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A PROBLEM IN MAGNETO-THERMOELASTICITY FOR A HOLLOW CIRCULAR CYLINDER WITH THERMAL RELAXATION

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مسألة في المرونة الحرارية المغناطيسية لإسطوانة دائرية مجوفة

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تم حل مسألة لاسطوانة دائرية مجوفة طولها لا نهائي سطح السطوانة الداخلي معزول حرارياً وسطحها الخارجي حرارته مثبته يؤثر على الوسط مجال مغنطيسي في إتجاه محور الاسطوانة . أسطح الاسطوانة خالية من الإجهاد . المسألة في مجال نظرية المرونة المغناطيسية الحرارية المعممة ذات الزمن الاسترخائي الواحد . إستخدم تحويل لابلاس بالنسبة للزمن إوجدت التحويلات العكسية باستخدام طريقة عددية تعتمد على مفكوك فوريير .

أجريت حسابات عددية لحساب قيمة الحرارة، الازاحة، والإجهاد مع المجال المغنطيسي والمجال الكهربائي الناشآن . كما تم تمثيل هذه الدوال بيانياً .

Key words : Generalized Thermoelasticity - Magneto Thermoelasticity.

ABSTRACT

The problem of an infinitely long annular cylinder whose inner surface is thermally insulated and whose outer surface is kept at a constant temperature is considered in the presence of an axial uniform magnetic field. The surfaces of the cylinder are taken to be traction free The problem is in the context of generalized magneto-thermoelasticity theory with one relaxation time. The Laplace transform with respect to time is used. The inversion process is carried out using a numerical method based on a Fourier series expansion.

Numerical computations for the temperature, displacement and stress distributions as well as for the induced magnetic and electric fields are carried out and represented graphically.

Biot [1] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equations for both theories are of the diffusion type predicting infinite speeds of propagation for heat waves contrary to physical observations. Lord and Shulmann [2] introduced the theory of generalized thermoelasticity with one relaxation time by postulating a new law of heat conduction to replace the classical Fourier's law. This law contains the heat flux vector as well as its time derivative. It contains also a new constant that acts as a relaxation time. The heat equation of this theory is of the wave-type, ensuring finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motion and the constitutive relations remain the same as those for the coupled and the uncoupled theories. This theory was extended by Dhaliwal and Sherief [3] to general anisotropic media in the presence of heat sources.

An increasing attention is being devoted to the interaction between magnetic fields and strain in a thermoelastic solid due to its many applications in the fields of geophysics , plasma physics and related topics. Usually, in these investigations the heat equation under consideration is taken as the uncoupled or the coupled equation not the generalized one. This attitude is justified in many situations since the solutions obtained using any of these equations differ little quantitively . However, when short time effects are considered, the full generalized system of equations has to be used or a great deal of accuracy is lost.

A comprehensive review of the earlier contributions to the subject can be found in [4]. Among the authors who considered the generalized magneto-thermoelastic equations are Nayfeh and Nemat-Nasser [5] who studied the propagation of plane waves in a solid under the influence of an electromagnetic field. They have obtained the governing equations in the general case and the solution for some particular cases. Choudhuri [6] extended these results to rotating media. Lately, Sherief [7] has solved a problem for a solid cylinder, while sherief and Ezzat [8] has solved a thermal shock half-space problem using asymptotic expansions.

FORMULATION OF THE PROBLEM

Let (r,ψ, z) be cylindrical polar coordinates with the z-axis coinciding with the axis of an annular infinitely long elastic circular cylinder of a homogeneous, isotropic material of finite conductivity whose inner and outer radii are R_1 and R_2. The surfaces of the cylinder are taken to be traction free. The inner surface is taken to be thermally insulated while the outer surface is kept at a constant temperature. A constant magnetic field of strength H_0 acts in the direction of the z-axis. Due to the effect of this magnetic field there arises in the medium an induced magnetic field **h** and an induced electric field **E** (both assumed to be small). Also, there arises a force **F** (the Lorentz Force). Due to the effect of this force points of the medium undergo displacement **u** which gives rise to a temperature T. This situation resembles that inside nuclear reactors and in some components of electronic devices.

The electromagnetic quantities satisfy Maxwell's equations

$$\mathbf{curl\, h} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} ,$$

$$\mathbf{curl\, E} = -\frac{\partial \mathbf{B}}{\partial t} ,$$

$$\mathbf{div\, h} = 0 \ , \ \mathbf{div\, E} = 0 ,$$

$$\mathbf{B} = \mu_0(\mathbf{H_0}+\mathbf{h}) \ , \ \mathbf{D} = \varepsilon_0 \mathbf{E} .$$

where **J** is the electric current density, μ_0 and ε_0 are the magnetic and electric permeabilities, respectively and **B**, **D** are the magnetic and electric induction vectors, respectively.

The elastic quantities satisfy the equations of motion in vector form

$$\sigma + \mathbf{F} = \int (\rho h_o) \frac{\partial^2 \mathbf{u}}{\partial t^2} , \qquad (5)$$

where σ is the stress tensor, **F** the external body force , which is here equal to the Lorentz force and $\int (rh_0)$ is the density.

The last field equation is the equation of energy balance, namely

$$\frac{\partial}{\partial t}\left[\rho c_E T + \gamma T_0 e\right] = -\mathbf{div q} , \qquad (6)$$

where **q** is the heat flux vector, c_E is the specific heat at constant strain, e = div **u** is the cubical dilatation, γ is a material constant equal to $(3\lambda+2\mu)\alpha_t$ where λ,μ are Lame's modulii and t is the coefficient of linear thermal expansion. T_0 is a reference temperature assumed to be such that $|(T-T_0)/T_0|<<1$.

The above field equations are supplemented by constitutive equations which consist first of Ohm´s law

$$\mathbf{J} = \sigma_0\left[\mathbf{E} + \mu_0 \frac{\partial \mathbf{u}}{\partial t} \times \left(\mathbf{H_0} + \mathbf{h}\right)\right] ,$$

where σ_0 is the electeric conductivity. The above equation can be linearized by neglecting small quantities of the second order giving

$$\mathbf{J} = \sigma_0\left[\mathbf{E} + \mu_0 \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H}_0\right]. \tag{7}$$

The second constitutive equation is the one for the Lorentz force which is

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{8}$$

The third constitutive equation is the Hooke-Duhamel-Neumann's law, namely

$$\sigma_{ij} = 2\mu\, e_{ij} + \lambda\, e\, \delta_{ij} - \gamma\,(T-T_0)\,\delta_{ij}, \tag{9}$$

where δ_{ij} is kronecker's delta tensor and e_{ij} is the strain tensor whose components are given by

$$e_{ij} = 1/2\,(u_{i,j} + u_{j,i}). \tag{10}$$

The last constitutive equation is the generalized Fourier's law of heat conduction which has the form

$$\mathbf{q} + \tau_0 \frac{\partial \mathbf{q}}{\partial t} = -\mathbf{k}\,\mathbf{grad}T. \tag{11}$$

Substituting from equation (9) into equation (5) and using equation (10), we arrive at the equations of motion in vector form

$$(\lambda+\mu)\nabla^2\mathbf{u} + \mu\,\mathbf{grad}\,\mathbf{div}\mathbf{u} - \gamma\,\mathbf{grad}T + \mathbf{F} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{12}$$

Applying the div operator to both sides of equation (12), we obtain

$$(\lambda+2\mu)\nabla^2 e - \gamma\nabla^2 T + \mathbf{div}\mathbf{F} = \rho\frac{\partial^2 e}{\partial t^2}, \tag{13}$$

where ∇^2 is Laplace's operator in cylindrical coordinates, given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

Applying the div operator to both sides of equation (11), then substituting from the resulting equation into equation (6) and its time derivative, we obtain the generalized heat equation

$$k\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\left(\rho c_E T + \gamma T_0 e\right). \tag{14}$$

Because of the cylindrical symmetry of the problem all the considered functions will be functions of r and t, also, components of the displacement vector will be of the form

$$u_r = u\,,\ u_\psi = u_z = 0.$$

The strain tensor components are thus given by

$$e_{rr} = \frac{\partial u}{\partial r}, e_{\psi\psi} = \frac{u}{r}, e_{zz} = e_{rz} = e_{r\psi} = e_{\psi z} = 0.$$

It follows that the cubical dilatation e is of the form

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} = \frac{1}{r}\frac{\partial (ru)}{\partial r}. \tag{15}$$

From equation (9) we obtain the components of the stress tensor as

$$\sigma_{rr} = 2\mu\frac{\partial u}{\partial r} + \lambda e - \gamma\,(T - T_0), \tag{16a}$$

$$\sigma_{\psi\psi} = 2\mu\frac{u}{r} + \lambda e - \gamma\,(T - T_0), \tag{16b}$$

$$\sigma_{zz} = \lambda e - \gamma\,(T - T_0), \tag{16c}$$

$$\sigma_{rz} = \sigma_{z\psi} = \sigma_{\psi r} = 0. \tag{16d}$$

The induced magnetic field **h** will have one component h in the z-direction while the induced electric field **E** will have one component E in the ψ- direction. From equation (7), it follows that the electric current density will have one component only in the ψ-direction, given by

$$\mathbf{J} = \sigma_0\left[\mathbf{E} - \mu_0 \mathbf{H}_0 \frac{\partial \mathbf{u}}{\partial t}\right]. \tag{17}$$

The vector equations (1) and (2), reduce to the following scalar equations

$$\frac{\partial \mathbf{h}}{\partial r} = -\left[\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right] \tag{18}$$

$$\frac{1}{r}\frac{\partial (rE)}{\partial r} = -\mu_0 \frac{\partial \mathbf{h}}{\partial t} \tag{19}$$

Eliminating J between equation (17) and (18), we obtain

$$\frac{\partial \mathbf{h}}{\partial r} = \sigma_0\mu_0 \mathbf{H}_0 \frac{\partial \mathbf{u}}{\partial t} - \left[\sigma_0 \mathbf{E} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right]. \tag{20}$$

Eliminating E between equations (18) and (19), we obtain

$$\left[\nabla^2 - \mu_0\sigma_0\frac{\partial}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right]\mathbf{h} = \mu_0\sigma_0\mathbf{H}_0\frac{\partial e}{\partial t}. \tag{21}$$

The Lorentz force has one component F in r-direction obtained from equations (8) and (17) as

$$\mathbf{F} = -\mu_0\mathbf{H}_0\left[\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{h}}{\partial r}\right]. \tag{22}$$

Substituting from equation (22) into equations (13), we obtain upon using equation (19)

$$(\lambda+2\mu)\nabla^2 e - \gamma\nabla^2 T + \mu_0^2\varepsilon_0\mathbf{H}_0\frac{\partial^2 \mathbf{h}}{\partial t^2} - \mu_0\mathbf{H}_0\nabla^2\mathbf{h} = \rho\frac{\partial^2 e}{\partial t^2}. \tag{23}$$

We shall use the following non-dimensional variables

$$r' = c_1\eta r\,,\ R_i' = c_1\eta R_i\,,\ u' = gc_1\eta u\,,\ e' = g\,e\,,\ \sigma_{ij}' = \frac{g\sigma_{ij}}{\mu}$$

$$t' = c_1^2\eta t\,,\ \tau_0' = c_1^2\eta\tau_0\,,\ \theta = \frac{T-T_0}{T_0}\,,\ F_i' = \frac{F_i - T_0}{T_0}$$

$$q' = \frac{q}{kT_0c_1\eta}\,,\ L_i' = \frac{L_i}{kc_1\eta}\,,\ h' = \frac{\eta g}{\sigma_0\mu_0 H_0}h\,,\ E' = \frac{\eta g}{\sigma_0\mu_0^2 H_0 c_1}E.$$

where $g = \frac{\gamma}{\rho c_E}$, $\eta = \frac{\rho c_E}{k}$ and $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$ is the speed of propagation of isothermal elastic waves.

In terms of these non-dimensional variables, the governing equations (18),(19), (21), (23) and (14) reduce to (dropping the primes for convenience)

$$\frac{\partial h}{\partial r} = \frac{\partial u}{\partial t} - \left[vE + V^2 \frac{\partial E}{\partial t} \right], \tag{24}$$

$$\frac{1}{r}\frac{\partial (rE)}{\partial r} = -\frac{\partial h}{\partial t}, \tag{25}$$

$$\left[\nabla^2 - v\frac{\partial}{\partial t} - V^2 \frac{\partial^2}{\partial t^2} \right] h = \frac{\partial e}{\partial t}, \tag{26}$$

$$\nabla^2 e - \varepsilon_1 \nabla^2 \theta - \varepsilon_2 v \left(\nabla^2 - V^2 \frac{\partial^2}{\partial t^2} \right) h = \frac{\partial^2 e}{\partial t^2}, \tag{27}$$

$$\nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right)(\theta + e), \tag{28}$$

where $1/v = \eta/\sigma_0\mu_0$ is a measure of magnetic viscosity, $V = c_1/c$ where c is the speed of light given by $c^2 = 1/\varepsilon_0\mu_0$, $\varepsilon_1 = bg/\beta^2$ is the thermoelastic coupling constant where $b = \gamma T_0/\mu$, $\beta^2 = (\lambda+2\mu)/\mu$ and $\varepsilon_2 = \mu_0 H_0^2/\rho c_1^2$ is the magnetoelastic coupling constant. We note that equation (15) retains its form.

The non-dimensional constitutive equations take the form

$$\sigma_{rr} = \beta^2 e - \frac{2u}{r} - \varepsilon_1 \beta^2 \theta, \tag{29a}$$

$$\sigma_{\psi\psi} = \beta^2 e - 2\frac{\partial u}{\partial r} - \varepsilon_1 \beta^2 \theta, \tag{29b}$$

$$\sigma_{zz} = (\beta^2 - 2)e - \varepsilon_1 \beta^2 \theta, \tag{29c}$$

$$\sigma_{rz} = \sigma_{z\psi} = \sigma_{\psi r} = 0. \tag{29d}$$

The initial conditions of the problem are taken to be homogeneous, while the boundary conditions are taken as follows

(1) The transverse components of the vector **E** are continuous across the surface of the cylinder, this gives

$$E(R_j,t) = E^j(R_j,t) \, , \, t > 0 \, , \, j = 1,2 \, , \tag{30}$$

where E^1 and E^2 are the components of the electric field intensities in the ψ-direction in free space inside and outside the cylinder, respectively.

(2) The transverse components of the vector **h** are continuous across the surface of the cylinder, this gives

$$h(R_j,t) = h^j(R_j,t) \, , \, t > 0 \, , \, j = 1,2 \, , \tag{31}$$

where h^1 and h^2 are the components of the induced magnetic field in the z-direction in free space inside and outside the cylinder, respectively.

(3) The surfaces of the cylinder are traction free, i.e.

$$\sigma_{rr}(R_j,t) = 0 \quad , \quad t > 0 \, , \, j = 1,2. \tag{32}$$

(4) The heat conduction boundary condition

$$\frac{\partial \theta}{\partial r} = 0 \, , \text{at } r = R_1 \, , \tag{33a}$$

$$\theta = \theta_0 \, H(t), \text{ at } r = R_2 \, , \tag{33b}$$

where θ_0 is a constant and H(t) is the Heaviside unit step function.

In order to utilize equations (30) and (31) above, we must obtain the induced fields E^j, h^j in the free space surrounding the medium. These quantities satisfy the following non-dimensional equations

$$\frac{\partial h^j}{\partial r} = -V^2 \frac{\partial E^j}{\partial t} \, , \, j = 1,2 \, , \tag{34}$$

$$\frac{1}{r}\frac{\partial (rE^j)}{\partial r} = -\frac{\partial h^j}{\partial t} \, , \, j = 1,2 \, . \tag{35}$$

SOLUTION IN THE LAPLACE TRANSFORM DOMAIN

Taking the Laplace transform with parameter s (denoted by a bar) of both sides of equations (24)-(35), we obtain the following set of equations

$$\frac{\partial \bar{h}}{\partial r} = s\bar{u} - (v + V^2 s)\bar{E}, \tag{36}$$

$$\frac{1}{r}\frac{\partial (r\bar{E})}{\partial r} = -s\bar{h} \, , \tag{37}$$

$$\left[\nabla^2 - vs - V^2 s^2 \right] \bar{h} = s\bar{e}, \tag{38}$$

$$(\nabla^2 - s^2)\bar{e} = \varepsilon_1 \nabla^2 \bar{\theta} + \varepsilon_2 v \left(\nabla^2 - V^2 s^2 \right) \bar{h}, \tag{39}$$

$$\nabla^2 \bar{\theta} = \left(s + \tau_0 s^2 \right)\left(\bar{\theta} + \bar{e} \right). \tag{40}$$

The non-dimensional constitutive equations (29) in the Laplace transform domain take the form

$$\bar{\sigma}_{rr} = \beta^2 \bar{e} - \frac{2\bar{u}}{r} - b\bar{\theta}, \tag{41a}$$

$$\bar{\sigma}_{\psi\psi} = \beta^2 \bar{e} - 2\frac{\partial \bar{u}}{\partial r} - b\bar{\theta}, \tag{41b}$$

$$\bar{\sigma}_{zz} = (\beta^2 - 2)\bar{e} - b\bar{\theta}, \tag{41c}$$

The boundary conditions in the Laplace trnsform domain become

$$\bar{E}(R_j,s) = \bar{E}^j(Rj,s) \quad , j = 1,2 \tag{42}$$

$$\bar{h}(R_j,t) = \bar{h}^j(Rj,t) \, , \, j = 1,2 \, . \tag{43}$$

$$\bar{\sigma}_{rr}(R_j,s) = 0 \, , \, j = 1,2 \, . \tag{44}$$

$$\frac{\partial \bar{\theta}}{\partial r} = 0 \, , \text{at } r = R_1 \, , \tag{45a}$$

$$\bar{\theta} = \frac{\theta_0}{s} \, , \text{at } r = R_2 \, . \tag{45b}$$

Equations (34) and (35) take the following form in the Laplace transform domain

$$\frac{\partial \bar{h}^j}{\partial r} = -V^2 s \bar{E}^j \ , \ j = 1,2 \ , \tag{46}$$

$$\frac{1}{r}\frac{\partial (r\bar{E}^j)}{\partial r} = -s\,\bar{h}^j \ , \ j = 1,2 \ . \tag{47}$$

Eliminating $\bar{h}$ and $\bar{\theta}$ between equations (38)-(40), we get the following sixth order didderential equation satisfied by $\bar{e}$

$$\left(\nabla^6 - A\nabla^4 + B\nabla^2 - C\right)\bar{e} = 0, \tag{48}$$

where

$$A = s[\varepsilon_1(s\tau_0 + 1) + \varepsilon_2 \nu + \nu + s\,(\tau_0 + V^2 + 1) + 1],$$

$$B = s^2\{\varepsilon_1(s\tau_0 + 1)\,(\nu + sV^2) + \varepsilon_2 \nu(s(\tau_0 + V^2) + 1) +$$
$$+\nu(s(\tau_0 + 1) + s(s(\tau_0\,(V^2 + 1) + V^2) + V^2 + 1)\},$$

$$C = s^4(s\tau_0 + 1)(\varepsilon_2 \nu\, V^2 + \nu + sV^2).$$

It should be noted that the above equation reduce to the usual equations of generalized thermoelasticity without electro-magnetic effects in the limit as ν, V and $\varepsilon_2 \to 0$.

Equation (48) can be factorized as

$$\left(\nabla^2 - k_1^2\right)\left(\nabla^2 - k_2^2\right)\left(\nabla^2 - k_3^2\right)\bar{e} = 0, \tag{49}$$

where k_1^2, k_2^2, k_3^2 are the roots of the characteristic equation

$$k^6 - A\,k^4 + B\,k^2 - C = 0$$

These roots are given by

$$k_1^2 = \tfrac{1}{3}\,[A + 2P \sin Q],$$

$$k_2^2 = \tfrac{1}{3}\,[A - P \sin Q - \sqrt{3}P \cos Q],$$

$$k_3^2 = \tfrac{1}{3}\,[A - P \sin Q + \sqrt{3}P \cos Q],$$

where

$$P = \sqrt{A^2 - 3B} \ , \ R = \frac{9AB - 2A^3 - 27C}{2P^3} \ , \ Q = \frac{1}{3}\sin^{-1}(R) \ .$$

The solution of equation (49) can be written as the sum

$$\bar{e} = \sum_{i=1}^{3} \bar{e}_i \ ,$$

where $\bar{e}_i$ is the solution of the equation

$$\left(\nabla^2 - k_i^2\right)\bar{e}_i = 0$$

Thus, the general solution of equation (49) has the form

$$\bar{e} = \sum_{i=1}^{3}\left(A_i I_0(k_i r) + B_i K_0(k_i r)\right), \tag{50}$$

where A_i and B_i are parameters depending on s only, i = 1,2,3 and I_0 and K_0 are the modified Bessel functions of order zero of the first and second kinds, respectively.

Eliminating $\bar{h}$, $\bar{e}$ and then $\bar{\theta}$, $\bar{e}$ between equations (38)-(40), we find that $\bar{\theta}$ and $\bar{h}$ satisfy the same equation as e, i.e.

$\left(\nabla^6 - A\nabla^4 + B\nabla^2 - C\right)\bar{\theta} = \left(\nabla^6 - A\nabla^4 + B\nabla^2 - C\right)\bar{h} = 0$

$\bar{\theta}$ and $\bar{h}$ are thus given by

$$\bar{\theta} = \sum_{i=1}^{3}\left(A_i' I_0(k_i r) + B_i' K_0(k_i r)\right), \tag{51}$$

$$\bar{h} = \sum_{i=1}^{3}\left(A_i'' I_0(k_i r) + B_i'' K_0(k_i r)\right), \tag{52}$$

where A_i' , B_i' , A_i'' , and B_i'' , are parameters depending on s only. The compatibility between equations (50)-(52) and equations (38) and (40) give

$$A_i = \frac{s(1 + \tau_0 s)}{k_i^2 - s(1 + \tau_0 s)} A_i \ , \ B_i = \frac{s(1 + \tau_0 s)}{k_i^2 - s(1 + \tau_0 s)} B_i, \tag{53}$$

$$A_i'' = \frac{s}{k_i^2 - \nu s - V^2 s^2} A_i \ , \ B_i'' = \frac{s}{k_i^2 - \nu s - V^2 s^2} B_i \ . \tag{54}$$

Substituting from equations (53) and (54) into equations (51) and (52), we obtain

$$\bar{\theta} = s(1 + \tau_0 s)\sum_{i=1}^{3}\frac{\left(A_i I_0(k_i r) + B_i K_0(k_i r)\right)}{k_i^2 - s(1 + \tau_0 s)}, \tag{55}$$

$$\bar{h} = s\sum_{i=1}^{3}\frac{\left(A_i I_0(k_i r) + B_i K_0(k_i r)\right)}{k_i^2 - \nu s - V^2 s^2} \ . \tag{56}$$

Substituting from equation (50) into the Laplace transform of equation (15) and integrating both sides with respect to r, we obtain

$$\bar{u} = \sum_{i=1}^{3}\frac{1}{k_i}\left(A_i I_1(k_i r) - B_i K_1(k_i r)\right). \tag{57}$$

In obtaining equation (57), we have used the following relations of the Bessel functions [9]

$$\int z I_0(z)\, dz = z I_1(z) \ , \ \int z K_0(z)\, dz = z K_1(z) \ .$$

Substituting from equations (56) and (57) into equation (36), and using the relations [9]

$$I_0'(z) = I_1(z), \ K_0'(z) = K_1(z),$$

we obtain

$$\bar{E} = -s^2 \sum_{i=1}^{3}\frac{\left(A_i I_1(k_i r) - B_i K_1(k_i r)\right)}{k_i\left(k_i^2 - \nu s - V^2 s^2\right)} \ . \tag{58}$$

Substituting from equations (50), (55) and (57) into equation (41a), we obtain

$$\bar{\sigma}_{rr} = \sum_{i=1}^{3}\left\{\beta^2\left[1 - \frac{\varepsilon_1 s(1 + \tau_0 s)}{k_i^2 - s(1 + \tau_0 s)}\right]\left(A_i I_0(k_i r) + B_i K_0(k_i r)\right)\right.$$
$$\left. - \frac{2}{r k_i}\left(A_i I_1(k_i r) - B_i K_1(k_i r)\right)\right\}, \tag{59}$$

Differentiating both sides of equqtion (57) with respect to r and using the relations [9]

$$\frac{dI_1(z)}{dz} = I_0(z) - \frac{1}{z} I_1(z), \quad \frac{dK_1(z)}{dz} = -K_0(z) - \frac{1}{z} K_1(z),$$

we obtain

$$\frac{\partial \bar{u}}{\partial r} = \sum_{i=1}^{3} \left[A_i \left(I_0(k_i r) - \frac{1}{k_i r} I_1(k_i r) \right) - B_i \left(K_0(k_i r) + \frac{1}{k_i r} K_1(k_i r) \right) \right]. \tag{60}$$

Substituting from equations (50), (55) and (60) into equation (41b), we obtain

$$\bar{\sigma}_{\psi\psi} = \sum_{i=1}^{3} A_i \left\{ \left[\beta^2 - 2 - \frac{\varepsilon_1 \beta^2 s(1+\tau_0 s)}{k_i^2 - s(1+\tau_0 s)} \right] I_0(k_i r) + \frac{2}{k_i r} I_1(k_i r) \right\} + \sum_{i=1}^{3} B_i \left\{ \left[\beta^2 + 2 - \frac{\varepsilon_1 \beta^2 s(1+\tau_0 s)}{k_i^2 - s(1+\tau_0 s)} \right] K_0(k_i r) + \frac{2}{k_i r} K_1(k_i r) \right\}, \tag{61}$$

In order to obtain the induced field in free space, we eliminate $\bar{E}^j$ between equations (46) and (47), to obtain

$$\left(\nabla^2 - V^2 s^2\right) \bar{h}^j = 0, \; j = 1,2. \tag{62}$$

$\bar{h}^1$ and $\bar{h}^2$ are the solutions of equation (62) which are bounded at the origin and at infinity, respectively. Thus, we have

$$\bar{h}^1 = A_0 I_0(Vsr), \tag{63}$$

$$\bar{h}^2 = B_0 K_0(Vsr), \tag{64}$$

where A_0 and B_0 are some parameters depending on s only.

Substituting from equations (63) and (64) into equation (46), we obtain

$$\bar{E}^1 = -\frac{A_0}{V} I_1(Vsr), \tag{65}$$

$$\bar{E}^2 = \frac{B_0}{V} K_1(Vsr). \tag{66}$$

We shall now use the boundary conditions of the problem to evaluate the unknown parameters of the problem, namely A_i and B_i , i =0,1,2,3. Equations (30) and (31) in the Laplace transform domain together with equations (56), (58), (63), (64), (65) and (66) immediately give

$$s^2 \sum_{i=1}^{3} \frac{\left(A_i I_1(k_i R_1) - B_i K_1(k_i R_1)\right)}{k_i \left(k_i^2 - vs - V^2 s^2\right)} - \frac{A_0}{V} I_1(VsR_1) = 0, \tag{67}$$

$$s^2 \sum_{i=1}^{3} \frac{\left(A_i I_1(k_i R_2) - B_i K_1(k_i R_2)\right)}{k_i \left(k_i^2 - vs - V^2 s^2\right)} + \frac{B_0}{V} K_1(VsR_2) = 0, \tag{68}$$

$$s \sum_{i=1}^{3} \frac{\left(A_i I_0(k_i R_1) + B_i K_0(k_i R_1)\right)}{k_i^2 - vs - V^2 s^2} - A_0 I_0(VsR_1) = 0, \tag{69}$$

$$s \sum_{i=1}^{3} \frac{\left(A_i I_0(k_i R_2) + B_i K_0(k_i R_2)\right)}{k_i^2 - vs - V^2 s^2} - B_0 K_0(VsR_2) = 0. \tag{70}$$

Equations (32) and (59) give

$$\sum_{i=1}^{3} \left\{ \beta^2 \left[1 - \frac{\varepsilon_1 s(1+\tau_0 s)}{k_i^2 - s(1+\tau_0 s)} \right] \left(A_i I_0(k_i R_1) + B_i K_0(k_i R_1)\right) - \frac{2}{R_1 k_i} \left(A_i I_1(k_i R_1) - B_i K_1(k_i R_1)\right) \right\} = 0, \tag{71}$$

$$\sum_{i=1}^{3} \left\{ \beta^2 \left[1 - \frac{\varepsilon_1 s(1+\tau_0 s)}{k_i^2 - s(1+\tau_0 s)} \right] \left(A_i I_0(k_i R_2) + B_i K_0(k_i R_2)\right) - \frac{2}{R_2 k_i} \left(A_i I_1(k_i R_2) - B_i K_1(k_i R_2)\right) \right\} = 0. \tag{72}$$

Equations (45) and (55) give

$$\sum_{i=1}^{3} \frac{k_i \left(A_i I_1(k_i R_1) - B_i K_1(k_i R_1)\right)}{k_i^2 - s(1+\tau_0 s)} = 0, \tag{73}$$

$$\sum_{i=1}^{3} \frac{\left(A_i I_0(k_i R_2) + B_i K_0(k_i R_2)\right)}{k_i^2 - s(1+\tau_0 s)} = \frac{\theta_0}{s^2 (1+\tau_0 s)}, \tag{74}$$

Equations (67)-(74) constitute a system of eight linear equations in the eight unknown parameters A_i, B_i , i = 0,1,2,3, whose solution (numerically) completes the solution of the problem in the Laplace transform domain.

INVERSION OF THE LAPLACE TRANSFORM

We shall now outline the numerical inversion method used to find the solution in the physical domain. Let $\bar{f}(s)$ be the Laplace transform of a function f(t) . The inversion formula for Laplace transforms can be written as

$$f(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st} \bar{f}(s) ds$$

where d is an arbitrary real number greater than all the real parts of the singularities of f(s). Taking s = d+iy, the above integral takes the form

$$f(t) = \frac{e^{dt}}{2\pi} \int_{-\infty}^{\infty} e^{ity} \bar{f}(d+iy) dy \quad .$$

Expanding the function h(t) = exp(-dt)f(t) in a Fourier series in the interval [0,2T], we obtain the approximate formula [10]

$$f(t) = f_\infty(t) + E_D,$$

where

$$f_\infty(t) = \frac{1}{2} c_0 + \sum_{k=1}^{\infty} c_k \quad \text{for } 0 \le t \le 2T, \tag{75}$$

and

$$c_k = \frac{e^{dt}}{T} \mathrm{Re}\left[e^{ik\pi t/T} \bar{f}(d + ik\pi / T)\right] \tag{76}$$

E_D is the discretization error, with can be made arbitrarily small by choosing d large enough [10].

Since the infinite series in equation (75) can only be summed up to a finite number N of terms, the approximate value of f(t) becomes

$$f_N(t) = \frac{1}{2}c_0 + \sum_{k=1}^{N} c_k \quad \text{for } 0 \le t \le 2T \,. \tag{77}$$

Using the above formula to evaluate f(t), we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the "Korrecktur" method is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and hence to accelerate convergence.

The Korrecktur method uses the following formula to evaluate the function f(t) :

$$f(t) = f_\infty(t) - e^{-2dT} f_\infty(2T+t) + \acute{E}_D,$$

where the discretization errors $|\acute{E}_D| << |E_D|$ [10]. Thus, the approximate value of f(t) becomes

$$f_{NK}(t) = f_N(t) - e^{-2dT} f_N(2T+t), \tag{78}$$

N´ is an integer such that N´ < N.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in equation (77). Let N be an odd natural number, and let

$$s_m = \sum_{k=1}^{m} c_k$$

be the sequence of partial sums of (77). We define the ε-sequence by

$$\varepsilon_{0,m} = 0, \varepsilon_{1,m} = s_m$$

and

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + 1/(\varepsilon_{p,m+1} - \varepsilon_{p,m}) \,, \; p = 1,2,3...$$

It can be shown that [10] the sequence

$$\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots \varepsilon_{N,1}$$

converges to $f(t) + E_D - c_0/2$ faster than the sequence of partial sums

$$s_m \,, \; m = 1,2,3,...$$

The actual procedure used to invert the Laplace transforms consists of using equation (78) together with the ε-algorithm. The values of d and T are chosen according to the criteria outlined in [10].

NUMERICAL RESULTS

The copper material was chosen for purposes of numerical evaluations. The constants of the problem were taken as

$R_1 = 1$, $R_2 = 2$, $\varepsilon_1 = 0.01\,68$, $\varepsilon_2 = 0.0008$, $\tau_0 = 0.02$,

$V = 1.39(10)^{-5}$, $\beta^2 = 4$, and $v = 0.008$.

The computations were carried out for two values of time, namely t = 0.1 and t = 0.2. The numerical technique outlined above was used to obtain the temperature, displacement and radial stress and transverse stress distributions as well as the induced magnetic and electric field distributions. In all figures, dashed lines represent the function when t =0.1 while solid lines represent the function when t=0.2. The temperature increment θ is represented by the graph in figure 1. The displacement u is shown in figure 2. The stress components σ_{rr} and $\sigma_{\psi\psi}$ are shown in figures 3 and 4, respectively while the induced fields h and E are shown in figures 5 and 6 respectively.

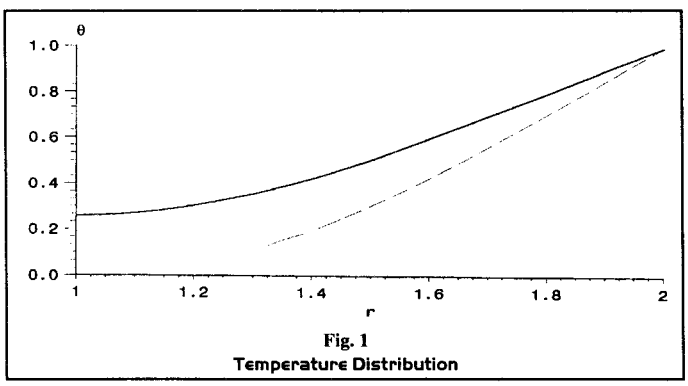

Fig. 1
Temperature Distribution

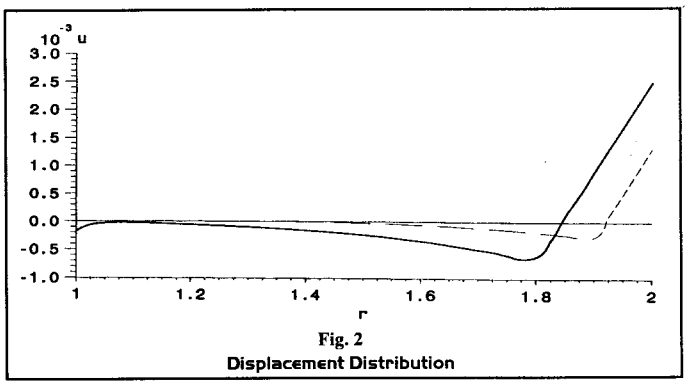

Fig. 2
Displacement Distribution

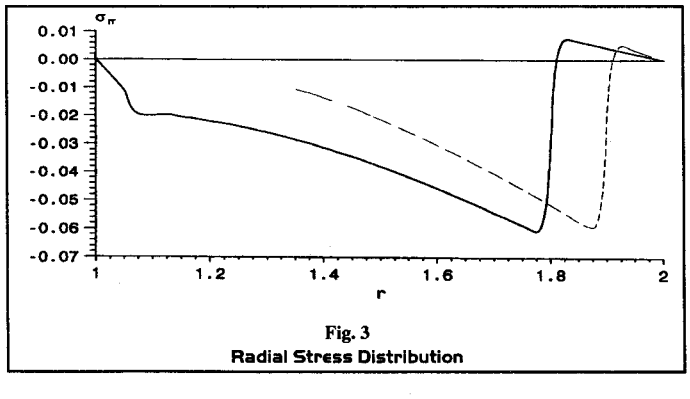

Fig. 3
Radial Stress Distribution

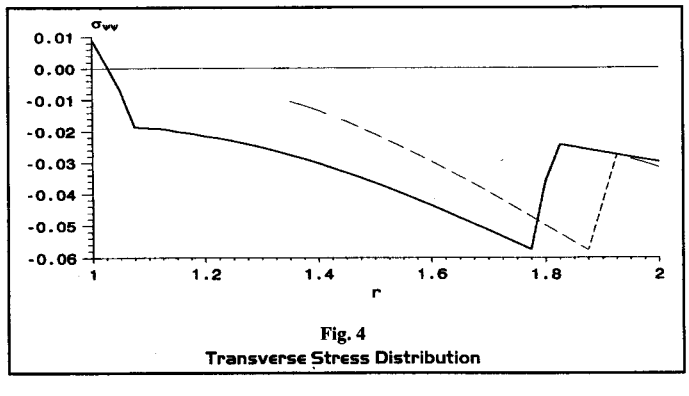

Fig. 4
Transverse Stress Distribution

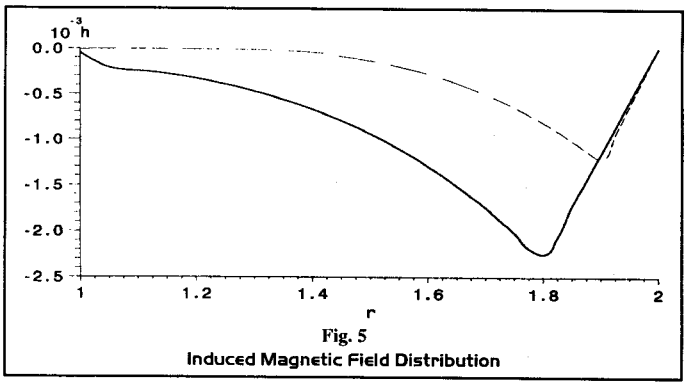

Fig. 5
Induced Magnetic Field Distribution

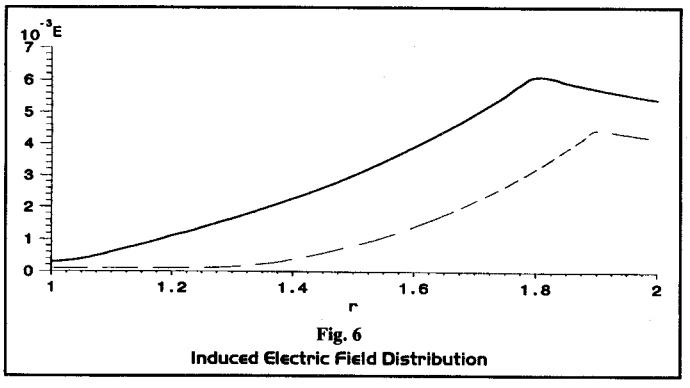

Fig. 6
Induced Electric Field Distribution

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