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Random number of units for K-out-of-n systems

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ABSTRACT

This paper takes up the reliability and preventive replacement problems for a K-out-of-n system, where K is a stochastic parameter provided. Firstly, we consider the case when K is predefined as constant numbers as is done with the traditional method, and obtain the system reliability $R(t)$, mean time to failure (MTTF), and replacement policies, in which the number n of units for replacement and replacement time T of operation are, respectively, optimized. Secondly, we focus on the above discussions again when K cannot be predefined constantly, but it is a random variable with an estimated probability function. Furthermore, we give approximate computations in an easier way for MTTF, optimal number n^* and replacement time T^*, respectively.

1. Introduction

The K-out-of-n $(1 \le K \le n)$ system, which can be operating if and only if at least number K of the total n units are operating [1], has been widely used in practical fields such as data transmission, redundant networks, redundant copies, etc. [2]. The reliability characteristics of a K-out-of-n system were investigated [3,4]. The number n of units that should be on-line to assure at least K units are available to complete an assignment for mass transit was determined [5]. A K-out-of-n system was also used as a self-checking checker for error detecting codes [6]. Surveys of multi-state and consecutive K-out-of-n systems [7,8] and parallel and consecutive-K-out-of-n: F systems [9] were conducted.

Other recent works on K-out-of-n systems, such as system with single vacation [10], computing tolerance limits for lifetime [11], the joint reliability importance (JRI) in K-out-of-n: G systems [12–14], lifetime and survival functions of two different multi-state K-out-of-n systems [15], binary decision diagrams (BDD) and multi-valued decision diagrams (MMDD) based method for binary and multi-state K-out-of-n: G systems [16], likelihood ratio order and hazard rate order for K-out-of-n systems [17], Monte-Carlo simulation algorithm of mean time to failure for the weighted K-out-of-n: G systems and linear consecutive weighted K-out-of-n: G systems [18] and the reliability evaluation methodology for multi-state weighted K-out-of-n system [19], have been surveyed.

When a K-out-of-n system is designed, both K and n should be decided at early designing phase to confirm the principal specifications. The fault tolerant computer is an example of a K-out-of-n system and is commonly applied to aircraft flight control systems and nuclear power plant controllers. The K is determined by considering a minimum constitution which can submit the correct outputs and is definitely decided by system design, and n is determined by considering the availability and maintenance cost of the system. When n is large, *i.e.*, the system has enough redundancy, fault tolerant ability of the

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system increases and maintenance cost of redundant subsystem also increases. In case of nuclear power plant controllers, the large n is for high damage tolerance because such systems are demanded to be high reliable.

The surface of an aluminum-skin airplane is composed of sheet metals which may be 3.8 m long and 1.5 m wide. Overlapping portions of sheets are called lap joints or lap splices and are joined together with three rows of hundreds of rivets typically. Lap joint is another example of K-out-of-n systems. When two sheet metals of an airframe is connected with n rivets, airframe should be designed to withstand maximum load during operation by $K(K < n)$ rivets because some rivets can be broken and loose their strength during flight. Airframe structures should retain their required residual strength for a period of operation after damage has occurred and such design concept is called damage-tolerance design. The damage tolerance design is required for civil aircraft developments by regulation after General Dynamics F-111 accident in 1969 and Dan Air Boeing 707 accident in 1977 [20].

Although the damage tolerance design has been inherited, additional requirements are usually settled by regulations after serious accidents. After Aloha accident in 1988, the widespread fatigue damage (WFD) was firstly introduced to regulations, and the full-scale fatigue test certificating damage tolerance for WFD was also introduced. The full-scale fatigue test was obligated to certify of freedom from WFD up to the service period [21]. WFD is defined as the simultaneous presence of cracks at multiple structural details that are of sufficient size and density whereby the structure will no longer meet its damage tolerance requirement [22]. From 2011, establishing a limit of validity (LOV) for future civil aircrafts are required [23]. LOV is a period of time when WFD will not occur in airframe structure.

To support LOV in design, parameters should be treated probabilistically because stress during each operation is changing and crack growth rates during each operation are also modifying. When we establish mathematical model of lap joints using K-out-of-n systems, both K and n may be regarded as stochastic parameters. However, the total number n, which depends on K, should be pre-specified at preflight checks as it is a number of normal bolts for an inspection standard. For this, one purpose of this paper is to determine how many n units should be provided according to the stochastic parameter K.

It has been supposed in most redundant systems that the number of units is constant and could be predetermined, e.g., a standard parallel system which consists of n identical units [2]. However, we might not know the exact number of units of whole system because real systems would be complex and large [24]. Such stochastic phenomena arise in order statistics when the sample size is random [25–29]. Reliability quantities such as MTTF, and replacement policies for such a parallel system with a random number n of units were discussed [30]. Similarly, for a K-out-of-n system with stochastic parameter K, another purposes of this paper are to observe the system reliability $R(t)$, to estimate mean time to failure (MTTF), and to model replacement policies.

The remainder of this paper is organized as follows: When K is predefined as constant numbers for the above K-out-of-n system, the system reliability $R(t)$ and MTTF are firstly obtained in Section 2. Optimal number n^* of units that should be provided for replacement and optimal replacement time T^* of operation are respectively discussed in Section 2. Sections 3–5 survey the models in Section 2 again when parameter K is a random variable with truncated Poisson distribution. Finally, conclusions of the paper are provided in Section 6.

2. Constant number *K* of units

2.1. Reliability and MTTF

Consider a K-out-of-n system $(n = 1, 2, \ldots)$ in which K is predefined as constant numbers such that $K = 1, 2, \ldots, n$. We suppose that each unit has an identical failure distribution $F(t)$ with finite mean $\mu \equiv \int_0^\infty \overline{F}(t)\mathrm{d}t < \infty$, density function $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$ and failure rate $h(t) \equiv f(t)/\overline{F}(t)$, where $\overline{F}(t) \equiv 1 - F(t)$, and the system fails when at least number $n - K + 1$ of units have failed. Then, the reliability of the system at time t is, i.e., the probability that at least number K of units are operating at time T is [1, p. 216], [31, p. 12]:

$$R(t) = \sum_{j=0}^{n-K} \binom{n}{j} [F(t)]^j [\overline{F}(t)]^{n-j} = \sum_{j=K}^{n} \binom{n}{j} [\overline{F}(t)]^j [F(t)]^{n-j}, \tag{1}$$

which decreases with t from 1 to 0, because;

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = nf(t) \sum_{j=K}^{n} \left\{ \binom{n}{j} [\overline{F}(t)]^j [F(t)]^{n-j-1} - \binom{n-1}{j-1} [\overline{F}(t)]^{j-1} [F(t)]^{n-j-1} \right\} = -nf(t) \binom{n-1}{j-1} [\overline{F}(t)]^{K-1} [F(t)]^{n-K} \le 0 .$$

The mean time to system failure (MTTF) is:

Table 1
Values of $\mu_{n,K}$, $\widetilde{\mu}^A_{n,K}=[\sum_{j=K}^n(1/j)]^{1/m}$ and $\widetilde{\mu}^B_{n,K}=\{\ln[n/(K-1)]\}^{1/m}$ when $F(t)=1-\exp(-t^m)\,(m=2,3)$ and $n=100$.

K	$m=2$			$m=3$		
	$\mu_{n,K}$	$\widetilde{\mu}^A_{n,K}$	$\widetilde{\mu}^B_{n,K}$	$\mu_{n,K}$	$\widetilde{\mu}^A_{n,K}$	$\widetilde{\mu}^B_{n,K}$
1	2.262	2.278	2.277	1.720	1.731	1.731
2	2.037	2.046	2.146	1.606	1.612	1.664
5	1.757	1.762	1.794	1.455	1.459	1.477
10	1.533	1.536	1.552	1.329	1.331	1.340
20	1.278	1.281	1.289	1.177	1.179	1.184
50	0.839	0.842	0.845	0.889	0.891	0.894
60	0.722	0.724	0.726	0.804	0.806	0.808
70	0.605	0.607	0.609	0.715	0.717	0.719
80	0.481	0.484	0.486	0.613	0.617	0.618
90	0.337	0.341	0.341	0.483	0.488	0.488
100	0.089	0.100	0.100	0.192	0.215	0.216

$$\mu_{n,K}\equiv\int_0^\infty R(t)\mathrm{d}t=\sum_{j=0}^{n-K}\binom{n}{j}\int_0^\infty[F(t)]^j[\overline{F}(t)]^{n-j}\mathrm{d}t$$
$$=\sum_{j=K}^{n}\binom{n}{j}\int_0^\infty[\overline{F}(t)]^j[F(t)]^{n-j}\mathrm{d}t\,, \tag{2}$$

which decreases with K from $\int_0^\infty\{1-[F(t)]^n\}\mathrm{d}t$ to $\int_0^\infty[\overline{F}(t)]^n\mathrm{d}t$.

In particular, when $F(t)=1-\mathrm{e}^{-\lambda t}$,

$$\mu_{n,K}=\frac{1}{\lambda}\sum_{j=K}^{n}\frac{1}{j}\,, \tag{3}$$

and it is approximately given by:

$$\widetilde{\mu}_{n,K}\equiv\frac{1}{\lambda}[\ln n-\ln(K-1)]=\frac{1}{\lambda}\ln\frac{n}{K-1}, \tag{4}$$

for large n, which is motivated from [24]. In case of $K=1$, $\widetilde{\mu}_{n,K}=(1/\lambda)(\ln n+\gamma)$ [30], where $\gamma\equiv 0.5772156649\ldots$ is Euler's constant [32].

In addition, when the failure time of each unit has a Weibull distribution, *i.e.*, $F(t)=1-\exp[-(\lambda t)^m]\,(m\geq 1)$, MTTF in (2) is approximately given by:

$$\widetilde{\mu}_{n,K}\approx\frac{1}{\lambda}\left(\sum_{j=K}^{n}\frac{1}{j}\right)^{1/m}\approx\frac{1}{\lambda}\left(\ln\frac{n}{K-1}\right)^{1/m}. \tag{5}$$

Denoting $\widetilde{\mu}^A_{n,K}=[\sum_{j=K}^n(1/j)]^{1/m}$ and $\widetilde{\mu}^B_{n,K}=\{\ln[n/(K-1)]\}^{1/m}$, Table 1 presents $\mu_{n,K}$ in (2), and its approximations $[\sum_{j=K}^n(1/j)]^{1/m}$ and $\{\ln[n/(K-1)]\}^{1/m}$ in (5), where $F(t)=1-\exp(-t^m)\,(m=2,3)$ and $n=100$. Obviously, $\{\ln[n/(K-1)]\}^{1/m}=(\ln n+\gamma)^{1/m}$ when $K=1$. It is much easier to compute approximations in (5), and Table 1 shows that the approximate $[\sum_{j=K}^n(1/j)]^{1/m}$ and $\{\ln[n/(K-1)]\}^{1/m}$ are very good approximations of the exact $\mu_{n,K}$.

2.2. Replacement policies

The system should be replaced with a new one immediately when it fails, we suppose that c_1 is a replacement cost for each unit and c_R is an assembly cost for the new system. Then, the expected cost rate until replacement at failure is [31, p. 12]:

$$C(n,K)=\frac{nc_1+c_R}{\mu_{n,K}}\qquad(n=K,K+1,\ldots)\,. \tag{6}$$

In particular, when $F(t)=1-\mathrm{e}^{-\lambda t}$, the expected cost rate is, from (3),

$$\frac{C(n,K)}{\lambda}=\frac{nc_1+c_R}{\sum_{j=K}^n(1/j)}\qquad(n=K,K+1,\ldots)\,. \tag{7}$$

We find optimal n^* to minimize $C(n, K)$ for given $K(K\geq 1)$. From the inequality $C(n+1,K)-C(n,K)\geq 0$,

$$(n+1)\sum_{j=K}^{n}\frac{1}{j}-n\geq\frac{c_R}{c_1}\qquad(n=K,K+1,\ldots)\,,$$

i.e.,

$$(n+1)\left(\sum_{j=K}^{n+1}\frac{1}{j}-1\right) \ge \frac{c_R}{c_1} \qquad (n=K, K+1, \dots), \tag{8}$$

whose left-hand side increases strictly with n to ∞. Thus, there exists a finite and unique minimum n^* $(K \le n^* < \infty)$ which satisfies (8) and increases with K.

Next, we suppose that the system is replaced preventively at a planned time $T(0 < T \le \infty)$ or correctively at system failure, whichever occurs first. Then, the expected cost rate is [2, p.12]:

$$C(T,K) = \frac{nc_1 + c_R\sum_{j=0}^{K-1}\binom{n}{j}[\overline{F}(T)]^j[F(T)]^{n-j}}{\sum_{j=K}^{n}\binom{n}{j}\int_0^T[\overline{F}(t)]^j[F(t)]^{n-j}\mathrm{d}t}, \tag{9}$$

where c_1 and c_R are given in (6).

We find an optimal T^* for given $K(1 \le K \le n)$ which minimizes $C(T, K)$.

Theorem 1. *If $Q(T, K)$ increases strictly with T to $Q(\infty, K)$, and*

$$Q(\infty,K)\mu_{n,K} > \frac{nc_1 + c_R}{c_R},$$

then there exists a finite and unique T^* $(0 < T^* < \infty)$ which satisfies:

$$Q(T,K)\sum_{j=K}^{n}\binom{n}{j}\int_0^T[\overline{F}(t)]^j[F(t)]^{n-j}\mathrm{d}t - \sum_{j=0}^{K-1}\binom{n}{j}[\overline{F}(T)]^j[F(T)]^{n-j} = \frac{nc_1}{c_R}, \tag{10}$$

where

$$Q(T,K) \equiv \frac{nh(T)\binom{n-1}{K-1}[\overline{F}(T)]^K[F(T)]^{n-K}}{\sum_{j=K}^{n}\binom{n}{j}[\overline{F}(T)]^j[F(T)]^{n-j}}.$$

Proof. Differentiating $C(T, K)$ with respect to T and setting it equal to zero, we have (10). Letting $L(T, K)$ be the left-hand side of (10), it can be easily shown that $L(0,K) = 0$, and

$$L(\infty,K) = Q(\infty,K)\mu_{n,K} - 1,$$
$$\frac{\mathrm{d}L(T,K)}{\mathrm{d}T} = \frac{\mathrm{d}Q(T,K)}{\mathrm{d}T}\sum_{j=K}^{\infty}\binom{n}{j}\int_0^T[\overline{F}(t)]^j[F(t)]^{n-j}\mathrm{d}t > 0,$$

which follows that $L(T, K)$ increases strictly with T from 0 to $L(\infty, K)$. Thus, if $Q(\infty,K)\mu_{n,K} > (nc_1 + c_R)/c_R$, then there exists a finite and unique T^* $(0 < T^* < \infty)$ which satisfies (10) to minimize $C(T, K)$ in (9). □

In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$,

$$Q(T,K) \equiv \frac{n\lambda\binom{n-1}{K-1}}{\sum_{j=K}^{n}\binom{n}{j}\left(\mathrm{e}^{\lambda T}-1\right)^{K-j}},$$

which increases strictly with T from 0 to $K\lambda$ for $K < n$ and is constant $n\lambda$ for $K = n$. Therefore, if $K < n$ and $(K/n)\sum_{j=K+1}^{n}(1/j) > c_1/c_R$, then there exists a finite and unique T^* $(0 < T^* < \infty)$ which satisfies (10). When $K = n$, $Q(T,K) = n\lambda$, and hence, $T^* = \infty$.

3. Random number ofunits

Suppose that the number K of units for a K-out-of-n system is a random variable with a probability distribution. In other words, suppose that K is a random variable for a specified $n(n \ge 1)$ and has a probability function $p_{k,n} \equiv P_r\{K = k\}$ $(k = 1, 2, \dots, n)$. Then, the reliability at time t is, from (1),

$$R(t) = \sum_{k=1}^{n}p_{k,n}\sum_{j=0}^{n-k}\binom{n}{j}[F(t)]^j[\overline{F}(t)]^{n-j} = \sum_{j=0}^{n-1}\sum_{k=1}^{n-j}p_{k,n}\binom{n}{j}[F(t)]^j[\overline{F}(t)]^{n-j}$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{j}p_{k,n}\binom{n}{j}[\overline{F}(t)]^j[F(t)]^{n-j}, \tag{11}$$

Table 2
MTTF $\mu_{n,p}$ and its approximation $\widetilde{\mu}_{n,p}$ when $F(t)=1-\exp(-t)$, $n=100$ and $E\{K\}=\omega$.

ω	θ	$\mu_{n,p}$	$\widetilde{\theta}$	$\widetilde{\mu}_{n,p}$
1	0.000	5.187	0	5.187
2	1.000	4.391	1	4.391
5	4.000	3.220	4	3.220
7	6.000	2.818	6	2.818
10	9.000	2.413	9	2.413
20	19.00	1.666	19	1.666
30	29.00	1.243	29	1.243
40	39.00	0.947	39	0.947

and the MTTF is:

$$\mu_{n,K}=\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}p_{k,n}\binom{n}{j}\int_0^\infty [F(t)]^j[\overline{F}(t)]^{n-j}\mathrm{d}t$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{j}p_{k,n}\binom{n}{j}\int_0^\infty [\overline{F}(t)]^j[F(t)]^{n-j}\mathrm{d}t\,. \tag{12}$$

Note that when K is a uniform distribution *i.e.*, $p_{k,n}=1/n$ $(k=1,2,\dots,n)$, $R(t)=\overline{F}(t)$ and $\mu_{n,K}=\mu$, which correspond to reliability and MTTF of the one unit system.

In particular, when $F(t)=1-\mathrm{e}^{-\lambda t}$,

$$\mu_{n,K}=\frac{1}{\lambda}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j}p_{k,n}=\frac{1}{\lambda}\sum_{k=1}^{n}p_{k,n}\sum_{j=k}^{n}\frac{1}{j}\,. \tag{13}$$

Clearly, from (4), MTTF is approximately given by:

$$\widetilde{\mu}_{n,K}=\frac{1}{\lambda}\sum_{k=1}^{n}p_{k,n}\ln\left(\frac{n}{k-1}\right), \tag{14}$$

for large n.

We next consider the following case of a truncated Poisson function for $p_{k,n}$: When $p_{k,n}=[\theta^{k-1}/(k-1)!]/\sum_{i=0}^{n-1}(\theta^i/i!)$ $(k=1,2,\dots,n)$ $(0<\theta<\infty)$ with:

$$E\{K\}=\frac{\theta\sum_{i=0}^{n-2}(\theta^i/i!)}{\sum_{i=0}^{n-1}(\theta^i/i!)}+1,$$

where $\sum_{i=0}^{-1}\equiv 0$, MTTF is:

$$\mu_{n,p}=\frac{1}{\lambda}\sum_{j=1}^{n}\frac{1}{j}\frac{\sum_{k=0}^{j-1}(\theta^k/k!)}{\sum_{k=0}^{n-1}(\theta^k/k!)}, \tag{15}$$

which decreases strictly with θ to $1/\lambda$.

For large n, $p_{k,n}=[\theta^{k-1}/(k-1)!]\mathrm{e}^{-\theta}$ with mean $\theta+1$, *i.e.*, $\sum_{i=0}^{n-1}(\theta^i/i!)\mathrm{e}^{-\theta}\to 1$, and;

$$\widetilde{\mu}_{n,p}=\frac{1}{\lambda}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=0}^{j-1}\frac{\theta^k}{k!}\mathrm{e}^{-\theta}. \tag{16}$$

Table 2 presents $\mu_{n,p}$ in (15) and its approximation $\widetilde{\mu}_{n,p}$ in (16) when $F(t)=1-\mathrm{e}^{-t}$, $n=100$ and $E\{K\}=\omega$. The MTTF $\mu_{n,p}$ and its approximation $\widetilde{\mu}_{n,p}$ can be calculated as follows: (i) Compute θ which satisfies:

$$\frac{\theta\sum_{i=0}^{n-2}(\theta^i/i!)}{\sum_{i=0}^{n-1}(\theta^i/i!)}+1=\omega.$$

Using θ, compute $\mu_{n,p}$ in (15). (ii) $\widetilde{\mu}_{n,p}$ in (16) is calculated when $\widetilde{\theta}+1=\omega$.

4. Optimal number of units

When K has a probability function $p_{k,n}$ $(k=1,2,\dots,n)$ and $F(t)=1-\mathrm{e}^{-\lambda t}$, the expected cost rate is, from (7) and (13),

$$\frac{C(n,p)}{\lambda}=\frac{nc_1+c_R}{\sum_{j=1}^{n}(1/j)\sum_{k=1}^{j}p_{k,n}}\qquad(n=1,2,\dots). \tag{17}$$

From the inequality $C(n+1,p)-C(n,p)\geq 0$,

$$(n+1)\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j}p_k-n\geq\frac{c_R}{c_1}\quad(n=1,2,\dots).\tag{18}$$

We find an optimal n_p^* which minimizes $C(n, p)$ in (17) when $p_{k,n}=[\theta^{k-1}/(k-1)!]/\sum_{i=0}^{n-1}(\theta^i/i!)\quad(k=1,2,\dots,n)$.

Theorem 2. *There exists a finite and minimum n_p^* $(1\leq n_p^*<\infty)$ which satisfies:*

$$\frac{n+1}{\sum_{i=0}^{n-1}(\theta^i/i!)}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=0}^{j-1}\frac{\theta^k}{k!}-n\geq\frac{c_R}{c_1}.\tag{19}$$

Proof. When $p_{k,n}=[\theta^{k-1}/(k-1)!]/\sum_{i=0}^{n-1}(\theta^i/i!)\quad(k=1,2,\dots,n)$, the expected cost rate in (17) is:

$$\frac{C(n,p)}{\lambda}=\frac{(nc_1+c_R)\sum_{i=0}^{n-1}(\theta^i/i!)}{\sum_{j=1}^{n}(1/j)\sum_{k=0}^{j-1}(\theta^k/k!)},\tag{20}$$

and from (18), we obtain (19). □

For any large $n>n_0$,

$$\begin{aligned}&\frac{n+1}{\sum_{i=0}^{n-1}(\theta^i/i!)}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=0}^{j-1}\frac{\theta^k}{k!}-n-\frac{n+1}{\sum_{i=0}^{n-1}(\theta^i/i!)}\sum_{j=1}^{n_0}\frac{1}{j}\sum_{k=0}^{j-1}\frac{\theta^k}{k!}+n_0\\&=\frac{n+1}{\sum_{i=0}^{n-1}(\theta^i/i!)}\sum_{j=n_0+1}^{n}\left(\frac{1}{j}\sum_{k=0}^{j-1}\frac{\theta^k}{k!}-\frac{1}{n+1}\sum_{i=0}^{n-1}\frac{\theta^i}{i!}\right)>0,\end{aligned}$$

as for large j,

$$\sum_{k=0}^{j-1}\left(\frac{\theta^k}{k!}-\frac{\theta^j}{j!}\right)>0.$$

Thus, the left-hand side of (19) goes to ∞ as $n\to\infty$, and there exists optimal n_p^* $(1\leq n_p^*<\infty)$ which satisfies (19).

When n is large, *i.e.*, $\sum_{i=0}^{n-1}(\theta^i/i!)e^{-\theta}\approx 1$, MTTF is given in (16), and the asymptotic expected cost rate is:

$$\frac{\widetilde{C}(n,p)}{\lambda}=\frac{(nc_1+c_R)}{\sum_{j=1}^{n}(1/j)\sum_{k=0}^{j-1}(\theta^k/k!)e^{-\theta}}.\tag{21}$$

From the inequality $\widetilde{C}(n+1,p)-\widetilde{C}(n,p)\geq 0$,

$$(n+1)\sum_{j=1}^{n}\frac{1}{j}\sum_{k=0}^{j-1}\frac{\theta^k}{k!}e^{-\theta}-n\geq\frac{c_R}{c_1},\tag{22}$$

where left-hand side increases strictly with n from $2e^{-\theta}-1$ to ∞. Thus, there exists a finite and unique $\widetilde{n}_p$ $(1\leq\widetilde{n}_p<\infty)$ which satisfies (22). Clearly, note that $2e^{-\theta}-1<0$ for $\theta>1$.

Table 3 presents optimal n^* in (8), n_p^* which satisfies (19) and $\widetilde{n}_p$ which satisfies (22) when $\omega=\theta+1$. In addition, compared (22) with (19), $\widetilde{n}_p\geq n_p^*$ This indicates that n^*, n_p^* and $\widetilde{n}_p$ increase with ω and c_R/c_1 and are almost the same.

5. Optimal replacement policy

When K has a probability function $p_{k,n}$ $(k=1,2,\dots,n)$, the expected cost rate in (9) is:

$$C(T,p)=\frac{nc_1+c_R\sum_{k=1}^{n}p_{k,n}\sum_{j=0}^{k-1}\binom{n}{j}[\overline{F}(T)]^j[F(T)]^{n-j}}{\sum_{k=1}^{n}p_{k,n}\sum_{j=k}^{n}\binom{n}{j}\int_0^T[\overline{F}(t)]^j[F(t)]^{n-j}dt}.\tag{23}$$

Differentiating $C(T, p)$ with respect to T and setting it equal to zero,

$$\begin{aligned}&Q(T,p)\sum_{k=1}^{n}p_{k,n}\sum_{j=k}^{n}\binom{n}{j}\int_0^T[\overline{F}(t)]^j[F(t)]^{n-j}dt\\&-\sum_{k=1}^{n}p_{k,n}\sum_{j=0}^{k-1}\binom{n}{j}[\overline{F}(T)]^j[F(T)]^{n-j}=\frac{nc_1}{c_R}.\end{aligned}\tag{24}$$

Table 3
Optimal n^* in (8), n_p^* in (19) and $\widetilde{n}_p$ in (22) when $F(t)=1-\mathrm{e}^{-t}$, $\omega=\theta+1$ and $c_R/c_1=50, 100$.

ω	n^*	$\frac{C(n^*)}{c_1}$	n_p^*	$\frac{C(n_p^*)}{c_1}$	$\widetilde{n}_p$	$\frac{C(\widetilde{n}_p)}{c_1}$
$c_R/c_1=50$						
1	19	19.45	19	19.45	19	19.45
2	26	26.63	24	24.84	24	24.84
5	40	41.00	38	38.93	38	38.93
7	48	48.79	46	46.89	46	46.89
10	59	59.43	57	57.70	57	57.70
20	91	91.21	89	89.69	89	89.69
30	120	120.81	119	119.35	119	119.35
40	149	149.51	148	148.09	148	148.09
$c_R/c_1=100$						
ω	n^*	$\frac{C(n^*)}{c_1}$	n_p^*	$\frac{C(n_p^*)}{c_1}$	$\widetilde{n}_p$	$\frac{C(\widetilde{n}_p)}{c_1}$
1	32	32.52	32	32.52	32	32.52
2	42	42.68	40	40.21	40	40.21
5	61	61.62	58	58.98	58	58.98
7	71	71.34	69	69.00	69	69.00
10	84	84.21	82	82.15	82	82.15
20	120	120.80	119	119.09	119	119.09
30	153	153.40	151	151.82	151	151.82
40	184	184.26	182	182.74	182	182.74

where

$$Q(T,p)\equiv\frac{nh(T)\sum_{k=1}^{n}p_{k,n}\binom{n-1}{k-1}[\overline{F}(T)]^k[F(T)]^{n-k}}{\sum_{k=1}^{n}p_{k,n}\sum_{j=k}^{n}\binom{n}{j}[\overline{F}(T)]^j[F(T)]^{n-j}}.$$

We find an optimal T_p^* which minimizes $C(T, p)$ in (23) when $p_{k,n}=[\theta^{k-1}/(k-1)!]/\sum_{i=0}^{n-1}(\theta^i/i!)$ $(k=1,2,\ldots,n)$ and $F(t)=1-\mathrm{e}^{-\lambda t}$.

Theorem 3. *If*

$$\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\left(\sum_{j=k+1}^{n}\frac{1}{j}-1\right)>\frac{nc_1}{c_R},$$

then there exists an finite T_p^* $(0<T_p^*<\infty)$ which satisfies:

$$\frac{\lambda\sum_{k=0}^{n-1}[(k+1)\theta^k/k!]H_{k+1}(T)}{\sum_{k=0}^{n-1}(\theta^k/k!)\sum_{j=k+1}^{n}H_j(T)}\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\sum_{j=k+1}^{n}\int_0^T H_j(t)\mathrm{d}t$$
$$-\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\sum_{j=0}^{k}H_j(T)=\frac{nc_1}{c_R}\sum_{k=0}^{n-1}\frac{\theta^k}{k!},\tag{25}$$

where

$$H_j(T)\equiv\binom{n}{j}\left(\mathrm{e}^{-\lambda T}\right)^j\left(1-\mathrm{e}^{-\lambda T}\right)^{n-j}\quad(j=0,1,\ldots,n).$$

Proof. When $p_{k,n}=[\theta^{k-1}/(k-1)!]/\sum_{i=0}^{n-1}(\theta^i/i!)$ $(k=1,2,\ldots,n)$ and $F(t)=1-\mathrm{e}^{-\lambda t}$, the expected cost rate in (23) is:

$$C(T,p)=\frac{nc_1\sum_{i=0}^{n-1}(\theta^i/i!)+c_R\sum_{k=0}^{n-1}(\theta^k/k!)\sum_{j=0}^{k}H_j(T)}{\sum_{k=0}^{n-1}(\theta^k/k!)\sum_{j=k+1}^{n}\int_0^T H_j(t)\mathrm{d}t},\tag{26}$$

and from (24), we obtain (25). □

Let $L(T)$ be the left-hand side of (25). If,

$$\frac{\sum_{k=0}^{n-1}[(k+1)\theta^k/k!]H_{k+1}(T)}{\sum_{k=0}^{n-1}(\theta^k/k!)\sum_{j=k+1}^{n}H_j(T)}$$

Table 4
Optimal T^* in (10), T_p^* in (25) and $\widetilde{T}_p$ in (27) when $F(t) = 1 - e^{-t}$ and $c_R/c_1 = 50, 75, 100$.

ω	T^*	$\frac{C(T^*)}{c_1}$	T_p^*	$\frac{C(T_p^*)}{c_1}$	$\widetilde{T}_p$	$\frac{C(\widetilde{T}_p)}{c_1}$
$c_R/c_1 = 50$						
1	4.48	26.9	4.48	26.9	4.50	27.2
2	3.59	31.8	3.78	32.3	3.80	32.7
5	2.68	41.1	2.74	42.6	2.76	43.3
7	2.37	46.2	2.41	48.0	2.43	49.0
10	2.05	53.1	2.08	55.4	2.09	56.9
20	1.43	75.8	1.45	79.9	1.47	83.0
30	1.07	101.8	1.09	108.1	1.12	113.9
40	0.81	134.7	0.85	144.2	0.89	154.5
$c_R/c_1 = 100$						
ω	T^*	$\frac{C(T^*)}{c_1}$	T_p^*	$\frac{C(T_p^*)}{c_1}$	$\widetilde{T}_p$	$\frac{C(\widetilde{T}_p)}{c_1}$
1	3.87	29.3	3.90	29.3	3.87	29.7
2	3.25	34.0	3.23	35.2	3.24	35.7
5	2.49	43.4	2.44	45.8	2.44	46.9
7	2.22	48.5	2.16	51.1	2.16	52.9
10	1.93	55.7	1.87	59.5	1.87	61.4
20	1.35	79.4	1.30	85.9	1.31	90.4
30	1.01	106.7	0.97	117.2	0.98	125.8
40	0.76	141.7	0.74	158.4	0.75	174.1

increases strictly with T, then $L(T)$ also increases strictly with T from 0 to:

$$L(\infty) = \lim_{T\to\infty} L(T) = \sum_{k=0}^{n-1} \frac{\theta^k}{k!}\left(\sum_{j=k+1}^{n} \frac{1}{j} - 1\right).$$

Therefore, if $L(\infty) > nc_1/c_R$, then there exists an optimal T_p^* $(0 < T_p^* < \infty)$ which satisfies (25).

For large n, when $p_k = [\theta^{k-1}/(k-1)!]e^{-\theta}$, from (25), approximate $\widetilde{T}_p$ satisfies:

$$\begin{aligned} &\frac{\lambda \sum_{k=0}^{n-1}[(k+1)\theta^k/k!]\widetilde{H}_{k+1}(T)}{\sum_{k=0}^{n-1}(\theta^k/k!)\sum_{j=k+1}^{n}\widetilde{H}_j(T)} \sum_{k=0}^{n-1}\frac{\theta^k}{k!}e^{-\theta}\sum_{j=k+1}^{n}\int_0^T \widetilde{H}_j(t)dt \\ &-\sum_{k=0}^{n-1}\frac{\theta^k}{k!}e^{-\theta}\sum_{j=0}^{k}\widetilde{H}_j(T) = \frac{nc_1}{c_R}, \end{aligned} \tag{27}$$

where

$$\widetilde{H}_j(T) \equiv \frac{(ne^{-\lambda T})^j}{j!}\exp(-ne^{-\lambda T}) \quad (j = 0, 1, 2, \ldots).$$

Table 4 presents optimal T^* in (10), T_p^* in (25) and its approximate value $\widetilde{T}_p$ in (27) when $n = 100$, $c_R/c_1 = 50, 100$, and $F(t) = 1 - e^{-t}$, which are computed as follows: (i) Optimal T^* satisfies:

$$\begin{aligned} &\frac{n\binom{n-1}{K-1}}{\sum_{j=0}^{K-1}\binom{n}{j}(e^T-1)^{K-j}}\sum_{j=K}^{n}\binom{n}{j}\int_0^T (e^{-t})^j(1-e^{-t})^{n-j}dt \\ &-\sum_{j=0}^{K-1}\binom{n}{j}(e^{-T})^j(1-e^{-T})^{n-j} = \frac{nc_1}{c_R}. \end{aligned}$$

(ii) Optimal T_p^* is computed as: First, compute θ that satisfies:

$$\frac{\theta\sum_{i=0}^{n-2}(\theta^i/i!)}{\sum_{i=0}^{n-1}(\theta^i/i!)} + 1 = \omega,$$

and when $\omega = 1$, $\theta = 0$. When $n = 100$, from (25), then compute T_p^* that satisfies:

$$\frac{\sum_{k=0}^{n-1}\frac{(k+1)\theta^k}{k!}H_{k+1}(T)}{\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\sum_{j=k+1}^{n}H_j(T)}\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\sum_{j=k+1}^{n}\int_0^T H_j(t)\mathrm{d}t$$
$$-\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\sum_{j=0}^{k}H_j(T)=\frac{nc_1}{c_R}\sum_{k=0}^{n-1}\frac{\theta^k}{k!}.$$

(iii) Approximation $\widetilde{T}_p$ is computed as: Compute $\theta=\omega-1$, and from (27), when $n=100$ and $\widetilde{H}_j(T)=[(ne^{-T})^j/j!]\exp(-ne^{-T})$, $\widetilde{T}_p$ satisfies:

$$\frac{\sum_{k=0}^{n-1}\frac{(k+1)\theta^k}{k!}\widetilde{H}_{k+1}(T)}{\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\sum_{j=k+1}^{n}\widetilde{H}_j(T)}\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\mathrm{e}^{-\theta}\sum_{j=k+1}^{n}\int_0^T \widetilde{H}_j(t)\mathrm{d}t$$
$$-\sum_{k=0}^{n-1}\frac{\theta^k}{k!}\mathrm{e}^{-\theta}\sum_{j=0}^{k}\widetilde{H}_j(T)=\frac{nc_1}{c_R}.$$

6. Conclusions

A K-out-of-n system with a stochastic parameter K has been modeled in this paper. When K is given as constant numbers, discussion of the system is a conventional way that may not meet the real requirement, and when K is estimated as random variables, we take up the system in an innovative way. We have obtained the system's reliability and MTTF for a K-out-of-n system in two cases when K are defined constantly and randomly. The number n of units that should be provided for replacement and replacement time T of operation have also been optimized, respectively. To make the computations easier, we have given approximate methods to compute MTTF, number n^*, and time T^*, respectively.

Obviously, it is a practical problem to consider a random K-out-of-n system when designing and maintaining aircraft systems, as indicated in Introduction. As future works, we may consider for a large K-out-of-n system that the total number n of units is also a random variable with some probability distribution [30] and the units have different failure probabilities and their failures have unexpected interdependences.

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