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Algebraic Approach to Formal Microstructure Sheaves and Formal Quantum Sheaves Over Projective Schemes

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نظرة جبرية على حزم البناء الدقيق الصورية على مستوى التدرج والتصفية والمعرفة على الفراغ الاسقاطي p (Y)

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في هذا البحث ندرس ونبرهن خواص جبرية ، كما في النماذج كاملة التصفية ، لحزم البناء الدقيق الصورية على مستوى التدرج والتصفية والمعرفة على الفراغ الاسقاطي (Y = P(Y) = Proj G (R)) كما نلاحظ أن البناء الدقيق للنماذج الكاملة بالنسبة لمثالي تكون حلا للحزم الصورية . واخيرا نثبت ان الحزم الصورية تكون قوية ومنسبطة التصفية .

Key words: *Formal filtered, Formal graded sheaves.*

ABSTRACT

The I-adic non-commutative filtrations have crucial applications in the study of Lie algebras and the integral group rings. The I-completion $\hat{R}$ of a commutative ring R is considered a useful tool in algebraic geometry. Here in this paper we study and prove certain algebraic properties of the filtered (Rees-graded) formal micro-sheaves over Y = $\mathrm{spec}^g(G(R))$ or p(Y) = proj(G(R)). We observed that the microlocalizations of I-adic completions appear as solutions for the formal sheaves over Y or p(Y). We prove that the formal sheaves over Y or p(Y) are strongly filtered flat sheaves.

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I. Introduction And Preliminaries

Sheaf theory provides a language for the discussion of geometry of objects of many different kinds. For a long time after its introduction by Lerary, (1945), sheaf theory was mainly applied to the theory of functions of several complex variables or to algebraic geometry, (Hartshorne 1966, 1977), until it became a basic tool for almost all mathematicians, (Schapira 1990, F. Vanoystaeyen 1981-.....). It is well known that the general purpose of sheaf theory is to obtain global information from local one, or else to define "obstructions" which characterize the fact that a local property does not hold globally any more, for example a manifold is not always orientable.

Constructing graded and filtered ring theory, Zariskian filtrations and microlocalizations of filtered objects by LiHuishi, Sallam, Verschoren and Vanoystaeyen, (1981-1993), gave Radwan, (1993-1999), the ability to build an almost non commutative sheaf theory over Schemes on filtered, graded and quantum levels. Many results concerning this theory had been obtained, by (Sallam, Radwan and Vanoystaeyen, (1993-1999). Here in this paper we try to obtain one, ([15] another), of essential objectives concerning this theory too and for the formal microstructure sheaves over projective schemes. At the same time, we mimic and construct sheaf theory on the fuzzy level, see [14].We recall and fix some basic notions while for full detail we have to refer to the references at the end of this note. It is hard to make this section more self-contained.

On graded and filtered ring theory, Zariskian filtrations and micro localization of filtered rings we may refer to the refs. [1], [4], [5], [8] and [17]. Some details on commutative (non commutative) algebraic geometry, for example, theory of schemes and coherent sheaves may be found in [3], [6] and [16]. Finally for the theory of coherent microstructure sheaves over schemes on the graded and filtered level one may handle [9], [10], [11], [12], [13], [17].

Most of the results in this paper are shown locally so we may work on $Y = spec^g(G(R))$, the graded prime spectrum of the commutative associated graded domain $G(R)$ for the Zariski strongly filtered R.

Y may have a base β of affine Noetherian basic open subsets in Y. The strongly graded condition on $G(R)$ gives rise that $Y = p(Y) = \{p \in Y : \oplus_{n>0} G(R)_n \not\subset p\}$ is possible, see [12]. Associating to $Y(f) \in b; f \in h(G(R))$ the Rees microlocalizations $\tilde{Q}^{\mu}_{\tilde{f}}(\tilde{R})$ respectively the strongly Zariski filtered microlocalizations $Q^{\mu}_{f}(R)$ we obtain the Rees coherent graded sheaves $\underline{\tilde{O}}^{\mu}_{Y}$ respectively the coherently filtered Zariski sheaves $\underline{O}^{\mu}_{Y}$ on Y having as the completed stalks at $p \in Y$ the Noetherian graded rings $\tilde{Q}^{\mu}_{p}(\tilde{R})$, respectively the Zariski filtered rings $Q^{\mu}_{p}(R)$. Replacement of R by $M \in R$-filt. with good filtration FM leads to the construction of coherent Rees $\underline{\tilde{O}}^{\mu}_{Y}$-Module $\underline{\tilde{M}}^{\mu}_{Y}$ and coherently filtered $\underline{O}^{\mu}_{Y}$-Module $\underline{M}^{\mu}_{Y}$ such that $(\underline{\tilde{M}}^{\mu}_{Y,p})^{\wedge} = \tilde{Q}^{\mu}_{p}(\tilde{M})$ and $(\underline{M}^{\mu}_{Y,p})^{\wedge} = Q^{\mu}_{p}(M)$. The graded objects $G(Q^{\mu}_{f}(R)) = Q^{g}_{f}(G(R))$ define the associated coherent graded sheaves $\underline{O}^{g}_{Y}$ on Y and the coherent graded $\underline{O}^{g}_{Y}$-Module $\underline{M}^{g}_{Y}$

such that $\underline{O}^g_{Y,p} = Q^g_p(G(R))$ and $\underline{M}^g_{Y,p} = Q^g_p(G(M))$. Restricting to parts of degree zero in filtrations and gradations gives the coherent sheaves of quantum sections over Y or p(Y). By using these sheaves we carried out, cf. [11] and [12], the formal theory of schemes on the graded level, the filtered level, the Ress level and the quantum level.

1.1. Lemma

For every $\underline{M}_y \in \underline{O}^\mu_y$-Filt, that is coherently filtered. It follows that $\tilde{\underline{M}}_y$ is $\underline{X}$-torsion free; see [13].

The aim of these notes is to show certain microstructure sheaves-like algebraic properties as in [13] of the formal microstructure sheaves over the projective scheme p(Y).

2. Filtered and Graded Formal schemes and Quantum Schemes.

Let $\underline{I}^\mu_Y$ be a coherently filtered sheaf of ideals in $\underline{O}^\mu_Y$ for good induced filtered ideal I in the Zariski filtered ring R. Then $\tilde{\underline{I}}^\mu_Y$ is a coherent graded sub sheaf of graded ideals in $\tilde{\underline{O}}^\mu_Y$. Put $I^g = G(I)$ and $\hat{Y} = V(I^g) \subset Y$ as a topological space. On $\hat{Y}$ we introduce the formal filtered microstructure sheaves $\underline{O}^\mu_{\hat{Y}}$; they are defined as:

$$\underline{O}^\mu_{\hat{Y}} = \underset{\overleftarrow{n}}{\lim}^F[\underline{O}^\mu_Y/(\underline{I}^\mu_Y)^n] = \underline{(\underset{\overleftarrow{n}}{\lim}^f R/I^n)^\mu}_Y$$

with

$$\underline{O}^\mu_{\hat{Y},p} = \underset{\overrightarrow{n}}{\lim}^f[\underline{O}^\mu_{Y,p}/(\underline{I}^\mu_{Y,p})^n] = [\underline{O}^\mu_{Y,p}]^{\wedge Q}; Q = \underline{I}^\mu_{Y,p}$$

and $p \in Y$, formal Rees microstructure sheaves $\hat{\underline{O}}^\mu_{\hat{Y}}$ which are defined as:

$$\tilde{\underline{O}}^\mu_{\hat{Y}} = \underset{\overleftarrow{n}}{\lim}^G[\tilde{\underline{O}}^\mu_Y/(\tilde{\underline{I}}^\mu_Y)^n] = \underline{[\underset{\overleftarrow{n}}{\lim}^g(\tilde{R}/\tilde{I}^n)]^\mu}_Y = \underline{[\underset{\overleftarrow{n}}{\lim}^f R/I^n]^{\sim\mu}}_Y$$

with $\tilde{\underline{O}}^\mu_{\hat{Y},P} = \underset{\overrightarrow{N}}{\lim}^G[\tilde{\underline{O}}^\mu_{Y,P}/(\tilde{\underline{I}}^\mu_Y)^n] = [\tilde{\underline{O}}^\mu_{Y,p}]^{\wedge\tilde{Q}}; \tilde{Q} = \tilde{\underline{I}}^\mu_{Y,p}$ and $p \in Y$ and formal quantum sheaves on $\tilde{Y}$ are defined as: $\underline{O}^{\mu,q}_{\hat{Y}} = F_0\underline{O}^\mu_{\hat{Y}}$ with $\underline{O}^q_{\hat{Y},p} = F_0[(\underline{O}^\mu_{Y,p})^{\wedge Q}]$, p and q are being as above. Replacement R by $M \in R$-filt. With good filtration FM leads to the construction of coherent Ress $\tilde{\underline{O}}^\mu_{\hat{Y}}$-Module $\tilde{\underline{M}}^{\mu,\Delta}_{\hat{Y}} = \underline{[\underset{\overleftarrow{n}}{\lim}^f(M/I^nM)]^{\sim\mu}}_Y$ with $\tilde{\underline{M}}^{\mu,\Delta}_{\hat{Y},p} = [\tilde{\underline{M}}^\mu_{Y,p}]^{\wedge\tilde{Q}}$, coherently filtered $\underline{O}^\mu_{\hat{Y}}$-Module $\underline{M}^{\mu,\Delta}_{\hat{Y}} = \underline{[\underset{\overleftarrow{n}}{\lim}^f(M/I^nM)]^\mu}_Y$ with $\underline{M}^{\mu,\Delta}_{\hat{Y},p} = (\underline{M}^\mu_{Y,p})^{\wedge Q}$ and coherently filtered $\underline{O}^{\mu,q}_{\hat{Y}}$-Module $\underline{M}^{\mu,q}_{\hat{Y}} = F_0\underline{M}^{\mu,\Delta}_{\hat{Y}}$ with $\underline{M}^{\mu,q}_{\hat{Y},p} = F_0[\underline{M}^\mu_{Y,p}]^{\wedge Q}$.

2.1, Lemma.

With considerations above, we may have that $\tilde{\underline{O}}^\mu_{\hat{Y}} \cong (\underline{O}^\mu_{\hat{Y}})^\sim$ and $\tilde{\underline{M}}^{\mu,\Delta}_{\hat{Y}} \cong (\underline{M}^{\mu,\Delta}_{\hat{Y}})^\sim$ as formal Ress Sheaves over $\hat{Y}$.

2.2. Remark.

a. We may regard the sheaves over $\widehat{Y}$ as sheaves on Y, and then this always means the extension by zero outside $\widehat{Y}$.

b. In this way, by using the formal sheaves defined above we may introduce formal "schemes" over Y on the filtered level, the Rees level and quantum level depending on the approach constructing the formal theory, this had been constructed in [11] and [12].

c. The sections of these sheaves can be calculated as; if $Y(f)\in\beta$ we have

$\underline{O}^{\mu}_{\hat{Y}}(Y(f))=(Q^{\mu}_f(R))^{\wedge Q}=Q^{\mu}_f(R^{\wedge I}),\tilde{\underline{O}}^{\mu}_{\hat{Y}}Y(f)=\tilde{Q}^{\mu}_f((R^{\wedge I})^{\sim})=(Q^{\mu}_f(R^{\wedge I}))^{\sim}$ and $(F_0\underline{O}^{\mu}_{\hat{Y}})Y(f))=F_0Q^{\mu}_f(R^{\wedge I})$. Similarly: $\underline{M}^{\mu,\Delta}_{\hat{Y}}(Y(f))=(Q^{\mu}_f(M^{\wedge I})$ as filtered complete $Q^{\mu}_f(R^{\wedge I})$-module, $\tilde{\underline{M}}^{\mu,\Delta}_{\hat{Y}}(Y(f))=(Q^{\mu}_f(M^{\wedge I}))^{\sim}$ as graded $(Q^{\mu}_f(R^{\wedge I}))^{\sim}$-module and $F_0\underline{M}^{\mu,\Delta}_{\hat{Y}}(Y(f))=F_0Q^{\mu}_f(M^{\wedge I})$ as filtered $F_0Q^{\mu}_f(R^{\wedge I})$-module.

2.3. Corollary.

With conventions above we have that the formal microstructure sheaves over Y or p(Y) are again microstructure sheaves over $\widehat{Y}$ or $p(\widehat{Y})$.

Proof. From definition above; since the process of microlocalizations of the I-adic complete good filtered modules $L=\underset{\longleftarrow n}{\lim}^{f}\dfrac{M}{I^nM}$ for any good filtered R-module M gives sections for our formal sheaves. Now, the element $X\in\tilde{R}$ determines a formal global section $\underline{X}^{\Delta}$ of $\tilde{\underline{O}}^{\mu}_{\hat{Y}}$ such that for each $Y(f)\in\beta$ we have $\underline{X}^{\Delta}|_{Y(f)}=X_{[(Q^{\mu}_f(R))^{\wedge Q}]^{\sim}}$ and for each $p\in Y$ we have $\underline{X}^{\Delta}_{\hat{Y},p}=X_{\tilde{\underline{O}}^{\mu}_{\hat{Y},p}}$. Denote by $\underline{I(X)}_{\hat{Y}}=\underline{X}^{\Delta}\tilde{\underline{O}}^{\mu}_{\hat{Y}}$ the formal sheaf of graded ideals in $\tilde{\underline{O}}^{\mu}_{\hat{Y}}$ such that, for each $Y(f)\in\beta,\underline{I(X)}_{\hat{Y}}(Y(f))=X\tilde{Q}^{\mu}_f([R^{\wedge I}]^{\sim})$ is a graded ideal in $\tilde{\underline{O}}^{\mu}_f([R^{\wedge I}]^{\sim})$. A formal graded sheaf $\underline{m}_{\hat{Y}}$ of $\tilde{\underline{O}}^{\mu}_{\hat{Y}}$-Modules is said to be $\underline{X}^{\Delta}$-torsionfree if and only if locally it is X-torsionfree.

2.4. Lemma. With convention above:

a. $\tilde{\underline{O}}^{\mu}_{\hat{Y}}\cong\tilde{\underline{O}}^{\mu}_{Y}$, as graded coherent sheaves over Y, if and only if $\tilde{R}$ is $\tilde{I}$-adic complete.

b. $\underline{X}^{\Delta}$ and $(\underline{I}-\underline{X}^{\Delta})$ act regularly on $\tilde{\underline{M}}^{\mu}_{\hat{Y}}$.

Proof.

a. $\tilde{R}$ is $\tilde{\underline{I}}$-adic complete if and only if $\tilde{R}^{\wedge \tilde{I}}=\underset{\longleftarrow n}{\lim}\tilde{R}/\tilde{I}^n\cong\tilde{R}$ Hence $\tilde{\underline{O}}^{\mu}_{\hat{Y}}=\tilde{\underline{R}}^{\mu}_{\hat{Y}}\cong(\underline{\tilde{R}^{\wedge I}})^{\mu}_{Y}=\tilde{\underline{O}}^{\mu}_{Y}$.

b. Locally, if $U=Y(f)\in\beta$ then the strict filtered morphisms $R\leftarrow Q^{\mu}_f(R)\leftarrow Q^{\mu}_f(R)^{\wedge Q})$ in R-filt induces the graded morphisms

$\tilde{R} \to (Q_f^{\mu}(R))^{\sim} \leftarrow (Q_f^{\mu}(R)^{\wedge Q})^{\sim}$ in $\tilde{R}-gr$. Hence $X \in \tilde{R}$ maps to the corresponding element $X_{(Q_f^{\mu}(R)^{\wedge Q})^{\sim}}$. Then the statement is checked locally since $\underline{X}^{\Delta}|_{Y(f)}$ $(\underline{X}^{\Delta}_{Y,p})$ acts regularly on the formal Rees module of sections (germs).

2.5. Theorem.

Let $F_{\underline{X}^{\Delta}}-Gr$ be the subcategory in $\underline{\tilde{O}}_Y^{\mu}-Gr$ of $\underline{X}^{\Delta}$- torsionfree graded coherent sheaves of modules over $\underline{\tilde{O}}_Y^{\mu}$. For every coherently filtered $\underline{m}_Y^{\Delta} \in \underline{O}_Y^{\mu}$-Filt. we have that $\underline{\tilde{m}}_Y^{\Delta}$ is $\underline{X}^{\Delta}$-torsionfree.

Proof.

From the above corollary 2.3 and lemma 1.1. Moreover; for each basic Noetherian affine open set $Y(f) \in \beta$, the section-wise calculation of the formal Ress sheaves is given $\Gamma(Y(f), \underline{\tilde{M}}_Y^{\mu,\Delta}) = \varprojlim_n{}^{g} \tilde{Q}_{\bar{f}}^{\mu}(\tilde{\bar{M}}); \bar{M} = \dfrac{M}{I^n M}$ and $\tilde{\bar{f}}$ is the image of $\tilde{f}$ module X^m for some m.

2.6. Theorem. With the same considerations:

a. $\underline{\tilde{M}}_Y^{\mu,\Delta} / \underline{X}^{\Delta} \underline{\tilde{M}}_Y^{\mu,\Delta} \cong G(\underline{M}_Y^{\mu,\Delta})$ and $\underline{\tilde{O}}_Y^{\mu} / \underline{X}^{\Delta} \underline{\tilde{O}}_Y^{\mu} \cong G(\underline{O}_Y^{\mu})$ as graded coherent sheaves.

b. $\underline{\tilde{M}}_Y^{\mu,\Delta} /(\underline{1}-\underline{X}^{\Delta} \underline{\tilde{M}}_Y^{\mu,\Delta} \cong \underline{M}_Y^{\mu,\Delta}$ and $\underline{\tilde{O}}_Y^{\mu} /(\underline{1}-\underline{X}^{\Delta} \underline{\tilde{O}}_Y^{\mu} \cong \underline{O}_Y^{\mu}$ as coherently filtered sheaves. Moreover $\underline{\tilde{M}}_Y^{\mu,\Delta} /(\underline{1}-\underline{X}^{\Delta} \underline{\tilde{M}}_Y^{\mu,\Delta} \cong \varinjlim_n (\underline{\tilde{M}}_Y^{\mu,\Delta})_n$ where the maps in the direct system of coherent sheaves of groups are given by the multiplication of $\underline{X}^{\Delta}$ and there are isomorphisms of coherent sheaves of additive groups:

$$\frac{(\underline{\tilde{M}}_Y^{\mu,\Delta})_n + (\underline{1}-\underline{X}^{\Delta})\underline{\tilde{M}}_Y^{\mu,\Delta}}{(\underline{1}-\underline{X}^{\Delta})\underline{\tilde{M}}_Y^{\mu,\Delta}} \cong F_n \underline{M}_Y^{\mu,\Delta}, n \in \mathbf{Z}.$$

Proof.

These statements are easily proved locally too. But we have to point out here briefly how the present proofs extend to this situation, e.g. in item a, it is enough to have for each $Y(f) \in \beta$ that

$$\varprojlim_n{}^{g} Q_{\bar{f}}^{g}(G(\bar{M})) = Q_{\bar{f}}^{g}(G \varprojlim_n{}^{f} \bar{M})); \bar{M} = \frac{M}{I^n M}.$$

When $\underline{\tilde{M}}_Y^{\mu,\Delta}, \underline{\tilde{N}}_Y^{\mu,\Delta}$ are formal Ress coherent graded $\underline{\tilde{O}}_Y^{\mu}$- Modules corresponding to $\underline{M}_Y^{\mu,\Delta}, \underline{N}_Y^{\mu,\Delta}$ for good filtered R-modules M, N then a formal Ress graded sheaf morphism $\tilde{\varphi}^{\Delta}: \underline{\tilde{M}}_Y^{\mu,\Delta} \to \underline{\tilde{N}}_Y^{\mu,\Delta}$ is determined by $\tilde{\varphi}^{\Delta}(Y(f)): (\tilde{Q}_{\bar{f}}^{\mu}(\tilde{M}))^{\wedge Q} \to (\tilde{Q}_{\bar{f}}^{\mu}(\tilde{N}))^{\wedge Q}; Y(f) \in \beta$ affine Noetherian basic open set in Y. For $Y(h) \subseteq Y(f)$ in β, the commutative diagrams:

$$\begin{array}{ccc} (\tilde{Q}^{\mu}_{f}(\tilde{M}))^{\wedge Q} & \longrightarrow & (\tilde{Q}^{\mu}_{f}(\tilde{N}))^{\wedge Q} \\ \downarrow & & \downarrow \\ (\tilde{Q}^{\mu}_{h}(\tilde{M}))^{\wedge Q} & \longrightarrow & (\tilde{Q}^{\mu}_{h}(\tilde{N}))^{\wedge Q} \end{array}$$

give rise, by using theorem 2.6, to commutative diagrams of strict filtered morphisms:

$$\begin{array}{ccc} (Q^{\mu}_{f}(M))^{\wedge Q} & \longrightarrow & (Q^{\mu}_{f}(N))^{\wedge Q} \\ \downarrow & & \downarrow \\ (Q^{\mu}_{h}(M))^{\wedge Q} & \longrightarrow & (Q^{\mu}_{h}(N))^{\wedge Q} \end{array}$$

The converse of this construction is also valid. All these lead to the following result:-

2.7. Theorem. With conventions above:

$$\mathrm{Hom}^{G}_{\tilde{Y}}(\tilde{\underline{M}}^{\mu,\Delta}_{\tilde{Y}}, \tilde{\underline{N}}^{\mu,\Delta}_{\tilde{Y}}) = \mathrm{Hom}^{F}_{\tilde{Y}}(\underline{M}^{\mu,\Delta}_{\tilde{Y}}, \underline{N}^{\mu,\Delta}_{\tilde{Y}});$$

where $\mathrm{Hom}^{G}_{\tilde{Y}}$ stands for the formal Ress graded sheaf morphisms and $\mathrm{Hom}^{F}_{\tilde{Y}}$ stands for the formal filtered sheaf morphisms. The R-modules M, N are good filtered modules with filtrations FM, FN respectively.

2.8. Theorem.

a. The functor $(\ )^{\sim,\Delta}$ determines an equivalence of categories of formal coherently filtered $\underline{O}^{\mu}_{\tilde{Y}}$-Modules and formal coherent graded $\underline{X}^{\Delta}$-torsionfree $\tilde{\underline{O}}^{\mu}_{\tilde{Y}}$-Modules in $F_{\underline{X}^{\Delta}} - \mathrm{Gr}$. In particular, every formal coherent

$\underline{X}^{\Delta}$- torsionfree $\tilde{\underline{O}}^{\mu}_{\tilde{Y}}$-Module $\tilde{\underline{m}}_{\tilde{Y}}$ is of the form $\tilde{\underline{M}}^{\mu,\Delta}_{\tilde{Y}}$ for some $\underline{M}^{\mu,\Delta}_{\tilde{Y}} \in \underline{O}^{\mu}_{\tilde{Y}}$ -Filt. with good filtered R-module M.

b. The localization of $\tilde{\underline{O}}^{\mu}_{\tilde{Y}}$ at the multiplicative closed set of formal global sections $\{\underline{1}, \underline{X}^{\Delta}, \ldots\}$ equals to $\underline{O}^{\mu}_{\tilde{Y}}[\underline{X}^{\Delta}, \underline{X}^{\Delta-1}]$ denoted by $(\tilde{\underline{O}}^{\mu}_{\tilde{Y}})_{\underline{X}^{\Delta}}$. Also $(\tilde{\underline{M}}^{\mu,\Delta}_{\tilde{Y}})_{\underline{X}^{\Delta}} = \underline{M}^{\mu,\Delta}_{\tilde{Y}}[\underline{X}^{\Delta}, \underline{X}^{\Delta-1}]$.

2.9. Remarks.

a. From the results 2.6 and 2.8 we say that $\underline{O}^{\mu}_{\tilde{Y}}$ is a dehogenization of its associated formal Ress sheaf $\tilde{\underline{O}}^{\mu}_{\tilde{Y}}$. In fact, these results are due the case of filtered modules over filtered ring.

b. Replacing Δ by q, i.e. going formal micro-sheaves to formal quantum sheaves, then the statements established above remain valid.

$\underline{m}_{\tilde{Y}} \in \tilde{\underline{O}}^{\mu}_{\tilde{Y}} - \mathrm{Gr}$ is said to be flat (see [6]) if and only if for any exact sequence $0 \to \underline{M}'_{\tilde{Y}} \to \underline{M}_{\tilde{Y}} \to \underline{M}''_{\tilde{Y}} \to 0$ of $\tilde{\underline{O}}^{\mu}_{\tilde{Y}}$-Modules the corresponding sequence

$$0 \to \underline{M}'_{\tilde{Y}} \otimes_{\tilde{\underline{O}}^{\mu}_{\tilde{Y}}} \underline{m}_{\tilde{Y}} \to \underline{M}_{\tilde{Y}} \otimes_{\tilde{\underline{O}}^{\mu}_{\tilde{Y}}} \underline{m}_{\tilde{Y}} \to \underline{M}''_{\tilde{Y}} \otimes \underline{m}_{\tilde{Y}} \to 0$$

is exact.

2.10. Lemma. With notions and assumptions as above

a. The functor $\tilde{M} \rightarrow \underline{\tilde{M}}_{\hat{Y}}^{\mu,\Delta}$ is exact functor from the category of finitely generated $\tilde{R}$ - modules to the category of formal graded coherent Rees $\underline{\tilde{O}}_{\hat{Y}}^{\mu,\Delta}$ -Modules.

b. If $\tilde{M} \in \tilde{R} - gr$. finitely generated then $\underline{\tilde{M}}_{\hat{Y}}^{\mu,\Delta} \cong \underline{\tilde{M}}_{\hat{Y}}^{\mu} \otimes_{\underline{\tilde{O}}_{Y}^{\mu}} \underline{\tilde{O}}_{\hat{Y}}^{\mu}$.

Now we are ready to study the flatness property for formal sheaves:-

2.11. Theorem.

With considerations and assumptions above, $\underline{\tilde{O}}_{\hat{Y}}^{\mu}$ is a flat $\underline{\tilde{O}}_{Y}^{\mu}$-Module.

Proof. This result follows functorially immediately by the above definition and lemma 2.10. Since the sequence of formal graded coherent $\underline{\tilde{O}}_{\hat{Y}}^{\mu}$-Modules

$$\begin{array}{ccccccccc} 0 & \rightarrow & \underline{\tilde{M}}_{\hat{Y}}^{\mu,\Delta} & \rightarrow & \underline{\tilde{N}}_{\hat{Y}}^{\mu,\Delta} & \rightarrow & \underline{\tilde{L}}_{\hat{Y}}^{\mu,\Delta} & \rightarrow & 0 \\ & & \cong & & \cong & & \cong & & \\ & & \underline{\tilde{M}}_{Y}^{\mu} \otimes \underline{\tilde{O}}_{\hat{Y}}^{\mu} & & \underline{\tilde{N}}_{Y}^{\mu} \otimes \underline{\tilde{O}}_{\hat{Y}}^{\mu} & & \underline{\tilde{L}}_{Y}^{\mu} \otimes \underline{\tilde{O}}_{\hat{Y}}^{\mu} & & \end{array}$$

is exact

2.12. Remarks.

a. The above result concerning the formal quantum Ress sheaves follows easily by restricting to (parts of degree zero) the affine Noetherian Y(f) using the strongly graded property.

b. For moment, if we consider that R is I-adic Zariski filtered ring with the I-adic filtration where I is a good filtered ideal in R as submodule with good filtration FI then $\underline{O}_{\hat{Y}}^{\mu} = (\underline{O}_{\hat{Y}}^{\mu})^{\wedge I}$; the $\underline{I}_{Y}^{\mu}$-adic filtered completion of $\underline{O}_{Y}^{\mu}$ and $\underline{M}_{\hat{Y}}^{\mu,\Delta} = (\underline{M}_{Y}^{\mu})^{\wedge I}$; the $\underline{I}_{Y}^{\mu}$-adic filtered completion of $\underline{M}_{Y}^{\mu}$. In this case, we obtain that $\underline{O}_{\hat{Y}}^{\mu}$ is Zariski filtered sheaf over $\hat{Y}$.Going back to Zariski filtered sheaf theory introduced in [9] and[10] leads us to define the notions of strongly filtered sheaves since, under our assumption that R is a Zariski strongly filtered ring, $Q_{f}^{\mu}(R)$ are strongly Zarski filtered rings. The following definition and results are due to F. Vanoystaeyen and L. Huishi, see [4], in case of filtered modules over strongly Zariski filtered ring. So one can easily modify the proofs in [4] to the case of coherent sheaves over the microstructure sheaf, e.g. let $\underline{O}_{Y}$ be a filtered sheaf of rings, then $\underline{O}_{Y}$ is said to be strongly filtered (or $F\underline{O}_{Y}$ is said to be strong) if $F_{n}\underline{O}_{Y}.F_{m}\underline{O}_{Y} = F_{n+m}\underline{O}_{Y}$ such that $F_{n}\underline{O}_{Y}(U).F_{m}\underline{O}_{Y}(U) = F_{n+m}\underline{O}_{Y}(U)$ and $\underline{F_{n}\underline{O}_{Y}}_{Y,p}.\underline{F_{m}\underline{O}_{Y}}_{Y,p} = \underline{F_{n+m}\underline{O}_{Y}}_{Y,p}$ for all $U \in \beta(Y)$ and $p \in Y$. These sheaves include the filtered sheaf of E-rings in sense of [4].

2.13. Theorem. Under our conventions above, the following properties are easily verified

a. $\underline{O}_{X}^{\mu}$ is a strongly filtered sheaf

b. $G\underline{O}_{X}^{\mu}$ is a strongly graded sheaf

c. Every coherently filtered sheaf $\underline{m}_Y \in \underline{O}_Y^{\mu}$-Filt. is strongly filtered in the sense $F_n\underline{O}_Y^{\mu}.F_m\underline{m}_Y = F_{n+m}\underline{m}_Y$. for all $n, m \in \mathbf{Z}$.

d. $\underline{\tilde{O}}_Y^{\mu}$ is a strongly graded sheaf

e. $(\underline{\tilde{O}}_Y^{\mu})_{\underline{X}} = \underline{O}_Y^{\mu}[\underline{X}, \underline{X}^{-1}]$ is a strongly filtered sheaf

f. Considering assumptions in Remark 2.12 leads that the above properties are valid for the formal level.

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