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ON EXISTENCE OF TWO REAL PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS OF RICCATI TYPE

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ABSTRACT

In this paper we consider the two equations

$$\dot{z} = z^2 + p(t)z + r_0(t), \qquad (*)$$

$$\dot{z} = z^2 + p(t)z + r_1(t), \qquad (**)$$

where $z \in C$, p, r_1 and r_0 are real, continuous and periodic with period T.

It was shown in[2]that if (*) has two T–periodic solutions and if $r_1(t) \leqslant r_0(t)$ for all $t \in [O, T]$, then (**) has two T–periodic solutions. In this note we extend this result by showing that if, moreover, the two T–periodic solutions of (*) are real then so are the T–periodic solutions of (**).

1. Preliminaries

This paper is concerned with the class H of differential equations

$$\dot{z} = z^2 + P(t)z + r(t) \qquad (z \in C, t \in R), \qquad (1)$$

where p and $r \in \mathbb{P}$ and $\mathbb{P}$ is the class of all continuous real–valued functions of period T (T being fixed throughout). The equation (1) is denoted by P and we regard H as the set $\mathbb{P} \times \mathbb{P}$ with norm

$$|P| = \max\{|p(t)|, |r(t)|; O \leqslant t \leqslant T\};$$

then (H, | . |) is a Banach Space.

The solution of P satisfying $z(t_o) = z_o$ is written $z_P(t; t_o, z_o)$ and the periodic solutions of P are determined by the zeros of

$$q_P : c \longrightarrow z_P(T; O, c) - c.$$

The function q_P is defined on an open subset Q_P of C.

To assist the reader we give precis of those definitions and results from [2], [3], and [4] which we shall need. The multiplicity of a periodic solution ϕ of P is defined as the multiplicity of $\phi(O)$ as a zero of q_P. It is shown in [4] that H has the following subsets

$B = \{ P \in H;$ P has a real solution which is unbounded both as t increases and as t decreases and is defined for a t−interval of length less than T $\}$,

$H_1 = \{ P \in H;$ P has two real T−periodic solutions and no other periodic solutions $\}$.

$H_2 = \{ P \in H;$ P has two T−periodic solutions, complex conjugate, and no other periodic solutions $\}$.

Account is always taken multiplicity in these definitions. Hence $P \in H_1$ may have only one periodic solution of multiplicity 2. Let H_{11} be the set of P which have exactly one real T−periodic solution. In [2] we proved that H_{11} is the boundary between H_1 and H_2, that is; $H_{11} = \bar{H}_1 \cap \bar{H}_2$ (where $\bar{H}_1$ and $\bar{H}_2$ are the closures of H_1 and H_2, respectively) and Lloyd in [4] proved that $H_1 \cup H_2$ is a component of $H \setminus B$.

In [2] we proved that H_2 and $H_1 \cup H_2$ are open subsets of H and H_1 is a closed subset of H.

2. Two Real T−Periodic Solutions

The method used in [2], [3] and [4] to study P was to look at the linear equation P^* :

$$\ddot{u} - p(t)\dot{u} + r(t)u = O, \qquad (2)$$

whose solutions are related to those of P by the transformation $z = -\dot{u}/u$. Let D be the set of P whose corresponding P^* are disconjugate on $[O, T]$. (Recall that a second order linear differential equation is disconjugate on an interval I if every non-trivial real solution has fewer than two zeros in I).

Lemma 2.1 $B \supseteq H \setminus D$.

(For the proof see [3]).

Directly from Theorem 7 of [1] we can prove the following lemma,

Lemma 2.2 $P = (p, r) \in D$ if and only if

$$\int_O^T \left(\exp - \int_O^t p(s)\, ds\right)(\dot{y}^2 - ry^2)\, dt > O$$

for all functions y which are piecwise continuously differentiable on $[O, T]$ and satisfy $y(O) = y(T) = O$.

Directly from Lemma 2.2 we can prove the following lemma,

Lemma 2.3 Let $(p, r_o) \in D$ and $r_1 \in P$. If $r_1(t) \leqslant r_o(t)$ for all $t \in [O, T]$, then $(p, r_1) \in D$.

Lemma 2.4 If ϕ is the unique T-periodic solution of $(p, r) \in H_{11}$, then

$$2\int_O^T \phi(t)\, dt = -\int_O^T p(t)\, dt.$$

(For the prove see [4]).

Lemma 2.5 If (p, r_o), $(p, r_1) \in H_{11}$, and $r_o(t_o) > r_1(t_o)$ for some $t_o \in [O, T]$, then there exists $t_1 \in [O, T]$ such that $r_o(t_1) \leqslant r_1(t)$.

Proof Suppose that $r_o(t) > r_1(t)$ for all $t \in [O, T]$ and $\phi_o, {}_1\phi$ are the periodic solutions of (p, r_o) and (p, r_1), respectively. We have two cases: (i) $\phi_i(t) > \phi_j(t)$ for all $t \in [O, T]$, (ii) $\phi_o(t_2) = \phi_1(t_2)$ for some $t_2 \in [O, T]$.

Case (i) In this case we have

$$\int_O^T \phi_i(t)\,dt > \int_O^T \phi_j(t)\,dt,$$

which cotradicts Lemma 2.4

Case (ii) Let $h(t) = \phi_o(t) - \phi_1(t)$. If $h(t_2) = O$ for some $t_2 \in [O, T]$, then $\dot{h}(t_2) = r_o(t_2) - r_1(t_2) > O$. Hence $h(t) \geqslant O$ over $[O, T]$ and $h(\dot{t}) > O$ for some $\dot{t}$ near t_2. Therefore

$$\int_O^T h(t)\,dt \geqslant O$$

and again we have a contradiction to Lemma 2.4.

Theorem 2.6 Suppose that $(p, r_o) \in H_1$. If $r_1 \in \mathbb{P}$ and $r_1(t) \leqslant r_o(t)$ for all $t \in [O, T]$, then $(p, r_1) \in H_1$.

Proof Let us assume that $r_1(t) < r_o(t)$ for all $t \in [O, T]$.

Since $(p, r_o) \in H_1$, then by lemma 2.3

$$L_1 = \{ (p, \lambda r_o + (1-\lambda) r_1);\ O \leqslant \lambda \leqslant 1 \} \leqslant D$$

Hence (p, r_o) and (p, r_1) are in the same component of $H \setminus B$. Hence $(p, r_1) \in H_1 \cup H_2$ (see Theorem 2 of [4]).

Let us assume that $(p, r_1) \in H_2$ and let $L_2 = \{ (p, \lambda r_1);\ O \leqslant \lambda \leqslant 1 \}$. It is clear that $L_1 \cap H_1 \neq \phi$, $L_1 \cap H_2 \neq \phi$, $L_2 \cap H_2 \neq \phi$ and $L_2 \cap H_1 \neq \phi$. Hence there exist λ_1 and λ_2 such that $(p, \lambda_1 r_o + (1 - \lambda_1) r_1)$ and $(p, \lambda_2 r_1) \in H_{11}$. But $\lambda_2 r_1 < \lambda_1 r_o + (1 - \lambda_1) r_1$ contradicts Lemma 2.4 Therefore $(p, r_1) \in H_1$.

Now suppose that $r_1(t) \leqslant r_o(t)$ for all $t \in [O, T]$. Let $s_n = r_1 - (1/n)$ $(n = 1, 2, \ldots)$. Hence $(p, s_n) \in H_1$ and $(p, s_n) \longrightarrow (p, r_1)$ as $n \longrightarrow \infty$.

Therefore $(p, r_1) \in H_1$, because H_1 is a closed subset of H.

Corollary Let $r \in P$ and $k \in R$. If $r(t) \leqslant k^2/4$ for all $t \in [O, T]$, then $(k, r) \in H_1$.

Proof It can be checked that $(k, b) \in H_1$, where $b = \max r(t)$. Hence by Theorem 2 – 6 $(k, r) \in H_1$.

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في وجــود حليــن حقيقييــن دورييــن لمعـادلات تفاضليـة من نـوع ريـكاتي

حسـن صـادق حسـن

في هذا البحث سندرس المعادلتين :

$$\dot{z} = z^2 + p(t)z + r_o(t), \qquad (*)$$

$$\dot{z} = z^2 + p(t)z + r_1(t) \qquad (**)$$

حيث z و C و r_o, r_1, p دوال حقيقية مستمرة ، دورية بدورة مقدارها T

سنبرهن إذا $(*)$ عندها حلين حقيقيين دوريين فان $(**)$ لها حلين حقيقيين دوريين إذا

$$r_1(t) \leqslant r_o(t) \qquad t \in [O, T]$$