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Axiomatic characterizations of the constrained probabilistic serial mechanism

Mustafa Oğuz Afacan[1]

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Abstract

Afacan (Games and Economic Behavior 110: 71-89, 2018) introduces an object allocation with random priorities problem. He proposes the constrained probabilistic serial (*CPS*) mechanism. This study, for the first time in the literature, provides axiomatic characterizations of *CPS*. The first result characterizes it via non-wastefulness, claimwise stability, constrained ordinal fairness, and surplus invariance to truncations. The other axiomatizes *CPS* via constrained stochastic efficiency, claimwise stability, and constrained ordinal fairness. The independence of the axioms is provided.

Keywords Matching · Constrained probabilistic serial · Characterization · Axiom · Claimwise stability · Constrained ordinal fairness

1 Introduction

Afacan (2018) considers an object allocation problem where the objects' priorities are random. Besides its richer theoretical scope, it has practical appeal. In many real-life object allocation problems, priorities come with large indifference classes.[1] Ties are first randomly broken, and then the outcome is calculated based on the obtained strict priorities. This, in turn, means that priorities are random ex-ante, thereby falling into Afacan (2018)'s setting.[2]

[1] For instance, the Boston and the New York City school districts—the two largest school placement organizations in the U.S.—are two examples of such problems (see Abdulkadiroğlu et al. (2009)).

[2] In practice, uniform tie-breaking rules are commonly used. However, non-uniform tie-breaking has useful applications, as discussed in detail in Afacan (2018).

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✉ Mustafa Oğuz Afacan
mafacan@qu.edu.qa

1 Department of Finance and Economics, College of Business and Economics, Qatar University, P.O. Box 2713, Doha, Qatar

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Afacan (2018) proposes a fairness notion, so-called "*claimwise stability*". He then introduces the *constrained probabilistic serial* (*CPS*) mechanism. It is mainly built on the well-known "probabilistic serial" (*PS*) mechanism of Bogomolnaia and Moulin (2001). Informally speaking, *CPS* lets agents consume objects in decreasing order of their preferences until either relevant claimwise stability constraints start binding or objects are exhausted (whichever occurs first). Afacan (2018) shows that *CPS* is claimwise stable and constrained stochastically efficient (henceforth, constrained sd-efficient) in the sense that its outcome is never stochastically dominated by another claimwise stable matching.[3][4] No claimwise stable and constrained sd-efficient mechanism is strategy proof; hence, in particular, *CPS* is not strategy-proof.[5] Afacan (2018) also shows that *CPS* satisfies a weaker version of equal treatment of equals.

In this study, we provide two characterizations of CPS: (*i*) A mechanism is *non-wasteful*, *claimwise stable*, *constrained ordinally fair*, and *surplus-invariant to truncations* if and only if it is *CPS*, and (*ii*) A mechanism is *constrained sd-efficient*, *claimwise stable*, and *constrained ordinally fair* if and only if it is *CPS*. We also obtain the independence of the axioms.

Non-wastefulness is a standard property that requires all preferred objects to be exhausted. *Claimwise stability* is a fairness property, motivated by both the usual fairness notion in object allocations and the proportional allocation principle in claim problems. In terms of the latter, claimwise stability advocates that objects' shares should be distributed in proportion to the probabilistic priorities. On the other hand, if we think of objects as perfectly divisible and interpret the assigned probabilities as the time-shares for which agents consume them, then claimwise stability rules out any time-share in which a pair of agents compete for the same object, but the lower priority one consumes it. In other words, it requires the usual fairness[6] over the whole consumption process (Bogomolnaia and Heo (2012)), where agents acquire objects' shares over the unit-time interval at the speed of one.

Constrained ordinal fairness is an adaptation of the ordinal fairness property of Kesten et al. (2011) and Hashimoto et al. (2014), which is used in their *PS* characterization, to our framework. Given a matching, let us call the agent *i*'s total share of the objects that are at least as good as object *a* "*surplus of agent i at object a*." If agent *i*'s surplus at object *a* is less than that of agent *j* while the latter obtains a positive share of object *a*, then constrained ordinal fairness guarantees that some claimwise stability constraint that is imposed on agent *i* for object *a* is binding at the matching. Hence, giving more of object *a* to agent *i* would violate claimwise

[3] A matching stochastically dominates (sd-dominates) another matching if each agent either unambiguously prefers the former to the latter or receives the same assignment under both, with the former holding for some agent.

[4] *CPS*'s outcome can be sd-dominated by a matching that is not claimwise stable. In other words, *CPS* is not sd-efficient. However, this is not a problem specific to *CPS* as there is a general incompatibility between claimwise stability and stochastic efficiency (see Afacan (2018)).

[5] A mechanism is strategy-proof if no agent ever benefits from misreporting his preferences.

[6] In standard object allocation problems, fairness eliminates matchings where an agent envies someone else while the former has a higher priority than the latter at the latter's assigned object (Gale and Shapley (1962)).

stability.[7] Therefore, constrained ordinal fairness can be deemed a fairness property in terms of surpluses but subject to claimwise stability. The last axiom—*surplus invariance to truncations*—is a mild invariance axiom. It imposes that after an agent truncates his preferences below an object a, his total share from the objects that are at least as good as object a remains the same. More formal motivations of the axioms are provided in the model.

This study is the first to offer an axiomatization of *CPS*. The most relevant works are the existing *PS* characterizations. Bogomolnaia and Moulin (2001) axiomatize *PS* for three-agent economies.[8] Then *PS* has been characterized for any number of agents by Kesten et al. (2011), Hashimoto and Hirata (2011), Bogomolnaia and Heo (2012), and Hashimoto et al. (2014). Besides, a recent related study is Balbuzanov (2022) where the author proposes a "*generalized constrained probabilistic serial mechanism*" (*GCPS*) to accommodate constraints over deterministic assignments into (random) matchings. In a priority-free setting, Balbuzanov (2022) characterizes constraints as upper-bound inequalities. *GCPS* works as *PS* except agents stop consuming an object whenever the relevant upper-bound constraints start binding. While our mechanisms *CPS* and *GCPS* have similar logic, the former addresses the claimwise stability constraints, which are not captured in Balbuzanov (2022).

2 The model

There are finite sets of agents N and objects $\hat{O}$. There is also a null-object, denoted by $\emptyset$. It represents receiving no object. Each agent $i \in N$ has a preference relation R_i, which is a complete, transitive, and antisymmetric binary relation over $\hat{O} \cup \{\emptyset\}$. For its asymmetric part, we write P_i, which is defined as $a\ P_i\ b$ whenever $a\ R_i\ b$ and $a \neq b$. Let $\mathcal{R}$ be the set of all preference relations. Object $a \in \hat{O}$ is acceptable to agent i if $a\ P_i\ \emptyset$, and otherwise, it is unacceptable. We write q_a for the number object a copies in the problem. In the rest of the paper, for ease of exposition, we assume that $q_a = 1$ for each $a \in \hat{O}$. However, as we discuss in Remark 7, the whole analysis carries over to the multi-copy case. The null-object is not scarce, that is, $q_\emptyset = |N|$.

For object $a \in \hat{O}$, we write $\succ_a$ for its strict priority ordering over N. Let $\succ = (\succ_a)_{a \in \hat{O}}$ and ζ be the priority profile and the set of all such profiles, respectively. By going beyond these priorities, we let the objects have a random priority profile Δ, which is a probability distribution over ζ. Let $\Delta(\succ)$ be the probability of $\succ = (\succ_a)_{a \in \hat{O}}$ under Δ. We write Δ_a for the (marginal) priority distribution of object $a \in \hat{O}$ under Δ.[9] There is no restriction on Δ; hence, the objects' priorities may be independent as well as correlated. We write $supp(\Delta) = \{\succ \in \zeta : \ \Delta(\succ) > 0\}$ for the support of Δ. We define $Pr_\Delta(i \rhd_a j) = \sum_{\succ \in \zeta:\ i \succ_a j} \Delta(\succ)$. In words, it is the probability

[7] Ordinal fairness eliminates cases where someone has a higher surplus than someone else at an object, while the former also receives a positive share of the object.

[8] Kesten et al. (2017) show that Bogomolnaia and Moulin (2001)'s axiomatization does not hold whenever the number of agents is at least five.

[9] That is, for any $\succ'_a$, $\Delta_a(\succ'_a) = \sum_{\succ \in \zeta:\ \succ_a = \succ'_a} \Delta(\succ)$.

that agent i has a higher priority than agent j at object a. In the rest of the paper, we fix the set of agents and objects and write (R, Δ) for the problem.

For ease of writing, let $O = \hat{O} \cup \{\emptyset\}$. A matching $\sigma = [\sigma_{i,a}]_{i \in N, a \in O}$ is a matrix such that (i) for each $i \in N$ and $a \in O$, $0 \leq \sigma_{i,a} \leq 1$, ($ii$) $\sum_{a \in O} \sigma_{i,a} = 1$, and ($iii$) for each $c \in \hat{O}$, $\sum_{i \in N} \sigma_{i,c} \leq 1$. Here, $\sigma_{i,a}$ represents the probability that agent i is matched with object a. Let $\sigma_i = [\sigma_{i,a}]_{a \in O}$ denote the matching of agent i. A matching σ is deterministic if $\sigma_{i,a} \in \{0, 1\}$ for each $i \in N$ and $a \in O$. Let $\mathcal{X}$ be the set of all matchings. We write $\mathcal{M}$ for the proper subset of $\mathcal{X}$ that consists of the deterministic matchings.

A probability distribution λ over $\mathcal{M}$ is called a lottery. Formally, $\lambda = (\lambda_\mu)_{\mu \in \mathcal{M}}$ is such that for each $\mu \in \mathcal{M}$, $0 \leq \lambda_\mu \leq 1$ and $\sum_{\mu \in \mathcal{M}} \lambda_\mu = 1$. We write σ^λ for the matching induced by λ, that is, for each agent–object pair (i, a), $\sigma^\lambda_{i,a} = \sum_{\mu \in \mathcal{M}:\ \mu_i = a} \lambda_\mu$.

Fact 1 (Birkoff-von Neumann and Kojima and Manea (2010)) Every matching is induced by a lottery λ over $\mathcal{M}$.

Because of the above well-known fact, in the rest of the paper, we consider matchings instead of lotteries. A mechanism ψ is a systematic procedure that produces a matching for each problem (R, Δ). We write $\psi(R, \Delta)$ for the outcome of ψ in problem (R, Δ).

2.1 The axioms

For an agent i and object a, let $SU(R_i, a) = \{c \in O :\ c\ P_i\ a\}$ (the strict upper contour set of agent i at object a), and $U(R_i, a) = SU(R_i, a) \cup \{a\}$ (the upper contour set of agent i at object a).

A matching σ is non-wasteful if, for any objects $a, b \in O$ and agent i, $a\ P_i\ b$ and $\sigma_{i,b} > 0$, then $\sum_{j \in N} \sigma_{j,a} = q_a$. Note that non-wastefulness implies individual rationality.[10] A matching σ induces a justified claim if there are a pair of agents i and j and object $a \in \hat{O}$ such that $\sigma_{j,a} > Pr_\Delta(j \triangleright_a i) + \sum_{c \in SU(R_i, a)} \sigma_{i,c}$. A matching is claimwise stable if it does not induce a justified claim.

To better understand the spirit of claimwise stability, by following Afacan (2018), let us utilize the consumption process representation of matchings by Bogomolnaia and Heo (2012). Each matching can be deemed as the outcome of a consumption process where, over the unit-time interval, agents continuously acquire objects' shares at the speed of one and in decreasing order of their preferences. Agents' attained shares are interpreted as time-shares during which they consume them in the consumption process.

Accompanied by Fig. 1, consider a pair of agents i, j, object a, and a matching σ. We can interpret $Pr_\Delta(j \triangleright_a i)$—orange region $R.1$—as the time-share in which agent j has a higher priority than agent i for object a. Similarly, $\sum_{c \in SU(R_i, a)} \sigma_{i,c}$—green region $R.2$—is the time-share in which agent i consumes objects that are better than

[10] A matching σ is individually rational if there are no agent i and object a such that $\emptyset\ P_i\ a$ and $\sigma_{i,a} > 0$.

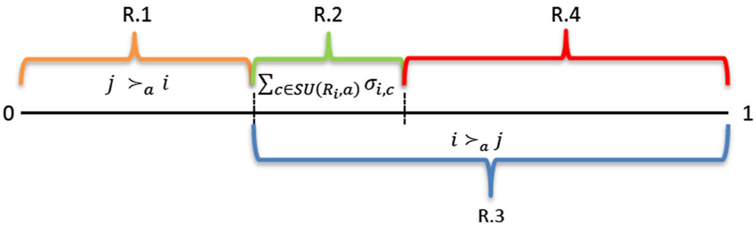

Fig. 1 Consumption process motivation of claimwise stability

object a. Therefore, it is unarguably fair to let agent j consume object a over these regions, which amounts to $Pr_\Delta(j \triangleright_a i) + \sum_{c \in SU(R_i,a)} \sigma_{i,c}$. However, once agent j's share exceeds it (more than $R.1+R.2$), he consumes object a during some time-share falling into the red region $R.4$. This is problematic on the fairness ground because over $R.4$, both agents i and j compete for object a (note that agent i would prefer consuming object a outside $R.2$), and agent i has a higher priority than agent j ($R.3$, the time-share in which agent i has a higher priority than agent j, includes $R.4$). That is, the standard fairness is violated for some time-share in $R.4$.

To make the motivation behind claimwise stability more concrete, let us consider a problem with three agents i, j, k, and two objects a, b. Let $Pr_\Delta(k \succ_a i) = 3/4$ and $Pr_\Delta(i \succ_b j) = 1$. Let the preferences be such that $P_i: \ a, b, \emptyset$; $P_j: \ b, \emptyset$; and $P_k: \ a, \emptyset$. Let us consider a matching σ given in the table below.

	a	b	∅
i	1/4	1/2	1/4
j	0	1/2	1/2
k	3/4	0	1/4

Here, $\sigma_{j,b} \ (= 1/2) > Pr_\Delta(j \succ_b i)(= 0) + \sum_{c \in SU(R_i,b)} \sigma_{i,c}(= 1/4)$, constituting a violation of claimwise stability (there is no other claimwise stability violation). Agent i desires to consume object b after having done with acquiring object a's share. As $\sigma_{i,a} = 1/4$, during a time-share of 3/4, agent i demands object b. On the other hand, $\sigma_{j,b} > 1/4$ means that in some positive time-share during which agent i demands object b, agent j manages to consume object b (note that this time-share has the mass of $\sigma_{j,b} - 1/4 > 0$). Moreover, in this time-share, agent i has a higher priority than agent j at object b.[11] This goes against the spirit of fairness that the higher priority agents should be treated favorably in receiving objects.

[11] Indeed, agent i has a higher priority than agent j at object b over the whole consumption process as $Pr_\Delta(i \succ_b j) = 1$.

Remark 1 Claimwise stability considers agent i's acquired shares of the objects that are better than object a. This is because agent i demands object a once he finishes consuming these better objects. That is, his object a demand is in the amount of $1-\sum_{c\in SU(R_i,a)}\sigma_{i,c}$. On the other hand, if we considered the agent i's object a share as well in calculating his demand, we would have underestimated the agent i's demand for object a (that is, $1-\sum_{c\in U(R_i,a)}\sigma_{i,c}$ might be less than the actual agent i's object a demand, which is $1-\sum_{c\in SU(R_i,a)}\sigma_{i,c}$). Thus, agent j would have been allowed to consume more than what is needed to ensure fairness over the whole consumption process.

Remark 2 Our model features randomness both in priorities and assignments. That is, it exhibits double uncertainty. This makes it hard to relate ex-post and ex-ante properties. Neither ex-post fairness nor claimwise stability does not easily imply the other. To see this, let us consider two agents i, j and one object a. Suppose each agent has a higher priority with the probability of 1/2. The only claimwise stable matching assigns 1/2 of object a to each agent. This matching is not ex-post fair irrespective of the priority realization (either of the agents has a higher priority ex-post, yet in the matching's lottery decomposition, it is always the case that the (ex-post) lower priority agent receives object a with the probability of 1/2). Similarly, any ex-post fair matching assigns the whole share to the (ex-post) higher priority agent. This assignment cannot be claimwise stable as each agent must receive 1/2 of object a under claimwise stability.

For a matching σ, agent i, and object a, let $F(R_i,a,\sigma_i)=\sum_{c\in U(R_i,a)}\sigma_{i,c}$. Matching σ stochastically (sd) dominates σ' if, for each agent i and object a, $F(R_i,a,\sigma_i)\geq F(R_i,a,\sigma'_i)$, where this strictly holds for some agent–object pair. Matching σ is sd-efficient if it is not sd-dominated. Afacan (2018) shows that there does not always exist a claimwise stable and sd-efficient matching. Thus, matching σ is constrained sd-efficient if it is not sd-dominated by a claimwise stable matching.

While constrained sd-efficiency is weaker than sd-efficiency, it still implies non-wastefulness, as formally stated below. All the proofs are relegated to Appendices.

Proposition 1 *In any problem (R,Δ), a constrained sd-efficient matching is non-wasteful.*

Remark 3 To see how constrained sd-efficiency implies non-wastefulness, suppose that object a is wasted at a constrained sd-efficient matching σ. This means that some agent, say j, would rather consume more of object a; yet this would violate claimwise stability (as, otherwise, σ could not have been constrained sd-efficient). That is, for some agent $i\neq j$, $\sigma_{j,a}=Pr_\Delta(j\succ_a i)+\sum_{c\in SU(R_i,a)}\sigma_{i,c}$. This implies that $\sigma_{i,a}<1-\sum_{c\in SU(R_i,a)}\sigma_{i,c}$.[12] Therefore, agent i also prefers to consume more of object a. However, for the same reason above, it would go against claimwise stability. This means that $\sigma_{i,a}=Pr_\Delta(i\succ_a k)+\sum_{c\in SU(R_k,a)}\sigma_{k,c}$ for some agent $k\neq i$.

[12] To see this inequality, assume for a contradiction that $\sigma_{i,a}\geq 1-\sum_{c\in SU(R_i,a)}\sigma_{i,c}$. Then, $\sigma_{j,a}+\sigma_{i,a}\geq Pr_\Delta(j\succ_a i)+\sum_{c\in SU(R_i,a)}\sigma_{i,c}+(1-\sum_{c\in SU(R_i,a)}\sigma_{i,c})$, implying that $\sigma_{j,a}+\sigma_{i,a}\geq 1$. This, however, contradicts our supposition that object a is wasted at σ.

Note that as $Pr_\Delta(i \succ_a j) + Pr_\Delta(j \succ_a i) = 1$, agent k have to be different from agent j (as, otherwise, $\sigma_{i,a} + \sigma_{j,a} = 1$, contradicting our supposition that object a is wasted). Thus, no pair of agents simultaneously prevent each other from consuming more of object a. In the proof of Proposition 1, we also observe that it holds for any collection of agents of size more than two. Thus, whenever an object is wasted at a constrained sd-efficient matching, we must have a chain of agents of the size of infinity, where each agent prevents the previous one from consuming more of a wasted object. This, however, is not possible, as there are finitely many agents.

By following Kesten et al. (2011) and Hashimoto et al. (2014), we refer to $F(R_i, a, \sigma_i)$ as "agent i's surplus at object a under σ." We adapt the ordinal fairness property of Kesten et al. (2011) and Hashimoto et al. (2014) to our setting as follows.[13]

Definition 1 In problem (R, Δ), matching σ is constrained ordinally fair if, for any pair of agents i,j, and object a, $F(R_i, a, \sigma_i) < F(R_j, a, \sigma_j)$ and $\sigma_{j,a} > 0$, then there exists an agent k such that $\sigma_{i,a} = Pr_\Delta(i \rhd_a k) + \sum_{c \in SU(R_k, a)} \sigma_{k,c}$.

In words, constrained ordinal fairness says that if agent i's surplus at object a is less than that of agent j while, at the same time, agent j receives a positive amount of object a—an ordinal fairness violation instance—then some claimwise stability constraint has to bind for agent–object pair (i, a). Therefore, giving more of object a to agent i would go against claimwise stability. In the presence of constrained ordinal fairness violation at the expense of agent i, we can increase agent i's surplus at object a (by a trade with agent j for an arbitrarily small amount of object a) while preserving claimwise stability. Hence, constrained ordinal fairness advocates equality in terms of surpluses to the extent that claimwise stability permits.

Remark 4 Ordinal fairness violation is not solely based on the object a's allotment, but on the total share of the objects that are at least weakly better than object a. Thus, it does not necessarily imply that agent i, at the expense of whom ordinal fairness is violated, receives a lesser amount of object a than agent j. However, the condition that agent j receives a positive amount of object a ensures that agent i's surplus can be increased by taking away some positive amount of object a from agent j.[14] However, under a constrained ordinally fair matching, doing that would go against claimwise stability.

Remark 5 Constrained ordinal fairness partly justifies ordinal fairness violations on the basis of allocation. That is, the condition of $\sigma_{i,a} = Pr_\Delta(i \rhd_a k) + \sum_{c \in SU(R_k, a)} \sigma_{k,c}$ depends on the agent k's share from $SU(R_k, a)$ at σ (the other term of $Pr_\Delta(i \rhd_a k)$ purely comes from the primitives). While perhaps sounds contriving at first glance, it could be supported via a myopic view. Suppose agent i objects the allocation because of the ordinal fairness violation against himself at object a. That is, $F(R_i, a, \sigma_i) < F(R_j, a, \sigma_j)$ and $\sigma_{j,a} > 0$ for some agent j. From a myopic perspective, it

[13] A matching σ is **ordinally fair** if there are no pair of agents i, j, and object a such that $F(R_i, a, \sigma_i) < F(R_j, a, \sigma_j)$ and $\sigma_{j,a} > 0$.

[14] Note that while agent i's surplus can be increased by giving more of the objects that are better than object a, agent j may not have a share from these objects.

is plausible to fix the other objects' assignments and look for a possible increase in the agent i's object a allotment, especially given the fact that agent j, for the sake of whom violation occurs, receives a positive amount of object a. Put another way, one can imagine that the assignments of all the objects except object a are taken as granted and search for a room to increase agent i's object a allotment. Constrained ordinal fairness ensures that any such increase is impossible under claimwise stability. Our approach to justifying a violation based on the allocation is common in the field.[15]

A mechanism ψ is *<non-wasteful, claimwise stable, constrained ordinally fair, constrained sd-efficient>* if, for each problem (R, Δ), $\psi(R, \Delta)$ is <non-wasteful, claimwise stable, constrained ordinally fair, constrained sd-efficient>

A preference relation R'_i is the truncation of R_i from some object $a \in \hat{O}$ if (i) $a \; P_i \; \emptyset$, (ii) for each pair of objects $c, c' \in \hat{O}$, $c \; R'_i \; c'$ if and only if $c \; R_i \; c'$; and (iii) $c \; P'_i \; \emptyset$ if and only if $c \in U(R_i, a)$.

Definition 2 A mechanism ψ is surplus-invariant to truncations if, for any problem (R, Δ) and any agent i, $F(R_i, a, \psi_i(R'_i, R_{-i}, \Delta)) = F(R_i, a, \psi_i(R, \Delta))$ where R'_i is the truncation of R_i from object a.[16][17]

The invariance axiom above requires no change in agent i's surplus at object a after he truncates his preferences from object a.[18] It is easy to see that surplus invariance to truncations is necessary for a mechanism to be strategy-proof; hence, it is desirable for the strategic concerns,[19] It is also of practical interest. In many real-life problems, there is a cap on the number of objects that can be ranked in preference lists.[20] Under such caps, it is natural for agents to truncate their preferences and submit a ranking only over their best objects. Our invariance property ensures that agents' surpluses at their last object under the truncated preferences would remain the same even in the absence of caps; thus, caps do not hurt agents in this sense.

[15] For instance, Ehlers et al. (2014) find the impossibility of obtaining the usual non-wastefulness in their setting and define constrained non-wastefulness that partly justifies the waste based on the allocation. Similarly, Troyan et al. (2020) weaken the standard fairness notion based on the whole allocation. The same approach is used by Ehlers and Morrill (2019) and Sönmez and Yenmez (2022).

[16] R_{-i} is the preference profile of all the agents except agent i.

[17] As R'_i is the truncation of R_i from object a, $F(R_i, a, \psi_i(R'_i, R_{-i}, \Delta)) = F(R_i, a, \psi_i(R, \Delta))$ if and only if $F(R'_i, a, \psi_i(R'_i, R_{-i}, \Delta)) = F(R_i, a, \psi_i(R, \Delta))$.

[18] This axiom is weak in that it allows changes in agent i's shares from the objects in $U(R_i, a)$ as long as his surplus at object a does not change. Similar axioms have been used for the characterization of *PS* or its variants in some studies, including Hashimoto and Hirata (2011), Bogomolnaia and Heo (2012), Heo (2014), Hashimoto et al. (2014), and Heo and Yılmaz (2015). The invariance properties in Hashimoto and Hirata (2011), Bogomolnaia and Heo (2012), Heo (2014), Hashimoto et al. (2014), and Heo and Yılmaz (2015) imply our invariance axiom, yet the converse is not true.

[19] A mechanism ψ is strategy-proof if, for each problem (R, Δ), agent i, and $R'_i \in \mathcal{R}$, either $\psi_i(R, \Delta)$ sd-dominates (with respect to R_i) $\psi_i(R'_i, R_{-i}, \Delta)$ or $\psi_i(R, \Delta) = \psi_i(R'_i, R_{-i}\Delta)$.

[20] For instance, students can rank at most 12, 4, and 3 schools in NYC, Rhode Island, and Cambridge, respectively (see Sönmez and Pathak (2013)).

2.2 Constrained probabilistic serial (CPS)

We relegate the formal description of *CPS* to Appendix *A*. Here, instead, we informally outline how it works and run it on a simple example. Let us consider the objects as perfectly divisible. In the course of *CPS*, over the unit-time interval and in decreasing order of their preferences, agents continuously acquire object shares at the speed of one. An agent stops consuming an object whenever the relevant claimwise stability constraints start binding, or the object is exhausted (whichever occurs first). The algorithm moves to the next step whenever an agent stops consuming an object, and it terminates when each agent has consumed a total share of one.

Below, we run *CPS* on a simple example. Let $N = \{i, j, k\}$ and $\hat{O} = \{a, b, c\}$. The preferences are given below.

R_i: $a, b, c, \emptyset$; R_j: $b, c, a, \emptyset$; R_k: $a, c, b, \emptyset$.

Let $supp(\Delta)$ contain the following deterministic priorities.

$\succ$			$\succ'$		
a	b	c	a	b	c
j	i	j	j	i	j
i	j	k	k	j	i
k	k	i	i	k	k

Let $\Delta(\succ) = 3/4$ and $\Delta(\succ') = 1/4$. Below demonstrates the working of *CPS* on the example.

Step 1. Each agent first attempts to consume his favorite object. The best object of both agents i and k is object a and $Pr_\Delta(i \rhd_a k) = 3/4$. Hence, because of the claimwise stability constraints that each of these agents imposes on the other one for object a, agents i and k can consume at most 3/4 and 1/4 of object a, respectively. As agent j attempts to consume object b, no constraint is imposed on him in this round. Hence, the agents start consuming their respective objects, and this round terminates at the time of $t^1 = 1/4$ when agent k stops consuming object a. By the end of this step, each agent has consumed 1/4 of his top object.

Step 2. Agent k now attempts to consume his second-best object, which is object c. The others continue attempting to consume their previous objects. As the agents' respective objects are all different, no claimwise stability constraint is imposed on the agents in this step. Hence, they start consuming their objects and keep consuming until time $t^2 = 3/4$ when object a has become exhausted. By the end of this step, each agent has consumed 1/2 of his object.

Step 3. Agent i now attempts to consume his second-best object, which is object b. Hence, both him and agent j desire to consume the same object in this round. We have $Pr_\Delta(i \rhd_b j) = 1$. However, agent i has already acquired 3/4 of his more preferred object a. Therefore, agent j can consume at most 3/4 of object b. However, as he has already consumed that much, he is not allowed to consume more of object b.

Therefore, he stops consuming object b, implying that the step terminates at the time of $t^3 = 3/4$, and no agent acquires a positive amount of any object.

Step 4. Agent j now attempts to consume his second-best object c. Therefore, both agents j and k attempt to consume object c in this round. We have $Pr_\Delta(j \triangleright_c k) = 1$. However, as agent j has already consumed 3/4 of his more preferred object b, agent k is allowed to consume at most 3/4 of object c. Note that agent k has already consumed 1/2 of object c up to this step. Hence, the agents start consuming their respective objects, and this step terminates at the time of $t^4 = 1$. In the course of this step, each agent has acquired 1/4 of his object. By the end of this step, all the agents have acquired a total share of one; hence, *CPS* terminates. Below is the *CPS* outcome:

	a	b	c	$\emptyset$
i	3/4	1/4	0	0
j	0	3/4	1/4	0
k	1/4	0	3/4	0

2.3 The results

We are now ready to state our results, whose proofs are relegated to Appendix B.

Theorem 1 *A mechanism ψ is CPS if and only if it is non-wasteful, claimwise stable, constrained ordinally fair, and surplus-invariant to truncations.*

Theorem 2 *A mechanism ψ is CPS if and only if it is constrained sd-efficient, claimwise stable, and constrained ordinally fair.*

The independence of the axioms is shown in Appendix C.

Remark 6 Constrained ordinal fairness can be seen as very tailored to *CPS*. However, as we see in the independence analyses (in Appendix *C*), each property is independent of others. Moreover, for constrained ordinally fair mechanisms different from *CPS*, we can think of *PS*; and a mechanism that applies the agent-proposing deferred acceptance (Gale and Shapley (1962)) whenever the priorities are deterministic, and otherwise *CPS*. As *PS* is ordinally fair (Kesten et al. (2011) and Hashimoto et al. (2014)), it is constrained ordinally fair. One can easily verify that any (deterministic) stable matching is constrained ordinally fair in the deterministic priority domain.[21] This, as well as the constrained ordinal fairness of *CPS*, implies that the second mechanism is also constrained ordinally fair.

[21] A (deterministic) matching σ is stable at (R, Δ), where Δ is deterministic, if it is non-wasteful, and for any agent pair i, j and object c, whenever $c = \sigma_i \, P_j \, \sigma_j$, $Pr_\Delta(i \succ_c j) = 1$.

Remark 7 We present the whole analysis for the case of unit-copy objects. In the case of multi-copy objects, claimwise stability is problematic, as extensively discussed in Afacan (2018). Its caveat is that even though an agent may already satisfy all of his demand with some object a, he may continue preventing someone else from consuming the object through claimwise stability. For instance, consider a problem with two agents i, j and one object a with the capacity of 2. Let $Pr_{\Delta}(i \succ_a j) = 1$, and both agents prefer consuming object a. Here, agent j cannot consume object a even if agent i receives 1 unit of object a (this is because $SU(R_i, a) = \emptyset$, hence $\sum_{c \in SU(R_i,a)} \sigma_{i,c} = 0$ at any matching σ). Afacan (2018) overcomes this caveat by transforming multi-quota problems into those with unit-quota. The transformation is such that each copy of the same object is treated as a different object with the same priority as the original. Agents' preferences over this extended set of objects are such that the ranking over the original objects is preserved while the same object's copies can be ranked in any matter. We can find a solution in the artificial problem and straightforwardly transform it to the original problem (each agent's share from an object is the sum of the acquired shares from its copies in the artificial problem). Similarly, any matching in the multi-copy problem can be written as a matching in the unit-copy problem in the same manner. Thus, we can only consider unit-copy problems without losing generality; thereby, our results can be invoked in a multi-copy object problem through its corresponding unit-copy object market.

Appendix A: the formal description of CPS

The description is taken from Afacan (2018). Let $|N| = n$ and $|O| = m$. For agent i and $O' \subseteq O$, let $top(i, O')$ denote the favorite object of agent i in O'. Given a $n \times m$ matrix σ, $A = (A_i)_{i \in N}$ where $A_i \subseteq O$, a priority order profile Δ, below we define Θ_i^a for each agent $i \in N$ and each object $a \in O$:

$\Theta_i^a(\sigma, \Delta, A) = min\{Pr_{\Delta}(i \rhd_a j) + \sum_{c \in SU(R_j,a)} \sigma_{j,c} - \sigma_{i,a} : \; j \in N \setminus \{i\} \;$ such that $a = top(j, A_j)\}$.

($\Theta_i^a(\sigma, \Delta, A) = 1$ if the above set is empty or a is the null-object).

Θ_i^a will keep track of how much more agent i can consume object a within each step of the algorithm without violating claimwise stability constraints.

We define $M(a, A, N) = \{i \in N : \; a = top(i, A_i)\}$. Let $A^0 = (A_i^0)_{i \in N}$ where $A_i^0 = O$ for each $i \in N$ and $\sigma^0 = [0]$ (matrix of zeros). Suppose that A^{s-1} and σ^{s-1} are already defined. Then, for each $a \in \bigcup_{i \in N} A_i^{s-1}$:

$$y^s(a) = \begin{cases} \dfrac{1 - \sum_{i \in N} \sigma_{i,a}^{s-1}}{|M(a, A^{s-1}, N)|} & \text{if } M(a, A^{s-1}, N) \neq \emptyset \text{ and } a \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Let us define the followings,

$$y^s = min\{y^s(a):\ a \in \cup_{i\in N} A_i^{s-1}\}$$
$$\theta^s = min\{\Theta_i^{top(i,A_i^{s-1})}(\sigma^{s-1}, \Delta, A^{s-1}):\ i \in N\}$$
$$c^s = min\{y^s, \theta^s\}$$

For each $i \in N$ and object $a \in O$:

$$\sigma_{i,a}^s = \begin{cases} \sigma_{i,a}^{s-1} + c^s & \text{if} a = top(i, A_i^{s-1}) \\ \sigma_{i,a}^{s-1} & \text{otherwise} \end{cases}$$
$$E = \{a \in \bigcup_{i\in N} A_i^{s-1} : a \neq \emptyset \text{ and } \textstyle\sum_{i\in N} \sigma_{i,a}^s = 1\}$$
$$A_i^s = \begin{cases} A_i^{s-1} \setminus (E \cup top(i, A_i^{s-1})) & \text{if } c^s = \Theta_i^{top(i,A_i^{s-1})}(\sigma^{s-1}, \Delta, A^{s-1}) \\ A_i^{s-1} \setminus E & \text{otherwise} \end{cases}$$

Note that A_i^s keeps track of the set of objects from which agent i can keep consuming in the next step. Here, E consists of the already exhausted objects. We also exclude the top object of agent i, $top(i, A_i^{s-1})$, whenever the relevant claimwise stability constraint starts binding.

The algorithm terminates whenever each agent has received a total object share of one. As the algorithm moves to the next step when an agent stops consuming an object and everything (i.e., both agents and objects) is finite, the algorithm terminates in a finite step. If the algorithm finishes by the end of step s, then σ^s, becomes the *CPS* outcome.

Appendix B: the proofs of proposition 1, Theorem 1, and Theorem 2

Proof of Proposition 1 Let σ be a constrained sd-efficient matching at problem (R, Δ). Assume for a contradiction that there exist an agent i and objects a, b such that $a\ P_i\ b$, $\sigma_{i,b} > 0$, $a \neq \emptyset$ (we will consider the null-object case later in the proof), and $\sum_{k\in N} \sigma_{k,a} < 1$. Then, by constrained sd-efficiency, we have $\sigma_{i,a} = Pr_\Delta(i \rhd_a j) + \sum_{c\in SU(R_j,a)} \sigma_{j,c}$ for some agent j. This is because, otherwise, we can give an arbitrarily small amount of object a to agent i in return of the same amount of object b. While this trade would benefit agent i, it would not violate claimwise stability, contradicting the constrained sd-efficiency of σ. Note that if $a = \emptyset$, then as no claimwise stability constraint is imposed for the allocation of null-object, the same kind of trade would directly go against the constrained sd-efficiency of σ. Hence, in the rest of the proof, we assume that $a \neq \emptyset$.

As $Pr_\Delta(i \rhd_a j) \geq 0$, we have $\sigma_{i,a} \geq \sum_{c\in SU(R_j,a)} \sigma_{j,c}$. This, along with our initial supposition $\sum_{k\in N} \sigma_{k,a} < 1$, shows that $\sigma_{j,a} < 1 - \sum_{c\in SU(R_j,a)} \sigma_{j,c}$. This, in turn, implies that there exists an object d such that $a\ P_j\ d$ and $\sigma_{j,d} > 0$. By the same argument as above, it implies that there exists an agent ℓ such that $\sigma_{j,a} = Pr_\Delta(j \rhd_a \ell) + \sum_{c\in SU(R_\ell,a)} \sigma_{\ell,c}$. We now claim that $\ell \neq i$. Assume for a contradiction that $\ell = i$. Then we have $\sigma_{i,a} \geq 1 - Pr_\Delta(j \rhd_a i)$ and $\sigma_{j,a} \geq 1 - Pr_\Delta(i \rhd_a j)$. These inequalities imply that $\sigma_{i,a} + \sigma_{j,a} \geq 2 - Pr_\Delta(i \rhd_a j) - Pr_\Delta(j \rhd_a i)$. From here, since $Pr_\Delta(i \rhd_a j) = 1 - Pr_\Delta(j \rhd_a i)$, we obtain $\sigma_{i,a} + \sigma_{j,a} \geq 1$, which contradicts our initial supposition ($\sum_{k\in N} \sigma_{k,a} < 1$); hence, $\ell \neq i$.

Next, by the same reasoning as before, there exists an object z such that $a\, P_\ell\, z$ and $\sigma_{\ell,z} > 0$. This in turn implies that there exists an agent h such that $\sigma_{\ell,a} = Pr_\Delta(\ell \triangleright_a h) + \sum_{c \in SU(R_h,a)} \sigma_{h,c}$. Then, by following the same steps in the previous paragraph, we can easily show that $h \neq j$. Furthermore, in what follows, we show that $h \neq i$ as well.

Assume for a contradiction that $h = i$. We already have $\sigma_{i,a} \geq 1 - Pr_\Delta(j \triangleright_a i)$ and $\sigma_{j,a} \geq 1 - Pr_\Delta(\ell \triangleright_a j)$. The last finding also implies that $\sigma_{\ell,a} \geq 1 - Pr_\Delta(i \triangleright_a \ell)$. Now, we can decompose $Pr_\Delta(i \triangleright_a \ell)$ as follows: $Pr_\Delta(i \triangleright_a \ell) = \sum_{\succ \in supp(\Delta):\, j \succ_a i \succ_a \ell} \Delta(\succ) + \sum_{\succ \in supp(\Delta):\, i \succ_a \ell \succ_a j} \Delta(\succ) + \sum_{\succ \in supp(\Delta):\, i \succ_a j \succ_a \ell} \Delta(\succ)$. The sum of the last two terms is less than or equal to $Pr_\Delta(i \triangleright_a j)$. On the other hand, we have $\sigma_{i,a} \geq Pr_{\Delta_a}(i \triangleright_a j)$. The first term, moreover, is less than or equal to $Pr_\Delta(j \triangleright_a \ell)$. Similar to above, we also have $\sigma_{j,a} \geq Pr_\Delta(j \triangleright_a \ell)$. These findings, therefore, show that $Pr_\Delta(i \triangleright_a \ell) \leq \sigma_{i,a} + \sigma_{j,a}$. This, together with $\sigma_{\ell,a} \geq 1 - Pr_\Delta(i \triangleright_a \ell)$, implies that $\sigma_{\ell,a} \geq 1 - \sigma_{i,a} - \sigma_{j,a}$. This, in turn, means that $\sigma_{i,a} + \sigma_{j,a} + \sigma_{\ell,a} \geq 1$, which contradicts our initial supposition. Hence, $h \in N \backslash \{i, j, \ell\}$.

If we continue in the same manner as above, we obtain an object u such that $a\, P_h\, u$ and $\sigma_{h,u} > 0$. This implies that there exists an agent l such that $\sigma_{h,a} = Pr_\Delta(h \triangleright_a l) + \sum_{c \in SU(R_l,a)} \sigma_{l,c}$. Then by following the same steps as above, we can easily show that $l \in N \backslash \{i, j, \ell, h\}$.

If we continue applying the same argument to agent l, then we find an agent who is different from all the previously considered agents. Each iteration gives us a different agent. This, however, is impossible because there are finitely many agents, yielding a contradiction. Hence, σ is non-wasteful. □

Proof of Theorem 1 "If" Part: From Afacan (2018), CPS is claimwise stable and constrained sd-efficient. Hence, by Proposition 1, it is non-wasteful as well. For constrained ordinal fairness, let us assume that at some problem (R, Δ), for a pair of agents i, k and object a, $F(R_i, a, CPS_i(R, \Delta)) < F(R_k, a, CPS_k(R, \Delta))$ and $CPS(R, \Delta)_{k,a} > 0$. As, in the course of CPS, agents consume objects in decreasing order of their preferences, it implies that agent i stops consuming object a before it is totally exhausted. By the definition of CPS, it happens only if some claimwise stability constraint imposed on agent i for object a starts binding. That is, for some agent j, $CPS(R, \Delta)_{i,a} = Pr_\Delta(i \triangleright_a j) + \sum_{c \in SU(R_j,a)} CPS(R, \Delta)_{j,c}$. Hence, CPS is constrained ordinally fair.

For surplus invariance to truncations, for some agent i, let R'_i be the truncation of R_i from some object a. By its definition, CPS works the same at the truncated preference profile until the time of $t = F(R_i, a, CPS_i(R, \Delta))$. Therefore, $F(R_i, a, CPS_i(R, \Delta)) = F(R_i, a, CPS_i(R'_i, R_{-i}, \Delta))$, showing that CPS is surplus-invariant to truncations.

"Only If" Part: Let ψ be a mechanism that is non-wasteful, claimwise stable, constrained ordinally fair, and surplus-invariant to truncations. Let (R, Δ) be any problem. For ease of notation, let $\sigma = CPS(R, \Delta)$ and $\sigma' = \psi(R, \Delta)$. We now claim that $\sigma = \sigma'$. We prove it through obtaining that $F(R_i, a, \sigma_i) = F(R_i, a, \sigma'_i)$ for every agent–object (i, a) pair. Assume for a contradiction that for some agent–object pair (i, a), $F(R_i, a, \sigma_i) \neq F(R_i, a, \sigma'_i)$. □

Claim 1 *There exists an agent–object pair* (i,a) *such that* $a \neq \emptyset$, $F(R_i, a, \sigma'_i) < F(R_i, a, \sigma_i)$, *and* $\sigma_{i,a} > \sigma'_{i,a}$.

First, it cannot be that for any agent–object pair (j, c), $F(R_j, c, \sigma'_j) \geq F(R_j, c, \sigma_j)$, with strictly holding for some agent–object pair. This is because otherwise σ' would sd-dominate σ. This, along with the claimwise stability of σ', would contradict the constrained sd-efficiency of *CPS* (Afacan (2018)). Hence, for some agent–object pair (i, a), $F(R_i, a, \sigma'_i) < F(R_i, a, \sigma_i)$ and $\sigma_{i,a} > \sigma'_{i,a}$ (the latter is true because whenever $F(R_i, a, \sigma'_i) < F(R_i, a, \sigma_i)$, there always exists an object $b \in U(R_i, a)$ such that $F(R_i, b, \sigma'_i) < F(R_i, b, \sigma_i)$ and $\sigma_{i,b} > \sigma'_{i,b}$). Moreover, $a \neq \emptyset$ because $F(R_i, \emptyset, \sigma_i) = F(R_i, \emptyset, \sigma'_i) = 1$ (due to the non-wastefulness of ψ and *CPS*), finishing the proof of Claim 1.

Let $Z = \{F(R_i, a, \sigma'_i) : (i, a) \in N \times \hat{O}$ such that $F(R_i, a, \sigma'_i) < F(R_i, a, \sigma_i)$ and $\sigma'_{i,a} < \sigma_{i,a}\}$. Moreover, define $Z' = \{F(R_i, a, \sigma'_i) \in Z :$ for any $F(R_k, c, \sigma'_k) \in Z$, $F(R_i, a, \sigma'_i) \leq F(R_k, c, \sigma'_k)\}$. From Claim 1, $Z \neq \emptyset$, thereby $Z' \neq \emptyset$. Let $F(R_i, a, \sigma'_i) \in Z'$.

Claim 2 *For every agent-object pair* (k,c) *where* $c \neq \emptyset$ *and* $F(R_k, c, \sigma'_k) < F(R_i, a, \sigma'_i)$, $F(R_k, c, \sigma'_k) = F(R_k, c, \sigma_k)$.

For each agent–object pair (k, c) with $F(R_k, c, \sigma'_k) < F(R_i, a, \sigma'_i)$, we have $F(R_k, c, \sigma'_k) \geq F(R_k, c, \sigma_k)$. This is because, otherwise, $F(R_k, c, \sigma'_k) < F(R_k, c, \sigma_k)$. This implies that for some object $d \in U(R_k, c)$, we have $F(R_k, d, \sigma'_k) < F(R_k, d, \sigma_k)$ and $\sigma'_{k,d} < \sigma_{k,d}$ (as σ' is non-wasteful, d is not the null-object, that is, $d \in \hat{O}$). Hence, $F(R_k, d, \sigma'_k) \in Z$. As $d \in U(R_k, c)$, $F(R_k, d, \sigma'_k) \leq F(R_k, c, \sigma'_k)$; hence, $F(R_k, d, \sigma'_k) < F(R_i, a, \sigma'_i)$. This, as well as $F(R_k, d, \sigma'_k) \in Z$, contradicts the fact that $F(R_i, a, \sigma'_i) \in Z'$.

Let $D = \{(k, c) \in N \times \hat{O} : F(R_k, c, \sigma'_k) < F(R_i, a, \sigma'_i)\}$. Note that if D is empty, then there is nothing to prove. Hence, suppose that $D \neq \emptyset$. Let $t = \max\{F(R_k, c, \sigma'_k) : (k, c) \in D\}$ (as both the sets of agents and objects are finite, this maximal selection–t– is well-defined).

In the course of *CPS*, agents consume objects in decreasing order of their preferences until claimwise stability constraints start binding or objects are totally exhausted. Hence, in particular, they do so until the time of t. This, as well as $F(R_k, c, \sigma'_k) \geq F(R_k, c, \sigma_k)$ for each $(k, c) \in D$, implies that $F(R_k, c, \sigma'_k) = F(R_k, c, \sigma_k)$ for each $(k, c) \in D$ (because, otherwise, it would imply that for some $(k, c) \in D$, agent k stops consuming object c in *CPS* before it gets exhausted or the relevant claimwise stability constraints start binding (the latter is due to fact that σ' is claimwise stable)). Therefore, for each $(k, c) \in D$, we have $F(R_k, c, \sigma'_k) = F(R_k, c, \sigma_k)$, proving Claim 2.

Claim 3 *There exists an agent* j *such that* $F(R_i, a, \sigma'_i) < F(R_j, a, \sigma'_j)$ *and* $\sigma'_{j,a} > 0$.

Assume for a contradiction that for each agent j with $\sigma'_{j,a} > 0$, we have $F(R_j, a, \sigma'_j) \leq F(R_i, a, \sigma'_i)$. If we define $N(a) = \{j \in N : \sigma'_{j,a} > 0\}$, then due to the non-wastefulness of ψ (note that $F(R_i, a, \sigma'_i) < 1$ implies that agent i consumes some positive amount of a less preferred object to object a at matching σ'), $\sum_{j \in N(a)} \sigma'_{j,a} = 1$. On the other hand, $\sigma_{i,a} > \sigma'_{i,a}$ implies that for some agent $k \in N(a)$, we have $\sigma_{k,a} < \sigma'_{k,a}$. From here, if $SU(R_k, a) \neq \emptyset$, then for any $b \in SU(R_k, a)$, we have $F(R_k, b, \sigma'_k) < F(R_i, a, \sigma'_i)$, therefore $F(R_k, b, \sigma_k) = F(R_k, b, \sigma'_k)$ (by Claim 2). This, along with $\sigma_{k,a} < \sigma'_{k,a}$, implies that $F(R_k, a, \sigma_k) < F(R_k, a, \sigma'_k) \leq F(R_i, a, \sigma'_i) < F(R_i, a, \sigma_i)$. Note that if $SU(R_k, a) = \emptyset$, then as $\sigma_{k,a} < \sigma'_{k,a}$, we directly obtain the above inequality. Hence, $F(R_k, a, \sigma_k) < F(R_i, a, \sigma_i)$, and moreover, $\sigma_{i,a} > 0$. Then, by constrained ordinal fairness of *CPS*, $\sigma_{k,a} = Pr_\Delta(k \triangleright_a s) + \sum_{c \in SU(R_s, a)} \sigma_{s,c}$ for some agent s. This implies that $\sum_{c \in SU(R_s, a)} \sigma_{s,c} \leq F(R_k, a, \sigma_k) < F(R_i, a, \sigma'_i)$, hence $\sum_{c \in SU(R_s, a)} \sigma_{s,c} = \sum_{c \in SU(R_s, a)} \sigma'_{s,c}$ (by Claim 2). As $\sigma_{k,a} < \sigma'_{k,a}$, we have $\sigma'_{k,a} > Pr_\Delta(k \triangleright_a s) + \sum_{c \in SU(R_s, a)} \sigma'_{s,c}$, contradicting the claimwise stability of ψ. Hence, there exists an agent j such that $F(R_i, a, \sigma'_i) < F(R_j, a, \sigma'_j)$ and $\sigma'_{j,a} > 0$, finishing the proof of Claim 3.

As $F(R_i, a, \sigma'_i) < F(R_j, a, \sigma'_j)$ and $\sigma'_{j,a} > 0$, by the constrained ordinal fairness of ψ, we have $\sigma'_{i,a} = Pr_\Delta(i \triangleright_a h) + \sum_{c \in SU(R_h, a)} \sigma'_{h,c}$ for some agent h. As $\sigma_{i,a} > \sigma'_{i,a}$, due to the claimwise stability of σ, we have $\sum_{c \in SU(R_h, a)} \sigma_{h,c} > \sum_{c \in SU(R_h, a)} \sigma'_{h,c}$. This also implies that a is not the top object of agent h (note that $a \, P_h \, \emptyset$, as otherwise due to the non-wastefulness of ψ and *CPS*, we would have $\sum_{c \in SU(R_h, a)} \sigma_{h,c} = \sum_{c \in SU(R_h, a)} \sigma'_{h,c} = 1$). Let object b be the least preferred object in $SU(R_h, a)$. Then we have $F(R_h, b, \sigma_h) > F(R_h, b, \sigma'_h)$.

To summarize our findings so far, we start with different matchings σ and σ' produced by *CPS* and ψ at (R, Δ), respectively. We then obtain that for some agent–object pair (i, a), $\sigma_{i,a} > \sigma'_{i,a}$, and moreover, with another agent h, we find the triplet $(i, h, a) \in N \times N \times O$ such that $\sigma'_{i,a} = Pr_\Delta(i \triangleright_a h) + \sum_{c \in SU(R_h, a)} \sigma'_{h,c}$. Moreover, $F(R_h, b, \sigma_h) > F(R_h, b, \sigma'_h)$ where b is the least preferred object by agent h among the objects in $SU(R_h, a)$.

Let us now consider the truncation R'_h of R_h from object b. Let $R' = (R'_h, R_{-h})$, $CPS(R', \Delta) = \nu$, and $\psi(R', \Delta) = \nu'$. As both *CPS* and ψ are surplus-invariant to truncations, we have $F(R_h, b, \nu_h) = F(R_h, b, \sigma_h) > F(R_h, b, \nu'_h) = F(R_h, b, \sigma'_h)$. Therefore, in particular, $\nu \neq \nu'$.

If we apply the same whole analysis above to the matchings ν and ν', we find another agent–object pair (j, d) such that $\nu_{j,d} > \nu'_{j,d}$, and moreover, with another agent k, we find a triplet $(j, k, d) \in N \times N \times O$ such that $\nu'_{j,d} = Pr_\Delta(j \triangleright_d k) + \sum_{c \in SU(R_k, d)} \nu'_{k,c}$. In this case, however, as $\sum_{c \in SU(R'_h, a)} \nu'_{h,c} = 1$ (due to the non-wastefulness of ψ and $\emptyset \in SU(R'_h, a)$), $(j, k, d) \neq (i, h, a)$.

Then, as the same as above, if we let agent k truncate his preferences from the least preferred object in $SU(R_k, d)$ and write R'_k for this truncated preferences, we obtain different CPS and ψ outcomes at $\tilde{R} = (R'_h, R'_k, R_{N\setminus\{h,k\}})$.[22,23]

If we keep applying the same steps above to the obtained different outcomes of CPS and ψ at the generated truncated preferences, then each iteration gives us a different triplet for which associated claimwise stability constraint binds. But then, as both agents and objects are finite, we eventually reach the preference profile R'' where for each agent i, the only acceptable object is his top choice at the original preference R_i. Moreover, if we let $CPS(R'', \Delta) = \tau$ and $\psi(R'', \Delta) = \tau'$, then we have $\tau \neq \tau'$.

As the same as above, at τ and τ', there exists an agent–object pair (s, e) such that $\tau_{s,e} > \tau'_{s,e}$ (where object $e \neq \emptyset$), and moreover, with another agent s', there exists a triplet (s, s', e) such that $\tau'_{s,e} = Pr_\Delta(s \rhd_e s') + \sum_{c \in SU(R''_{s'}, e)} \tau'_{s',c}$. Due to the definition of R'' and the non-wastefulness of ψ and CPS, we have $\sum_{c \in SU(R''_{s'}, e)} \tau'_{s',c} = \sum_{c \in SU(R''_{s'}, e)} \tau_{s',c}$. This, along with the fact that $\tau_{s,e} > \tau'_{s,e}$, implies $\tau_{s,e} > Pr_\Delta(s \rhd_e s') + \sum_{c \in SU(R''_{s'}, e)} \tau_{s',c}$, contradicting the claimwise stability of CPS. Hence, for any agent–object pair (i, a), $F(R_i, a, \sigma_i) = F(R_i, a, \sigma'_i)$, implying that $\sigma = \sigma'$, which finishes the proof.

Proof of Theorem 2 "If" Part. Afacan (2018) shows that CPS is constrained sd-efficient. This, as well as Theorem 1, shows this part.

"Only If" Part. Let ψ be a constrained sd-efficient, claimwise stable, and constrained ordinally fair mechanism. Assume for a contradiction that $\psi \neq CPS$. This means that for some problem (R, Δ), $CPS(R, \Delta) = \sigma$, $\psi(R, \Delta) = \sigma'$, and $\sigma \neq \sigma'$.

For the proof, we invoke the consumption process representation of matchings by Bogomolnaia and Heo (2012): Any matching can be obtained as the outcome of a consumption process where, over the unit-time interval, agents continuously acquire probability shares of objects in decreasing order of their preferences at the speed of one.

We now claim that for some agent–object pair (i, a), $F(R_i, a, \sigma_i) > F(R_i, a, \sigma'_i)$. This holds, as otherwise, σ' sd-dominates σ, implying that σ cannot be constrained sd-efficient. This, however, contradicts the fact that CPS is constrained sd-efficient (Afacan (2018)).

Let $Z = \{F(R_i, a, \sigma_i) : (i, a) \in N \times O \text{ and } F(R_i, a, \sigma_i) > F(R_i, a, \sigma'_i)\}$. From above, $Z \neq \emptyset$. Let (i, a) be such that $F(R_i, a, \sigma_i) \in Z$ and $F(R_i, a, \sigma_i) \leq F(R_j, b, \sigma_j)$ for each $F(R_j, b, \sigma_j) \in Z$. For each $(k, c) \in N \times O$ with $F(R_k, c, \sigma_k) < F(R_i, a, \sigma_i)$, we have $F(R_k, c, \sigma'_k) \geq F(R_k, c, \sigma_k)$. **(1)**

In CPS, each agent continues acquiring probability shares of objects until claimwise stability constraints start binding or the objects are exhausted (whichever occurs first). This, as well as the claimwise stability of ψ and (1), implies that for each (k, c) with $F(R_k, c, \sigma_k) < F(R_i, a, \sigma_i)$, we have $F(R_k, c, \sigma_k) = F(R_k, c, \sigma'_k)$. **(2)**

As $F(R_i, a, \sigma_i) > F(R_i, a, \sigma'_i)$, for some object $c \in U(R_i, a)$, we have $\sigma_{i,c} > \sigma'_{i,c}$

[22] By the same previous arguments, object d is not the top object of agent k, and $d \, P_k \, \emptyset$.

[23] $R_{N\setminus\{h,k\}}$ is the preference profile of the agents except agents h and k.

and $F(R_i, c, \sigma_i) > F(R_i, c, \sigma'_i)$. Without loss of generality, let $\sigma_{i,a} > \sigma'_{i,a}$. By (2), under both σ and σ', the consumption process works the same until time $t = F(R_i, a, \sigma'_i)$. This, as well as $\sigma_{i,a} > \sigma'_{i,a}$, implies that under σ', agent i stops consuming object a before it gets exhausted. Moreover, as σ' is constrained sd-efficient (hence, by Proposition 1, it is non-wasteful), for some agent j, $\sigma'_{j,a} > 0$ and $F(R_i, a, \sigma'_i) < F(R_j, a, \sigma'_j)$. Then by (2) and the constrained ordinal fairness of ψ, some agent, say k, starts consuming object a in the consumption process by time $t = F(R_i, a, \sigma'_i)$, and the claimwise stability constraint on agent i (for object a) due to agent k starts binding. Therefore, without loss of generality, we can take agent k as agent j, and we have $\sigma'_{i,a} = Pr_\Delta(i \rhd_a j) + \sum_{c \in SU(R_j, a)} \sigma'_{j,c}$. Thus, so far we have a group of agents $\{i, j\}$ and object a, where $\sigma'_{j,a} > 0$, and for some object d, $\sigma'_{i,d} > 0$ and $a\ P_i\ d$ (the latter is because $F(R_i, a, \sigma'_i) < F(R_i, a, \sigma_i) \leq 1$).

We have $\sigma'_{i,a} \geq \sum_{c \in SU(R_j, a)} \sigma'_{j,c}$. Therefore, for agent j, if object b is the least preferred object to object a, then $F(R_j, b, \sigma'_j) \leq F(R_i, a, \sigma'_i) < F(R_i, a, \sigma_i)$ (note that if object a is the best alternative of agent j, then we take $a = b$). If $F(R_j, b, \sigma_j) < F(R_i, a, \sigma_i)$, then by (2), $F(R_j, b, \sigma_j) = F(R_j, b, \sigma'_j)$. This, as well as $\sigma'_{i,a} < \sigma_{i,a}$, implies that $\sigma_{i,a} > Pr_\Delta(i \rhd_a j) + \sum_{c \in SU(R_j, a)} \sigma_{j,c}$, contradicting the claimwise stability of σ. Therefore, we have $F(R_j, b, \sigma_j) \geq F(R_i, a, \sigma_i)$, implying that $F(R_j, b, \sigma'_j) < F(R_j, b, \sigma_j)$.

Let $t' = F(R_j, b, \sigma'_j) < F(R_i, a, \sigma_i)$. As $F(R_j, b, \sigma'_j) < F(R_j, b, \sigma_j)$, for some object $d \in U(R_j, b)$, $\sigma'_{j,d} < \sigma_{j,d}$. Without loss of generality, let $\sigma'_{j,b} < \sigma_{j,b}$. From (2), under both σ and σ', the consumption process works the same until time t'. This, as well as $\sigma'_{j,b} < \sigma_{j,b}$, implies that under σ', agent j stops consuming object b before it gets exhausted. Moreover, as σ' is constrained sd-efficient, for some agent k, $\sigma'_{k,b} > 0$ and $F(R_j, b, \sigma'_j) < F(R_k, b, \sigma'_k)$. Then as the same as above, by (2) and the constrained ordinal fairness of ψ, some agent, say h, starts consuming object b in the consumption process by time t', and the claimwise stability constraint on agent j (for object b) due to agent h starts binding. Therefore, without loss of generality, let $k = h$, and $\sigma'_{j,b} = Pr_\Delta(j \rhd_b k) + \sum_{c \in SU(R_k, b)} \sigma'_{k,c}$. Hence, we now have a group of agents $\{i, j, k\}$ and objects a, b, d such that $\sigma'_{j,a} > 0$, $\sigma'_{k,b} > 0$, $\sigma'_{i,d} > 0$, $a\ P_i\ d$, and $b\ P_j\ a$.

For agent k, let object e be the least preferred object to object b (if object b is the top object, then we take $e = b$). From above, $F(R_k, e, \sigma'_k) < F(R_j, b, \sigma'_j) < F(R_i, a, \sigma_i)$. If $F(R_k, e, \sigma_k) < F(R_i, a, \sigma_i)$, then by (2), we have $F(R_k, e, \sigma'_k) = F(R_k, e, \sigma_k)$. This, as well as $\sigma'_{j,b} < \sigma_{j,b}$, implies that $\sigma_{j,b} > Pr_\Delta(j \rhd_b k) + \sum_{c \in SU(R_k, b)} \sigma_{k,c}$, contradicting the claimwise stability of σ. Therefore, we have $F(R_k, e, \sigma_k) \geq F(R_i, a, \sigma_i)$. This implies that $F(R_k, e, \sigma'_k) < F(R_k, e, \sigma_k)$. We then apply the same arguments above for the agent–object pair (k, e). As both agents and objects are finite, continuing in the same manner gives us a set of agents $\{i_1, .., i_m\}$ and set of objects $\{c_1, .., c_m\}$ such that for each $k = 1, .., m$, $\sigma'_{i_k, c_k} > 0$ and $c_{k+1}\ P_{i_k}\ c_k$, where $c_{m+1} = c_1$. For an arbitrarily small amount, each agent i_k can receive c_{k+1} in return of giving the same amount c_k to agent i_{k-1}. We keep everything else the same as under σ'. By this way, we can obtain a matching that sd-dominates σ'. Moreover, as these agents receive better objects, the claimwise stability constraints imposed on others (because of themselves) relax. This means that for an arbitrarily small amount of cyclic trade, we can preserve claimwise stability. This, however, contradicts constrained sd-efficiency of σ'. Hence, $\sigma' = \sigma$, finishing the proof. □

Appendix C: the independence of the axioms for Theorem 1, and Theorem 2

We provide four examples below to show the independence of the axioms. The table below summarizes the properties that are simultaneously obtained and those that are not in each example.

	Claim. St	NW	Const. Sd-Eff	Const. OF	Surp. In
Ex 1.	×	✓	✓	✓	✓
Ex 2.	✓	×	×	✓	✓
Ex 3.	✓	✓	×	✓	×
Ex 4.	✓	✓	✓	×	✓

Example 1 From Hashimoto et al. (2014), we know that *PS* is non-wasteful, ordinally fair, hence constrained ordinally fair. It is easy to verify that *PS* is also surplus-invariant to truncations. *PS* is sd-efficient, hence constrained sd-efficient. Yet, it is not claimwise stable whenever $|N| \geq 2$ (*PS* is defined for problems without priorities, hence it ignores Δ).

Example 2 The mechanism that always leaves each agent unassigned is claimwise stable, constrained ordinally fair, surplus-invariant to truncations, yet it is not non-wasteful. By Proposition 1, it is not constrained sd-efficient either.

Example 3 Let $N = \{i,j\}$ and $O = \{a,b\} \cup \{\emptyset\}$. Consider the following preference and deterministic priority profile:

R_i: $a, b, \emptyset$ and $R_j : b, a, \emptyset$.

$\succ_a : j, i$ and $\succ_b : i, j$.

Consider a mechanism ψ that gives the deterministic matching μ where $\mu_i = b$ and $\mu_j = a$ at the above problem and coincides with *CPS* at all other problems. Note that the *CPS* outcome at the above problem is such that both agents receive their top choices. Hence, $\psi \neq CPS$. Notice that *CPS* sd-dominates ψ; hence, ψ is not constrained sd-efficient. As μ is claimwise stable and non-wasteful, ψ is claimwise stable and non-wasteful. Moreover, it is easy to verify that constrained ordinal fairness is satisfied at μ. This, along with the constrained ordinal fairness of *CPS*, shows that ψ is constrained ordinally fair as well. However, ψ is not surplus-invariant to truncations. To see this, let $R'_j : b, \emptyset$; and let σ be the outcome of ψ at the truncated profile (R_i, R'_j). Note that σ is deterministic such that $\sigma_j = b$. Hence, $F(R_j, b, \mu_j) = 0$ and $F(R_j, b, \sigma_j) = 1$, showing that ψ fails to be surplus-invariant to truncations.

Example 4 Let us consider a problem where $N = \{i, j, k\}$ and $O = \{a, b\} \cup \{\emptyset\}$. Consider the following problem (R, Δ);

$$R_i = R_j = R_k : \ a, b, \emptyset.$$

$\succ$		$\succ'$		$\succ''$	
a	b	a	b	a	b
i	i	j	k	j	i
j	j	i	i	i	j
k	k	k	j	k	k

Let $\Delta(\succ) = 1/2$, $\Delta(\succ') = 1/10$, and $\Delta(\succ'') = 2/5$. Let ψ be the mechanism that gives the following matching σ at the above instance and coincides with *CPS* at every other instance.

	a	b	$\emptyset$
i	1/2	1/6	1/3
j	1/2	7/30	4/15
k	0	3/5	2/5

Note that the *CPS* outcome at the above problem is different from σ as, for at least (j, b) pair, $CPS(P, \Delta)_{j,b} = 1/6$. Hence, $CPS \neq \psi$. It is easy to verify that ψ is constrained sd-efficient (hence, non-wasteful), claimwise stable, and surplus-invariant to truncations. However, $F(R_i, b, \sigma_i) < F(R_j, b, \sigma_j)$ and $\sigma_{j,b} > 0$, and for every agent $k \neq i$, $\sigma_{i,b} < Pr_\Delta(i \rhd_b k) + \sum_{c \in SU(R_k, b)} \sigma_{k,c}$. Hence, ψ fails to be constrained ordinally fair.

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Conflict of interest The author declares that he has no conflict of interest relevant to this work.

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