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A CLASS OF FUNCTIONS STRONGER THAN M-PRECONTINUITY, PREIRRESOLUTE AND A-FUNCTIONS

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ABSTRACT

We propose a definition for a new class of functions that may be stronger than the previous types. Some of its characterizations, connections with known ones and the effect of them on some topological spaces are studied .

1 - INTRODUCTION

In recent years, one type of functions between topological spaces has been defined by three independent authors. In 1982, Mashhour *et al.* (3) called it M-precontinuous, in 1985, Reilly and Vamanamurthy (6) named it a preirresolute function and in 1988, Rose (8) called it A-function depending on the concept of almost-open set (8).

Therefore, let A be a subset of a topological space X and its closure and interior are denoted by cl(A) and int(A), respectively. A is called preopen (2) or almost open (8) if $A \subset$ int (cl (A)) and (X-A) is called preclosed or almost closed, respectively. The intersection (union) of all preclosed (preopen) sets which contain (contained in) A, is called preclosure (3) (preinterior (3)) and denoted by p.cl(A)(p.int)A)). The preboundary of A (p.b(A)) is the intersection of p.cl(A) and p-cl(X-A). The collection of all preopen sets in X will be denoted by PO(X). A function $f:X\rightarrow Y$ is called M-precontinuous (3) (M.P.C) or preirresolute (6) (A-function(8)) if $f^{-1}(V)$ is preopen (almost open) in X for each preopen (almost open) V in Y. f is called preopen (2) (preclosed (5)) if the image of each open (closed) is preopen (preclosed). A space X is called strongly compact (4) (S-closed (9), strongly Lindelof (4)) if for each preopen (semi-open, preopen) cover of X, there exists a finite subcover (a finite subcover, countable subcover) covers (its closure covers) X. Also, X is called a

resolvable space (1) if there is a subset D of X such that D and X-D are both dense in X. Spaces pre-T_i (i = 0,1,2) defined likewise the spaces T_i (i = 0,1,2,) except that open sets are replaced by preopen ones.

2 - DEFINITION AND CHARACTERIZATIONS

Definition 2.1: A function f:X→ Y is called strongly M-precontinuous (abbreviated as SMPC), if the inverse image of each preopen in Y is open in X.

Some characterizations of SMPC are established throughout the next result.

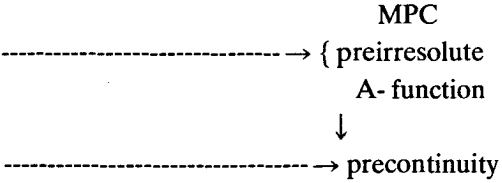
Theorem 2.1: Let f:X→ Y be a function, then the following statements are equivalent.

(i) f is SMPC.
(ii) For each $x \in X$ and each $V \in PO(Y)$ containing f(x), there exists an open set U of X containing x such that $f(U) \subset V$.
(iii) the inverse image of each preclosed set in Y is closed in X.
(iv) $cl(f^{-1}(B)) \subset f^{-1}(p.cl(B))$. , for each $B \subset Y$.
(v) $f(cl\ A) \subset p.cl(f(A))$, for each $A \subset X$.
(vi) $f^{-1}(p.int(B)) \subset int(f^{-1}(B))$, for each $B \subset Y$.
(vii) $b(f^{-1}(B)) \subset f^{-1}(p.b(B))$, for each $B \subset Y$.
(Viii) $f(b(A)) \subset p.b(f(A))$, for each $A \subset X$.

Proof: (i) ⟷ (ii) and (i) ⟷ (iii) are obvious.
(iii) → (iv) Since $B \subset p.cl(B)$, for each $B \subset Y$. so., $f^{-1}(B) \subset f^{-1}(p.cl(B))$, but $f^{-1}(p.cl(B))$ is closed, hence $cl(f^{-1}(B)) \subset f^{-1}(p.cl(B))$.
(iv) ⇒ (v). Replacing f(A) instead of B in (iv).
(v) ⇒ (iii). Let $F \subset Y$ be preclosed, then $f^{-1}(F) \subset X$ and so $f(clf^{-1}(F)) \subset p\text{-}cl(ff^{-1}(F)) \subset p\text{-}cl(F) = F$. This gives $clf^{-1}(F) \subset f^{-1}(F)$. Hence $f^{-1}(F)$ is closed.
(i) ⇒ (vi). For any $B \subset Y$, $p.int(B) \in PO(Y)$ and so $f^{-1}(p\text{-}int(B)) \in \tau(X)$. Hence $f^{-1}(p.int(B)) = int(f^{-1}(p.int(B))) \subset int(f^{-1}(B))$.
(vi) ⇒ (i). Follows directly by taking $B \in PO(Y)$.
(vi) ⇒ (vii). Let $B \subset Y$, since $b(f^{-1}(B)) = cl(f^{-1}(B)) - int(f^{-1}(B)) \subset f^{-1}(p.cl(B)) - f^{-1}(p.int(B)) = f^{-1}(p.b(B) \cup (p.int(B)))$. $f^{-1}(p.int(B)) = f^{-1}(p.b(B))$.
(vii) ⇒ (viii). By putting f(A) instead of B, the result follows immediately.
(viii) ⇒ (ii). Let $B \subset Y$ be preclosed, then $f^{-1}(B) \subset X$
by (viii) $f(b(f^{-1}(B))) \subset p.b(ff^{-1}(B))\ p.b(B) \subset p.cl(B) = B$
and so $b(f^{-1}(B)) \subset int\ f^{-1}(B)) \subset f^{-1}(B)$ i.e. $cl\ f^{-1}(B) \subset f^{-1}(B)$. Hence $f^{-1}(B)$ is closed and therefore f is SMPC.

3. SOME CONNECTIONS AND PROPERTIES OF SMPC

Next diagram shows the implications between SMPC and other known types of functions.



These implications are not reversible as shown throughout the following examples.

Example 3.1: Let X = {a, b, c, d } having τ_1 = { X, ϕ, {a, b}, {a, b, d}} and τ_2 = {X, $\emptyset$, {a,c}, {a, b, c}}.

The function f: (X, τ_1) → (X, τ_2) defined as f(a) = f(b) = a, f(c) = b and f(d) = c is continuous but not SMPC.

Example 3.2: Let X = {a, b, c, d } with τ_1 be an indiscrete topology and τ_2 = {X, $\emptyset$, {a}, {b}, {a, b}}. The identity function from (X, τ_1) (X, τ_2) is M-precontinuous but not SMPC.

Theorem 3.1: The following holds for any function f: X → Y

(i) If X is submaximal, then f is SMPC iff it is M-precontinuous.
(ii) SMPC coincides on continuity if Y is submaximal.
(iii) If both X and Y are submaximal, then all types in previous diagram are equivalent.

Proof: Follows by using Theorem (4) (7).

Theorem 3.2: A function f: X→ Y, where Y = FUG denote the Hewitt-representation, cl (G) is open and {Y} $\notin$ PO(Y) for each Y $\in$ int. (F.) Then f is SMPC iff f : X → PO(Y) is continuous.

Proof: Under these assumptions and by Theorem (5) of (1), we have f : X → Y is SMPC iff $f^{-1}(V)$ is open in X, for each V $\in$ PO(Y), that is iff f: X → PO(Y) is continuous.

Corollary 3.1: Any function f : X → Y, where Y is resolvable and each open set in it is closed. Then f is SMPC iff f : X → PO(Y) is continuous.

Proof: Since resolvabilty of any space in which each open set is closed is considered as one of the applications of the properties on Y in previous Theorem ((1), Corollary (2)). Then f: X → Y is SMPC iff f : X → PO(Y) is continuous.

The following two results concerning compositions and restriction can be shown, therefore their proofs are an immediate consequence of their definitions.

Theorem 3.3: The following holds for functions f : X→Y and g:Y→Z.

(i) If both f and g gave were SMPC, then their composition is also SMPC.
(ii) If gof is SMPC and f is open (preopen), then g is SMPC (preirresolute.)
(iii) If g is SMPC and gof is preopen (preclosed), then f is open (closed).
(iv) If f is SMPC and g is precontinuous, then gof is continuous.
(v) gof is SMPC if one of the next is verified.
  (1) f is SMPC and g is preirresolute.
  (2) f is continuous and g is SMPC.

Theorem 3.4: Two statements are true for a function f: X→ Y
(i) For any A⊂ X, f/A is SMPC if f is SMPC.
(ii) If $\{U_\alpha, \alpha \in \nabla\}$ is an open cover of X, then f is SMPC, If f/U_α is SMPC, for each $\alpha \in \nabla$.

Lemma 3.1 (5): Let $\{ X_i : i \in I \}$ be a family of topological spaces, $X = \prod X_i$ the product space and A_i be a nonempty subset of X_i for $i \in I$

each $i \in I$. For a positive integer n if $A = \prod^n A_{ij} \times \prod_{j=i \quad i=j} X_i$, then

$A \in PO(X)$ iff $A_{ij} \in PO(X_{ij})$, $j \in \{ 1, 2, 3, \ldots\ldots\ldots n \}$.

Theorem 3.5: The function f: X→ Y is SMPC if the graph function g : X→ XxY of f which defined by g(x) = (x,f(x)), for each x ∈ X is SMPC.

Proof: Let g be SMPC, x ∈ X and V∈ PO(Y) containing f(x). Then by Lemma (3.1) XxV∈ PO(XxY) containing g(x), so there exists an open set U of X containing x such that g(U)⊂ XxV and hence f(U)⊂ XxV. Therefore f is SMPC.

Theorem 3.6: Let $\{X_\alpha: \alpha \in \nabla\}$ be any family of spaces and $f:X \to \prod X_\alpha$ be SMPC. Then P_α of: $X \to X_\alpha$ is SMPC, for each $\alpha \in \nabla$, where P is the projection of $\prod X_\beta$ onto X_α, for each $\alpha \in \nabla$.

Proof: We shall consider a fixed $j \in \nabla$ and let $V_j \in PO(X_j)$, from projection properties, $P_j^{-1}(V) = V_j \times \prod_{\alpha \neq j} X_\alpha \in PO(\prod X_\alpha)$ {see lemma 3.1}

and so $(P_j of)^{-1}(V_j) = f^{-1}(P_j^{-1}(V_j)) = f^{-1}(V_j \times \prod_{\alpha \neq j} X_\alpha) \in \tau(X)$.

Hence Pof is SMPC, for each $j \in \nabla$.

Theorem 3.7: Let $f_\alpha : X_\alpha \to Y_\alpha$, ∇Y_α, $\alpha \in \nabla$ be a family of functions and $f : \prod X_\alpha \to \prod Y_\alpha$ defined as $f\{x_\alpha\} = \{f_\alpha(x_\alpha)\}$. Then f is SMPC iff f_α is SMPC, for each $\alpha \in \nabla$.

Proof: let f be SMPC, for a fixed $j \in \nabla$ and $V_j \in PO(Y_j)$.

Then $P_j^{-1}(V_j) = V_j \times \pi Y_\alpha \in PO(\pi Y_\alpha)$, Lemma (3.1) shows
$\alpha \neq j$

this fact and so $f^{-1}(P_j^{-1}(V_j)) = f_j^{-1}(V_j) \times \pi X_\alpha \in \tau(\pi X_\alpha)$
$\alpha \neq j$

Hence $f^{-1}(V_j) \in \tau(X_j)$. Therefore f_j is SMPC, for each $j \in \nabla$

Conversely, let f_α is SMPC, for each $\alpha \in \nabla$ and $V \in PO(\pi Y_\alpha)$. Then

there exists finite ∇_o of ∇ such taht $V = \pi V_j \times \pi Y_\alpha$ by Lemma
$J \in \nabla_o \quad \alpha \neq j$

(3-1) $V_j \in PO(Y_j)$ for each $j \in \nabla_o$ and so $f_j^{-1}(V_j) \in \tau(X_j)$ for each $j \in \nabla_o$ this leads to $f^{-1}(V) = \pi f_j^{-1}(V_j) \times \pi X_\alpha \quad (\pi X_\alpha)$. Then f is SMPC.
$j \in \nabla_o \quad \alpha \neq j$

Theorem 3.8: If $f_\alpha : X \to X_\alpha$ is defined $f_\alpha(x) = x_\alpha$ and $f: X \to \pi X_\alpha$ is given by $f(x) = \{f_\alpha(x_\alpha)\}$ for each $x \in X$, $\alpha \in \nabla$. Then f is SMPC iff f_α is SMPC for each $\alpha \in \nabla$.

Proof: Necessity, it is enough to show this fact at one fixed $j \in \nabla$. So, let $V_j \in PO(X_j)$, by Lemma (3.1) and projection function one can show that $V = V_j \times \pi X_\alpha \in PO(\pi X_\alpha)$ and by SMPC of f we get $f^{-1}(V) = f^{-1} V_j \times \pi X_\alpha)$
$\alpha \neq j$
$= f_j^{-1}(V_j) \times X \in \tau(X)$ which gives

$f_j^{-1}(V_j) \in \tau(X_j)$. Therefore f_j is SMPC for each $j \in \nabla$.

Sufficiently, let f_α b SMPC, for each $\alpha \in \nabla$ and $V \in PO(\pi X_\alpha)$. So, for a finite ∇_o of ∇ we get $V = \pi V_j \times \pi X_\alpha$, hence by Lemma (3.1)
$j \in \nabla_o \quad \alpha \neq j$

$V_j \in PO(X_j)$ for each $j \in \nabla_o$ and therefore $f_j^{-1}(V_j) \in \tau(X)$ for each $j \in \nabla_o$

and so, $f^{-1}(V) = \pi f_j^{-1}(V_j) \times X \in \tau(X)$. Hence f is MPC.
$j \in \nabla_o$.

4- SMPC and some spaces

Theorem 4.1: The image of compact space under a surjective SMPC function is strongly compact.

Proof Let $f: X \to Y$ be surjective SMPC. Let also X be compact and $\{V_i : i \in I\}$ be a

preopen cover of Y. By SMPC of f, $f^{-1}(V_i) \in \tau(X)$ for each $i \in I$. Hence $\{f^{-1}(V_i) : i \in I\}$ is open cover of X which is compact, then there exists a finite subfamily I_o of I such that $X = \cup \{ f^{-1}(V_i): i \in I_o \}$ So, $Y = f(X) = \cup\{V_i : i \in I_o\}$. Hence Y is strongly compact.

Corollary 4.1: Strongly compactness is preserved under a surjective SMPC function.

Corollary 4.2: For any SMPC surjective function, the image of a compact space is nearly compact and almost compact.

Theorem 4.2: Let $f : X \to Y$ be SMPC surjective function and X be S- closed, then Y is almost compact.

Proof: If $\{V_i : i \in I\}$ is preopen cover of Y, then $\{f^{-1}(Vi) : i \in I\}$ is open cover of X. Since X is S-closed, then there exists a finite $I_o \subset I$ suchthat $X = \cup\{ cl\, f^{-1}(V_i): i \in I_o\}$. Thus, $Y = \cup \{f(cl\, f^{-1}(V_i)) : i \in I_o\} \subset \cup \{ p.cl(V_i) : i \in I_o\} \subset \cup\{ cl(V_i): i \in I_o\}$ and so Y is almost compact.

Theorem 4.3: If $f:X \to Y$ is SMPC surjective and X is Lindelof, then Y is strongly Lindelof.

Proof: Let $\{ V_i : i \in I \}$ be preopen cover of Y, then $\{f^{-1}(V_i) : i \in I\}$ is open cover of X. Since X is Lindelof, there exists a countable subcover with $X = \cup \{ f^{-1}(V_i) : i \in I_o$ (countable)$\}$. Hence $Y = f(X) = \cup \{V_i : i \in I_o\} \subset \cup \{cl(V_i) : i \in I_o\}$, therefore Y is strongly Lindelof.

Theorem 4.4: The inverse image of pre-T_i, (i=0, 1,2) under an injective SMPC function is T_i, (i = 0, 1, 2).

Proof: We prove this result in one case (say i= 0. So, let $f:X \to Y$ be SMPC injective, Y be pre-T_o and x_1, x_2 be two distinct points of X. Then $f(x_1) \neq f(x_2)$. Hence for each preopen set $V \subset Y$ containing one of $f(x_j)$, $j \in \{1, 2\}$, there exists $U_j \in \tau(X)$ containing a corresponding point x_j, (j = 1,2) such that $f(U_j) \subset V_j$. Then X is a T_o-space. The other parts of the Proof follows similarly.

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مجموعة الدوال الأقوى من
M - Precantinuity, Preirresalute and A-Functions

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