 2 \, E_{p,q}(S)$$

which finishes the proof.

Naturally, we would ask whether the converse of the above theorem holds? In case $p = q$ this is true and has been proved in [1]. Our answer to the proposed question is negative in case $p \neq q$ as shown by the following theorem. First we give a lemma.

Lemma 4 [1]. Let $S \in L(l_2, l_2)$ such that $S\{\xi_n\} = (\sigma_n \xi_n)$ and $\{\sigma_n\} \in c_0$. If we define

$$E(\epsilon) := \max \{ n \, ; e_n(S) > \epsilon \} \quad \text{for } 0 < \epsilon < \sigma_1$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant 0$. Then $E(2\epsilon) \leqslant 1 + \sum_{\sigma_k > \epsilon} \log(8\sigma_k/\epsilon)$.

Theorem 4. Let $q \leqslant p$ and $S \in L(l_2, l_2)$ such that $S\{\xi_n\} = \{\sigma_n \xi_n\}$ and $\{\sigma_n\} \in c_o$. Then

$$S^{app}_{q,q}(l_2, l_2) \subset \zeta_{p,q}(l_2, l_2) \quad \text{for each } S \in S^{app}_{q,q}(l_2, l_2)$$

and

$E_{p,q}(S) \leqslant c\, \sigma_{q,q}(S)$, where c is some positive constant.

Proof. Let $S \in \zeta_{p,q}(l_2, l_2)$. Then

$$2^{-q}(E_{p,q}(S))^q = 2^{-q} \sum_{n=1}^{\infty} n^{q/p-1}\, e_n(S)^q.$$

Since $q \leqslant p$, we have

$$2^{-q}(E_{p,q}(S))^q \leqslant 2^{-q} \sum_{n=1}^{\infty} e_n(S)^q$$

$$= 2^{-q} \sum_{n=1}^{\infty} n\, [e_n(S)^q - e_{n+1}(S)^q]$$

$$\leqslant 2^{-q} \int_0^{\sigma_1} E(\epsilon)\, d\epsilon^q$$

$$\leqslant \sigma_1^q + \int_0^{\sigma_1} \sum_{\sigma_k > \epsilon} \log(8\,\sigma_k/\epsilon)\, d\epsilon^q$$

$$= \sigma_1^q + \sum_{i=1}^{\infty} \int_{\sigma_{i+1}}^{\sigma_i} \sum_{(\sigma_k > \epsilon} \log(8\,\sigma_k/\epsilon)\, d\epsilon^q$$

$$= \sigma_1^q + \sum_{i=1}^{\infty} \sum_{k=1}^{i} \int_{\sigma_{i+1}}^{\sigma_i} \log(8\,\sigma_k/\epsilon)\, d\epsilon^q$$

$$= \sigma_1^q + \sum_{k=1}^{\infty} \int_0^{\sigma_k} \log(8\,\sigma_k/\epsilon)\, d\epsilon^q$$

$$2^{-q}(E_{p,q}(S))^q = \sigma_1^q + 8^q/q \int_0^{8^{-q}} \log(1/t)\, dt \sum_{k=1}^{\infty} \sigma_k^q$$

This shows that

$$E_{p,q}(S) \leqslant c\, \sigma_{q,q}(S), \qquad p \geqslant q$$

which completes the proof.

As a consequence of theorems 3 and 4 we obtain

Theorem 5. Let $0 < q \leqslant p < \infty$. Then

$$S^{app}_{q,q}(l_2, l_2) \subset \zeta_{p,q}(l_2, l_2) \subset S^{app}_{p,q}(l_2, l_2)$$

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