Generalized Airy's- and Taylor Integral Functions

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ABSTRACT

In this paper a generalized form of Airy's function has been estimated and an asymptotic expansion for one of its types is obtained. Furthermore, an improved formula could be induced to acquire the asymptotic expansion of Toylor* integral function directly.

* : By 'Taylor integral functions' we mean functions which can be expressed in form of Taylor integral.

Introduction

The differential equation

$$y'' + p(x)\,y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

plays an important role in the field of linear differential equations. The second order differential equation of the general form $u'' + a(x)\,u' + b(x)\,u = 0$ can be reduced to the form (1) by using the transformation $u(x) = \exp(-\frac{1}{2}\int a(x)\,dx)$ and hence replacing $p(x)$ by $b(x) - \frac{1}{4}a^2(x) - \frac{1}{2}a'(x)$.

The behaviour of the solution of (1) in its non-homogeneous form was studied by Abramovich in [1]. In [4], we have solved the equation (1), considering particular $p(x)$ for large values of x.

For the important special case, $p(x) = x$, one gets Airy's differential equation, which is satisfied by any linear combination of the following Airy's functions [3] :

$$Ai(x) = \frac{1}{\pi}\int_0^{\infty} \cos(t^3/3 + xt)\,dt,$$

and

$$Bi(x) = \frac{1}{\pi}\int_0^{\infty} (e^{-t^3/3} + xt + \sin(t^3/3 + xt))\,dt, \quad x > 0.$$

For $p(x) = \lambda x^n$, where $\lambda = \pm 1$, $n \in R$ and $x \in C$ the solution of (1) has been obtain and expressed in the form of the contour integral [5]:

$$y = \frac{x}{2\pi i}\oint_{\Gamma} (t^3 + \lambda)^{\frac{x}{2n+4}} \exp\left(\frac{-2n}{n+2}\sqrt{x^{n+2}}\right) dt$$

where Γ is any contour containing all the poles $\pm 1, \infty$. Moreover, by making use of the relations governing Bessel function, hypergeometric series and continued fractions, the solution of $y'' + x^n y = 0$ has been represented also in [5] as a linear combination of $\Phi(x)$ and $\psi(x)$, which are

$$\Phi(x) = A\,\frac{\sqrt{|x}}{\Gamma\left(\frac{n+1}{n+2}\right)} \exp\left\{\frac{2}{n+2}\sqrt{x^{n+2}} + \int\left[0;\ \frac{+k}{\Upsilon - \xi + k}\,\xi\right] dx\right\}$$

and

$$\psi(x) = \frac{1}{A} \cdot \frac{1}{\Gamma(\frac{n+1}{n+2})} \cdot \exp\left\{ \frac{2}{n+2} \sqrt{x^{n+2}} + \int \left[0; \frac{\delta + k}{2\delta - \xi + k} \xi \right] dx \right\}$$

where

$$A = \left(\frac{i}{n+2}\right)^{\frac{1}{n+2}}, \beta = \frac{n+4}{2n+4}, \left[0; \frac{a(k)}{b(k)} \right] = \cfrac{a(1)}{b(l) + \cfrac{a(2)}{b(2) + \ldots}}$$

$$\Upsilon = \frac{n+4}{n+1}, \ \delta = \frac{n}{2n+4} \ \text{and} \ \xi = \frac{2x}{n+2}.$$

In this paper, we make use of the fixed point theorem to estimate a generalized form of Airy's function, which is the solution of the differential equation (1) for arbitrary p(x). Furthermore, an asymptotic expansion for a type of that function has been obtained in the neighbourhood of infinity. Lastly, it could then be induced a formula, which is useful to get the asymptotic expansion of Taylor integral functions directly.

§ (1) Estimating a Generalized Airy's Function

To estimate a generalized form of Airy's function, satisfying the differential equation (1) for large values of x, put $x = \frac{1}{t}$ to obtain.

$$v^{\cdot} = \frac{2}{t} v + \frac{1}{t^4} p\left(\frac{1}{t}\right) v^2 + 1,$$

where

$$v = \frac{y}{y^{\cdot}}, \ \left({}^{\cdot} = \frac{d}{dt} \right)$$

Let us suppose that

$$A(v) = v_o + \int_{t_o}^{t} \left(\frac{2}{t} v + \frac{1}{t^2} P\left(\frac{1}{t}\right) v^2 + 1 \right) dt$$

since it can be proved that the conditions of the fixed point theorem [6] are satisfied by A (v) on $[t_o, t]$, we then have

$$d\left[A(v),\ A(w)\right] < \propto d\ (v,w)$$

where

$\propto < 1$ and d is the distance function.

Making use of the condition $\propto < 1$, one can prove that the considered generalized Airy's function y(x), and hence its derivative y'(x), have the following estimations for $p(x) \geqslant \frac{2x-1}{x^2}$

$$\exp \int \frac{2x^2}{2x+1}\ p(x)\,dx < y\,(x) < \exp \int \frac{2x^2}{2x-1}\ p(x)\,dx \qquad (2)$$

and

$$\frac{2x^2P(x)}{2x+1} \exp \quad \frac{2x^2}{2x+1}\ p(x)\,dx < y\ \ (x) < \frac{2x^2P(x)}{2x-1} \exp \int \frac{2x^2}{2x-1}\ p(x)\,dx \qquad (3)$$

For $p(x) < \frac{2x-1}{x^2}$ it can be shown that the estimations of y(x) and y'(x) are obtained from the inequalities (2) and (3) by exchanging the two sides in every one.

Corallary

For $p(x) = x^n$ in equation (1), one gets

$$\prod_{s=1}^{n+2} \exp \left\{ (-\tfrac{1}{2})^{n-s+2} \frac{x^s}{s} \right\} < y(x) < \prod_{s=1}^{n+2} \exp \left\{ (\tfrac{1}{2})^{n-s+2} \frac{x^s}{s} \right\}$$

hence, it follows immeddiately that Airy's function takes its values for large values of x on the interval $\left(\exp\left\{\frac{x^3}{3} - \frac{x^2}{4} + \frac{x}{4}\right\},\ \exp\left\{\frac{x^3}{3} + \frac{x^2}{4} + \frac{x}{4}\right\}\right)$.

∮ (2) Asymptotic Expansion for a Particular Form of the considered Airy's Function

Consider the following theorem [3] :

Theorem

Let g (t) and h (t) be functions on the interval $[\alpha, \beta]$ for which the integral

$$f(x) = \int_{\alpha}^{\beta} g(t) \exp(xh(t))\, dt,$$

exists for large x>o; let h (t) be real continuous differentiable and h'<o on $[a, a+\eta]$ and $h(t) \leq h(\alpha) - \varepsilon$ for $a + \eta \leq \varepsilon < \beta$, $h \sim -a(t-a)^{\nu-1}$*

$g(t) \sim b(t-a)^{\mu-1}$ as $t \rightarrow a$, such as $\mu, \nu > 0$ then $f(x) \sim \frac{b}{\nu}\Gamma(\frac{\mu}{\nu}) . \exp(xh(a))$.

Applying this theorem on a particular form of the considered generalized Airy's function, namely when $p(x) = x^n$, which is given in [4] as :

$$y = x_1 \int^{\infty} (t^2-1)^{\frac{-n}{2n+4}} \exp\left(\frac{-2t}{n+2} \sqrt{x^{n+2}}\right) dt \tag{4}$$

one obtains its asymptotic expansion in the form

$$y \sim A_o x^{\frac{-n}{4}} \exp\left(\frac{-2}{n+2} \sqrt{x^{n+2}}\right) \tag{5}$$

where

$$A_o = 2^{\frac{-n}{2n+4}} \Gamma\left(\frac{n+4}{2n+4}\right) \left(\frac{n+2}{2}\right)^{\frac{n+4}{2n+4}}$$

Now we are in a position to induce a formula which represents a useful tool to get directly the asymptotic expansion of Taylor integral functions. This formula could be stated through the following theorem.

Theorem

For an integral of the form

$$f(x) = a \int^{\infty} \exp(-xt)\, g(t)\, dt, \tag{6}$$

*: f(x) ~ g(x) means that the functions f(x) and g(x) have the same asymptotic expansion to N terms.

where $a \in R$; assuming that

1. g(t) is real and continuously differentiable at t=a.

2. $g(t) \sim b(t-a)$ at $\mu \rightarrow a$, where λ (>0) and b are arbitrary constants, then it follows for large positive values of x :

$$f(x) \sim b\,\Gamma(\lambda)\exp(ax)\,x^{-\lambda} \tag{7}$$

The proof of this theorem follows by using the transformation u=t−a in the integral (6) and considering the second condition, namely $g(t) \sim b(t-a)^{\lambda}$.

Corollary

That asymptotic expansion (5) can directly be obtained in the frame-work of this approach.

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دوال ايرى المعممة والتي على صـورة تكامل تيلور

محمــد كـــامل جبــر و شــــــعبان ســـالم

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ملخـــــــــص

في هذا البحث تم الحصول على تقدير لصورة معممة لدالة ايرى وكذلك المفكوك التقريبي لاحدى صورها .. بالاضافة إلى ذلك استنبطت باستخدام نظرية النقطة الثابتة المفكوك التقريبي للدوال التي على صورة تيلور .