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# STABILITY OF SQUARE INTERVAL MATRICES FOR DISCRETE TIME SYSTEMS

**Nabil S. Rousan**
Mutah University
Karak, Jordan
Email: nrousan@yahoo.com

## ABSTRACT

New sufficient conditions for the Schur stability of interval matrices are provided. Moreover, a time saving version of a necessary and sufficient condition for the Schur stability of a class of interval matrices is introduced. Improvement over conditions in the literature is shown by examples.

## NOMENCLATURE

$|A|$ If $A=[a_{ij}]$, then $|A|=[|a_{ij}|]$ which is non-negative.

$\rho[A]$ Spectral radius of $A=\max_i |\lambda_i|$ where $\lambda_i$ is an eigenvalue of $A$

$\|A\|_\infty$ infinity norm of the matrix $A$.

$A^T$ Transpose of $A$

$\pm A$ $A$ Or $-A$

$O$ Null matrix with zero elements

$I$ Identity matrix

## 1. INTRODUCTION

The problem of maintaining the stability of a nominally stable system subjected to perturbations has been an active area of research for some time. There is considerable literature on this topic for continuous time systems [1-6, 8,16]. Recently, the robust stability of discrete time systems also received considerable interest [5-17]. Motivated by the results obtained for Hurwitz stability of interval matrices, we consider here the Schur stability of interval matrices for discrete time systems.

An interval matrix is a real square matrix in which all elements are known only within certain closed intervals. In mathematical terms, an $n \times n$ interval matrix $A_I = [B,C]$ is a set of real matrices defined by

$$A_I = \left\{A = \left[a_{ij}\right]; b_{ij} \le a_{ij} \le c_{ij}; i,j = 1,\ldots,n\right\} \tag{1}$$

The set $A_I$ is described geometrically as a hyperrectangle in the space $\Re^{n\times n}$ of the coefficients $a_{ij}$. We say that a set $A_I$ is Schur stable if every $A \in A_I$ is Schur stable. Associated with the set $A_I$ we define the average matrix $V$ at the center of the uncertainty hyperrectangle and the deviation matrix $D$ as

$$V = \left[v_{ij}\right] = \frac{C+B}{2}, \qquad D = \left[d_{ij}\right] = \frac{C-B}{2} \tag{2}$$

The interval matrix $A_I$ can be represented using the matrices $V$ and $D$ as follows:

$$A_I = V + E, \qquad |E| \le D \tag{3}$$

Where $|E|$ denotes the modulus of the perturbation matrix $E$ and $\le$ denotes the inequality of the corresponding elements of matrices under consideration.

Utilizing special type of matrices, in particular, a Morishima matrix, this paper provides an easy to use version of a necessary and sufficient condition to check the Schur stability of a class of interval matrices. The equivalent version, reported in [1, corollary 1.3] does not provide a methodology for constructing the extreme vertex of the hyperrectangle that corresponds to the test matrix, but rather gives, in some cases, a large bundle of extreme vertices which are needed to be checked individually. Furthermore, the sufficient conditions introduced in this paper have eliminated the constraint $\max_{i,j}\left\{\left|b_{ij}\right|,\left|c_{ij}\right|\right\} < 1$ that was required by most conditions in the literature [5-7,9,13,17]. These sufficient conditions have shown to be conclusive about the stability of some interval matrices where conditions in the literature have failed to do so.

In what follows; some definitions, lemmas, corollaries, and theorems are introduced. The results are related to those reported in the literature when it is appropriate and warranted.

## 2. MAIN RESULTS

1- A matrix $A$ is called a Morishima matrix if there exists a diagonal matrix, $S$, of the form

$$S = [s_{ij}] = \begin{cases} \pm 1 & i = j \\ 0 & i \neq j \end{cases} \qquad i,j = 1,2,\ldots\ldots,n \tag{4}$$

such that $SAS = |A|$.

2- A matrix $A$ is Schur stable Morishima if it is both Schur stable and Morishima.

### Lemma 1 [18]

If $\rho[A] < p$, then $pI \pm A$ is a non-singular matrix. $\rho[A]$ is the spectral redius of a expressed as $\rho[A] = \max_i |\lambda i|$ and $\lambda i$ is an eigen value of A.
With $S$ being the same as in (4) for both $B$ and $C$ in the interval matrix $A_I = [B,C]$, we construct the matrix $W = [w_{ij}]$, $i,j = 1,\ldots\ldots n$ where
$w_{ij} = \max\limits_{i,j} \{(SBS)_{ij}, (SCS)_{ij}\}$
This constructed matrix and the average matrix $V$ defined in (2) are used often in the following discussion.

### Theorem 1

If $\pm V$ is Schur stable Morishima, then $A_I$ is Schur stable if and only if $W$ is Schur stable.

### Proof

Since $\pm V$ is a Morishima matrix, then there exists $S$ of the form (4) such that $S(\pm V)S = |V|$ is the average matrix of the interval matrix $SA_IS$. If $A_I$ is Schur stable, then $SA_IS$ is Schur stable. But $W \in SA_IS$, then $W$ is Schur stable.

If $W$ is Schur stable, then for every $A \in A_I$, we have $\rho[A] = \rho[SAS] \leq \rho[W] < 1$ which implies that $A_I$ is Schur stable. This completes the proof.

This theorem concludes that the search for a Morishima matrix is by only testing the average matrix or its negative. While in the version provided in [1], it is needed to look among all extreme vertices of the interval matrix for a Morishima that corresponds to the test matrix. In our case, the test matrix $W$ is determined directly. For matrices of large size, this improvement or version of the condition might be appreciated significantly. Furthermore, it may be important to indicate that if $\pm V$ are not both Schur stable Morishima, then the theorem provided fails to conclude any thing about the stability of $A_I = [B,C]$.

## Corollary 1

If $\pm H$ is Morishima, then the interval matrix $[-|H|,|H|]$ is Schur stable if and only if $H$ Schur stable.

## Proof

The null matrix $O$ is the average matrix for the interval $[-|H|,|H|]$. It is also Schur stable Morishima that satisfies Theorem 1 with $S$ such that $S(\pm H)S = |H|$.

## Corollary 2

$A_I$ is Schur stable if $|V| + D$ is Schur stable.

## Proof

$A_I$ is a subset of the interval matrix $\left[-(|V|+D),(|V|+D)\right]$. But if $|V|+D$ is Schur stable, then it is Schur stable Morishima. Therefore, the Schur stability of $A_I$ follows from Corollary 1.

These two corollaries are stated as the main theorem and a corollary, respectively, in [1] while Corollary 2 coincides with the main result reported by Rachid [7]. Moreover, for interval matrices that do no have a Morishima matrix in them, we introduce more sufficient conditions for their Schur stability. These conditions are shown, through examples, to improve the condition in corollary 2 and at the same time refrain from the constraint $\max_{i,j}\left\{|b_{ij}|,|c_{ij}|\right\} < 1$.

## Theorem 2

If the average matrix $V$ is such that $|V|$ is Schur stable, then $A_I$ is Schur stable if $\rho\left[(I-|V|)^{-1}D\right]<1$.

## Proof

The interval matrix $A_I$ can be represented in the form $V+E$ where $|E|\le D$. Using the equality

$$zI-(V+E)=(zI-V)^{-1}\left[I-(zI-V)^{-1}E\right]$$

and Lemma 1, it is clear that the Schur stability of $V+E$ is satisfied if $\rho\left[(zI-V)^{-1}E\right]<1$ for all $|z|\ge 1$ which is also true if

$$\rho\left[(zI-V)^{-1}E\right]\le\rho\left[\left|(zI-V)^{-1}\right|D\right]\le\rho\left[(I-|V|)^{-1}D\right]<1$$

for all $|z|\ge 1$.

## Theorem 3

Let $V\in A_I$ be Schur stable average matrix of $A_I$, then $A_I$ is Schur stable if $M$ has no eigenvalues on the unit circle and $\left\|(I-V)^{-1}\right\|_\infty<\frac{1}{\|D\|_\infty}$ where $D$ is the deviation matrix and

$$M=\begin{bmatrix} V-\|D\|_\infty^{\;2}V^{-T} & V^{-T} \\ -\|D\|_\infty^{\;2}V^{-T} & V^{-T}\end{bmatrix}$$

## Proof

The interval matrix $A_I$ can be represented in the form $V+E$ where $|E|\le D$. Using the equality

$zI-(V+E)=(zI-V)^{-1}\left[I-(zI-V)^{-1}E\right]$

and Lemma 1, it is clear that the Schur stability of $V+E$ is satisfied if $\rho\left[(zI-V)^{-1}E\right]<1$ for all $|z|\ge 1$ which is also true if

$\left\|(zI-V)^{-1}E\right\|_\infty\le\left\|(zI-V)^{-1}\right\|_\infty\|D\|_\infty<1$

for all $|z| \geq 1$. But if $M$ has no eigenvalues on the unit circle and $\left\|(I-V)^{-1}\right\|_{\infty} < \frac{1}{\|D\|_{\infty}}$ [19], then $\left\|(zI-V)^{-1}\right\|_{\infty} < \frac{1}{\|D\|_{\infty}}$. This completes the proof.

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## Example 1

Consider the interval matrix of the discrete time systems $A_I = [B,C]$ with

$$B = \begin{bmatrix} 0.4 & -0.6 \\ -0.6 & 0.4 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.5 & -0.4 \\ -0.4 & 0.5 \end{bmatrix}$$

The average matrix $V$ is a Schur stable Morishima with

$$V = \begin{bmatrix} 0.45 & -0.5 \\ -0.5 & 0.45 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Applying Theorem 1, we conclude that

$$W = \begin{bmatrix} 0.5 & 0.6 \\ 0.6 & 0.5 \end{bmatrix}$$

is not Schur stable. Therefore, $A_I$ is not Schur stable. Notice that $SWS \in A$ is not Schur stable.

## Example 2

Consider the interval matrix of the discrete time systems $A_I = [B,C]$ with

$$B = \begin{bmatrix} 0.15 & 0 \\ 0.4 & -0.45 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.55 & 0 \\ 0.8 & 0.25 \end{bmatrix}$$

The average and deviation matrices are

$$V = \begin{bmatrix} 0.35 & 0 \\ 0.6 & -0.1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.35 \end{bmatrix}$$

Since $\pm V$ is not Morishima and $|V|$ is Schur stable, we use Theorem 2 to get $\rho\left[(I-|V|)^{-1}D\right]<0.3889$. Therefore $A_I$ is Schur stable. Notice also that $H=\begin{bmatrix}0.55 & 0\\ 0.8 & 0.45\end{bmatrix}$ is a Schur stable Morishima matrix, which implies that, $[-|H|,|H|]\supset A_I$ is Schur stable by corollary 1 and corollary 2. Hence, $A_I$ is Schur stable.

## Example 3

Consider the interval matrix of the discrete time systems $A_I=[B,C]$ with

$$B=\begin{bmatrix}0.5 & -0.6\\ 0.4 & 0.5\end{bmatrix},\qquad C=\begin{bmatrix}0.5 & -0.4\\ 0.6 & 0.5\end{bmatrix}$$

The average and deviation matrices are

$$V=\begin{bmatrix}0.5 & -0.5\\ 0.5 & 0.5\end{bmatrix},\qquad D=\begin{bmatrix}0 & 0.1\\ 0.1 & 0\end{bmatrix}$$

Notice that $\pm V$ is not Morishima, and $|V|$ is not Schur stable. Corollary 2 and the condition in [6] are inconclusive about the Schur stability of this interval matrix. However, using Theorem 3, we conclude that $A_I$ is Schur stable.

## 3. CONCLUSIONS

New sufficient conditions for the Schur stability of interval matrices are provided. The conditions are shown to improve and sometimes coincide with results reported in the literature. One disadvantage of the sufficient condition in Theorem 3 is giving a blind eye to the structure of the perturbation of $E$ and dealing only with the infinity norm of $E$. The necessary and sufficient condition that is provided in Theorem 1 improves the necessary and sufficient condition in [1, corollary 1.3] in the sense of its easiness to check the Schur stability of interval matrices.

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