Engineering Journal of Qatar University, Vol. 3, 1990.

"ONSET OF CONVECTION IN A LIQUID FILM FLOWING DOWN AN INCLINED HEATED PLATE"

By

M. A. Hassab
Department of Mechanical Engineering
Faculty of Engineering, Qatar University
Doha, Qatar - Arabian Gulf

ABSTRACT

The thermal instability of a vicsious liquid film flowing down an inclined plane heated from below is studied analytically. The effects of Biot number at the free surface, the inclination, the Prandtl number and the Reynolds number on instability for the longitudinal and traverse rolls have shown that the longitudinal rolls have priority of occurence over the transverse rolls. For small Biot numbers with Pr = 7 and 100, the thermal entry region is more unstable than the fully developed region, but for large Biot numbers this situation is reversed.

INTRODUCTION

The thermal instability of a viscous liquid film flowing down an inclined plane and heated from below is the main concern in this study. Benjamin [1] was one of the earlier investigators who studied the hydrodynamic stability of free-surface flow down a vertical plane. For liquid layer flowing down an inclined plate, Yih [2] analytically solved the stability problem by including the surface tension effects and compared his results with those for confined plane Poiseulle flow and showed that free-surface flow is more suitable than the Poiseuille flow. Bankoff [3] extended the Yih-Benjamin analysis of hydrodynamic stability by taking into account the effects of evaporation or condensation. Wassan et al. [4] considered the instability of laminar flow of water over heated and cooled flat plates. The numerical results obtained from the solution of the modified Orr-Sommerfeld equation, which included variation of viscosity with temperature, demonstrated that heating of the profiles, whereas cooling results in destabilizing effects. That is, increasing the wall temperature from 60 to 150°F, increases the critical Reynolds number from 151 to 14800. This conclusion is opposite to that for corresponding gas flow, because viscosity of liquids decreases with temperature, whereas viscosity of gases increases with temperature.

For fully developed laminar flow between two isothermal infinite horizontal plates, Nakayama et al. [5] studied theoritically the convective instability associated with longitudinal vortex rolls under the situations of a constant axial temperature gradient with different or identical wall temperatures. The theoritical results when compared with the experimental results of Akiyama et al. [6], in which the onset of secondary flow was determined by a direct flow-visualiation technique using cigarette smoke, were found in good agreement. Later, Hwang and Cheng [7] analyzed the stability of longitudinal rolls for fully developed flow heated from below by including axial heat conduction. The graphs of their results for the entrance region, indicate that the critical Rayleigh number characterizing the onset of instability increases as Prandtl number increases and reaches the limiting value $R_a = 1708$ in the fully developed region for all values of Prandtl numbers regardless of the presence of forced flow.

The thermal instability of the above problem in the thermal boundary layer region was examined experimentally in air by Kamotani and Ostrach [8]. The critical Rayleigh number determined in this study was found to be almost an order of magnitude higher than the theoritical results by Hwang and Cheng. However, this discripancy between the experimental and the actual result of the above channel flow problem was also observed between the experimental [9] and theoritical [10-13] results concerning longitudinal vortex disturbances of natural convection flow on inclined surfaces. The difference, as interpreted later by Cheng and Wu, was attributed to infinitesimal disturbances in theory and measurable disturbances in the experiment.

The exact analysis of the Graetz problem with axial conduction being included, were described by Hsu [14] and Wu et al. [15]. They stressed that for low Peclet number uniform fluid temperature at the inlet section is no longer valid because the heat conducted upstream at x = 0 causes a transverse variation on the fluid temperature at $x \leqslant 0$. Therefore, they considered the fluid temperature to be uniform upstream at $x = -\infty$. These interesting results, encouraged Cheng and Wu [16] to carry out the stability analysis for the above Graetz problem and to investigate the effects of axial heat conduction and the transverse variations of the fluid temperature at $x \leqslant 0$ on the onset of instability for longitudinal and transverse vortex rolls. They found that for the case of typical upstream fluid temperature and upper plate temperature, the increasing Peclet number has a stabilizing effect on flow in the entrance region and that transverse rolls are dominant over the longitudinal vortex rolls for $R_e < 1$ and $P_r \geqslant 1$. They also based on the assumption that the fluid temperature is uniform at x = 0, and found that both results were in good agreement for $P_e \geqslant 50$ except the regions near the inlet section x = 0, in which Rayleigh number based on the earlier work becomes infinite.

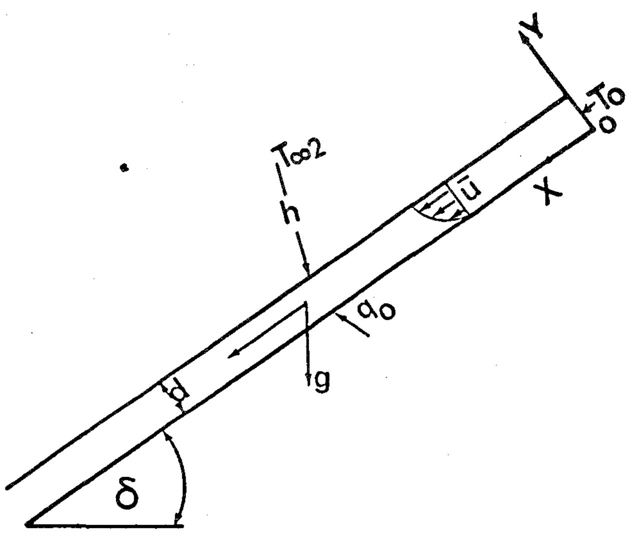

Figure 1: Flow of a Viscous Liquid Film under the Influence of Gravity Over Heated Plane.

The purpose of the present study is to investigate the thermal instability of a vascous liquid film flow a heated inclined plane for the case where there are longitudinal and transverse vortex distubances.

ANALYSIS

Consider a film of viscous liquid flowing down along an inclined flat plate as shown in Figure 1. It is assumed that the fluid enters the upper end of the surface at constant temperature T_o, is subjected to a constant positve heat flux, q_o from the inclined surface and exchanges heat by convection to or from the surroundings at a specified temperature $T_{\infty 2}$. The fluid is assumed to be in compressible, laminar, Newtonian, and have constant physical properties except for the density which appears in the body forces (i.e., Boussinesq approximation). Viscous dissipation, work of compression and radiation are neglected.

The purpose of this study is the determination of the conditions at which the fluid becomes thermally unstable. The base flow and the disturbance analysis needed for this purpose are developed as described below.

Base Flow Analysis

The governing equations of motion and energy for the thermal entrance region problem are written in the dimensioless form as:

x – momentum: $$\frac{d^2\bar{u}}{dy^2} + 2\sin\delta = 0 \quad \text{in } 0 < y < 1 \tag{1a}$$

Energy : in $$\bar{u}\frac{\partial\bar{\theta}}{\partial x} = \frac{\partial^2\bar{\theta}}{\partial y^2} \quad \text{in} \quad 0 < y < 1,\ x > 0 \tag{2a}$$

subject to the boundary conditions :

$$\bar{u} = 0 \quad \text{at} \quad y = 0, \quad \frac{d\bar{u}}{dy} = 0 \qquad y = 1 \tag{1b}$$

$$\frac{\partial\bar{\theta}}{\partial y} = -1 \qquad y = 0, \quad \frac{\partial\bar{\theta}}{\partial y} + H\bar{\theta} = 0 \qquad y = 1 \tag{2b}$$

$$\bar{\theta} = \theta_o \qquad x = 0, \quad y > 0 \tag{2c}$$

where, H is the Biot number at the upper surface. Here, in the x-momentum equation (1a), the bouyancy force is ignored as its magnitude is very small compared to the gravity force. In the energy equation (2a), the effect of axial backward cinduction, discussed in references [14-15] for the case of liquid metals, is not included because the present analysis is concerned with liquids of high Prandtl numbers. The solution for u(y) from equation (1) is:

$$\frac{\bar{u}(y)}{\sin\delta} = \bar{u}_v(y) = y(2 - y) \tag{3}$$

where $u_v(y)$ is the dimensionless velocity of the fluid at the vertical position.

To solve for 0, a superposed solution is constructed as:

$$\bar{\theta}(x,y) = \theta_{fd}(y) + \theta_e(x,y) \tag{4}$$

where, $\theta_{fd}(y)$ is the dimensionless temperature profile in the fully developed region given as :

$$\theta_{fd}(y) = (1 + \frac{1}{H} - y) \tag{5}$$

The solution for the excess temperature $\theta_e(x,y)$ in the thermally developing region is constructed as:

$$\theta_e(x,y) = \sum_{m=1}^{N} A_m\theta_m(y)\exp(-\beta_m^2 x/\sin\delta) \tag{6}$$

where the summation is taken over all the eigenvalues and $\theta_m(y)$ are the eigenfunctions of the following eigenvalue system :

$$\frac{d^2\theta_m}{dy^2} + 2\beta_m^2 \cdot y(1 - y/2)\theta_m(y) = 0 \text{ in } 0 < y < 1 \tag{7a}$$

$$\frac{d\theta_m}{dy} = 0 \qquad \text{at } y = 0 \tag{7b}$$

$$\frac{d\theta_m}{dy} + H\theta_m = 0 \qquad \text{at } y = 1 \tag{7c}$$

The exact analytical solution of this eigenvalue problem could not be found in the literature; but it could be solved by purely numerical methods [17-18]. A similar equation subject to boundary conditions of the first kind has been solved both by the Galerkin method and numerically. It has also been shown that the solution by the Galerkin method is highly accurate [19]. Therefore, we prefer to solve the above eigenvalue problem (7) by the Galerkin method. The function $\theta_m(y)$ is constructed in a series of orthogonal functions that satisfy its boundary conditions (7b,c) as

$$\theta_m(y) = \sum_{i=1}^{N} a_i^{(m)} \Phi_i(y), \quad \Phi_i(y) = \cos \mu_i y \tag{8a}$$

where μ_i's are the positive roots of the equation

$$\mu \sin \mu - H \cos \mu = 0 \tag{8b}$$

The orthogonality condition for the functions $\Phi_i(y)$ is established as :

$$\int_0^1 \Phi_i(y)\Phi_j(y)dy = \frac{1}{2}\left[1 + \frac{H}{\mu_i^2 + H^2}\right]\delta_{ij} \equiv \delta_{ij} N_i \tag{8c}$$

where δ_{ij} is the Kroneker Delta. When equation (8a) is introduced into equation (7a) and the orthogonality condition (8c) is applied, the following infinite set of linear homogeneous equations are obtained for the coefficients $a_i^{(m)}$'s

$$\sum_{i=1}^{N} a_i^{(m)} [I_1(j,i) + \beta_m^2 I_2(j,1)] = 0 \tag{9a}$$

$$(i = 1,2, \ldots, j = 1,2, \ldots)$$

where :

$$I_1(j,i) = \int_0^1 \frac{d^2\Phi_i}{dy^2} \Phi_j(y)dy \tag{9b}$$

$$I_2(j,i) = \int_0^1 2y(1 - y/2)\Phi_i(y)\Phi_j(y)dy \tag{9c}$$

The solution of (7a) is possible if and only if the determinant of the coefficient of a_i^{m}'s vanishes. This requirement leads to the following secular equation for the determination of the eigenvalues, β_m^{2}'s

$$\left|\left| I_1(j,i) + \beta_m^2 I_2(j,i) \right|\right| = 0 \tag{10}$$

$$(i = 1,2, \ldots, N; \; j = 1,2, \ldots N)$$

Hence, the system (9a), for every evaluated as a rood, contains a set of N-1 linearly independent equations for $a_i^{(m)}$'s which may be solved only in term of one of them, say $a_{i(m)}$. Then equations (9a) yields the followingequations for the determination of the coefficients, $c_i^{(m)} = \dfrac{a_i^{(m)}}{a_1^{(m)}}$:

$$[I_1(j,i) + \beta_m^2 I_2(j,i)]C_i^{(m)} = -[I_1(j,1) + \beta_m^2 I_2(j,1)] \tag{11}$$

$$(i = 2,3, \ldots, N, \; J = 2, 3, \ldots, N)$$

the approximate eigenfunctions $\theta_m(y)$ become :

$$\theta_m(y) = a_1^{(m)} \sum_{i=1}^{N} C_i^{(m)} \cos \mu_i y, \text{ with } C_1^{(m)} = 1 \tag{12}$$

Now, by substituting equation (12) into (6), we obtain $\theta_e(x,y)$ as:

$$\theta_e(x,y) = \sum_{i=1}^{N} A_m^* \left[\sum_{i=1}^{N} C_i^{(m)} \cos \mu_i y \right] e^{-\beta_m^2 x/\sin \delta} \tag{13}$$

$$\text{with } A_m^* = a_1^{(m)} \cdot A_m$$

The constants A_{m^*}'s are determined by satisfying the boundary conditions (2c) since the finite set $\{\cos \mu_i y\}$ satisfies the necessary regularity conditions, A_{m^*} can be determined by the application of an extension of the Woierstrass theorem. Equation (13) is multiplied by $\theta_n(y)$ and integrated over $0 \leqslant y \leqslant 1$,

$$\sum_{m=1}^{N} A_m^* \, I(m,n) = L^*(n), \quad j = 1,2, \ldots, N \tag{14}$$

where the integrals are defined as:

$$I(m,n) = \int_0^1 \left[\sum_{i=1}^{N} C_i^{(m)} \cos \mu_i y \right] \left[\sum_{i=1}^{N} C_i^{(n)} \cos \mu_i y \right] dy \tag{15a}$$

$$L^*(n) = \int_0^1 \left[\theta_o - (1 + \frac{1}{H} - y) \right] \left[\sum_{i=1}^{N} C_i^{(n)} \cos \mu_i y \right] dy \tag{15b}$$

Since $C_i^{(m)}$'s are previously determined, equations (14) represent a nonhomogeneous system of N equations in N unknows, A_{m^*}, which has a unique solution. The integrals I(m,n) and L*(n) are integrated analytically. Then the solution for the dimensionless base flow temperature is given as:

$$\bar{\theta}(x,y) = (1 + \frac{1}{H} - y) + \sum_{m=1}^{N} A_m^* \left(\sum_{j=1}^{N} C_i^{(m)} \cos \mu_i y \right) e^{-\beta_m^2 x / \sin \delta} \tag{16}$$

Stability Analysis

To derive the perturbation equations needed for the analysis of stability, small quantities on the velocity components, pressure and temperature are defined as:

$$U = \bar{U}(y) + u', \quad V = v', \quad W = w', \quad P = p', \quad T = \bar{T}(X,Y) + T' \tag{17}$$

where quantities with prime denote the small perturbations. Introducing these variables into the conservation equations of mass, momentum and energy using the Boussinesq approximation and subtracting out the basic flow equations, the resulting perturbation equations, after neglecting nonlinear terms, are expressed inthe dimensionless form as:

$$\frac{\partial \hat{u}}{\partial x'} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} = 0 \tag{16}$$

$$\frac{\partial \hat{u}}{\partial \tau} + Re\, \bar{u}_v \frac{\partial \hat{u}}{\partial x'} \sin\delta + Re \frac{d\bar{u}_v}{dy} \hat{v} \sin\delta = -\frac{\partial \hat{p}}{\partial x'} + \nabla^2 \hat{u} - \hat{\theta} \sin\delta \tag{17a}$$

$$\frac{\partial \hat{v}}{\partial \tau} + Re\, \bar{u}_v \frac{\partial \hat{v}}{\partial x'} \sin\delta = -\frac{\partial \hat{P}}{\partial y} + \nabla^2 \hat{v} + \hat{\theta} \cos\delta \tag{17b}$$

$$\frac{\partial \hat{w}}{\partial \tau} + Re\, \bar{u}_v \frac{\partial \hat{w}}{\partial x'} \sin\delta = -\frac{\partial \hat{P}}{\partial z} + \nabla^2 \hat{w} \tag{17c}$$

$$Pr \frac{\partial \hat{\theta}}{\partial \tau} + Pc\, u_v \frac{\partial \hat{\theta}}{\partial x'} \sin\delta + \frac{Ra}{Pc} \frac{\partial \bar{\theta}}{\partial x} \hat{u} + Ra \frac{\partial \bar{\theta}}{\partial y} \hat{v} = \nabla^2 \hat{\theta} \tag{18}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The above equations are to be subjected to the following boundary conditions:

$$\hat{u} = \hat{v} = \hat{w} = \frac{\partial \theta}{\partial y} = 0 \quad \text{at} \quad y = 0 \ \text{(rigid wall)}$$

$$\frac{\partial \hat{u}}{\partial y} = \hat{v} = \hat{w} = \frac{\partial \hat{\theta}}{\partial y} + H\hat{\theta} = 0 \quad \text{at} \quad y = 1 \ \text{(free surface)} \tag{19}$$

Here, various dimensionless quantities are defined as:

$$H = hd/k, \ \hat{P} = P'/(\mu\hat{u}_o/d), \ Pe = Pr \cdot Re, \ Pr = \nu/\alpha, \ Ra = \frac{\gamma g q_o d^4}{k\alpha\nu}$$

$$Re = \frac{\bar{U}_o d}{\nu}, \ U_o = \frac{gd^2}{2\nu}, \ (\hat{u},\hat{v},\hat{w}) = (u',v',w')/\hat{u}_o, \ \hat{u}_o = \frac{\gamma g q_o d^3}{\nu k}, \ \bar{u} = \bar{U}/\bar{U}_o$$

$$(x',y,z) = (X,Y,Z)/d, \ x = x'/Pe, \ \bar{\theta} = \frac{T - T_{\infty 2}}{(q_o d/k)}, \ \hat{\theta} = \frac{T'}{(q_o d/k)}, \ \tau = \frac{\nu t}{d^2}.$$

where P_e, P_r, R_a and the Peclet number, Prandtl number, Rayleigh number and the Reynolds number at vertical position of the system respectively.

In accordance with linear stability theory, the solutions for the perturbations $\hat{u},\hat{v},\hat{w},\hat{p}$ and $\hat{T}$ may be expressed in the form:

$$F(x',y,z,\tau) = F^*(y)\exp[c^*x' + i(a_1x' + a_2z) + c\tau] \quad (20)$$

where C. is the amplification factor; c is, in general , a complex number; a_1 and a_2 are the wave numbers in the x and z directions respectively. When the disturbances are considered stationary, (c = 0) as verified in references are (7,8,10). Furthermore, if attention is confined to neutral stability, one sets $c^* = 0$. When the solution (20) is introduced into the system (16) to (18), one obtains the following stability equations for the three-dimensional distubances as:

$$\frac{dv^*}{dy} + ia_1u^* + ia_2w^* = 0 \quad (21a)$$

$$i a_1 \text{ Re } \bar{u}_v \; u^* \sin \delta + \text{Re} \frac{d\bar{u}_v}{dy} v^* \sin \delta$$

$$= - i a_1 p^* + (D^2 - a^2) u^* - \theta^* \sin \delta \qquad \text{(21b)}$$

$$i a_1 \text{Re } \bar{u}_v \, v^* \sin \delta = - \frac{dp^*}{dy} + (D^2 - a^2) v^* + \theta^* \cos \delta \qquad \text{(21c)}$$

$$i a_1 \text{ Re } \bar{u}_v \; w^* \sin \delta = - i a_2 p^* + (D^2 - a^2) w^* \qquad \text{(21d)}$$

$$i a_1 \text{Re } \bar{u}_v \, \theta^* \sin \delta + \frac{\text{Ra}}{\text{Pe}} \frac{\partial \bar{\theta}}{\partial x} u^* + \text{Ra} \frac{\partial \bar{\theta}}{\partial y} v^* = (D^2 - a^2) \theta^* \qquad \text{(21e)}$$

together with the boundary conditions

$$u^* = v^* = w^* = \frac{d\theta^*}{dy} = 0 \qquad y = 0 \quad \text{(at rigid wall)}$$

$$\frac{du^*}{dy} = v^* = w^* = \frac{d\theta^*}{dy} + H^* = 0 \quad y = 1 \text{ (at free surface)} \qquad \text{(22)}$$

where, $$D^2 \equiv \frac{d^2}{dy^2} \, , \quad a^2 = a_1^2 + a_2^2$$

The onset of instability for the **above system** is associated by only one of the following two dimensional disturbances **[16]**.

Longitudinal Rolls: The governing equations for the analysis of the longitudinal rolls are obtained by setting $a_1 = 0$ in the system (21). Then, if p^* and w^* are eliminated fromthe resulting equations, the following equations are obtained.

$$(D^2 - a_2^2) \, u^+ = \text{Re} \frac{d\bar{u}_v}{dy} v^+ + \theta^+ \qquad \text{(23a)}$$

$$(D^2 - a_2^2)^2 v^+ = a_2^2 \theta^+ \cos \delta \qquad (23b)$$

$$(D^2 - a_2^2)\, \theta^+ = \frac{Ra}{Pr} \frac{\partial \bar{\theta}}{\partial x^+} u^+ + Ra \frac{\partial \bar{\theta}}{\partial y} v^+ \qquad (23c)$$

These equations must be solved to the following boundary conditions;

$$u^+ = v^+ = \frac{dv^+}{dy} = \frac{d\theta^+}{dy} = 0 \text{ at } v = 0 \quad \text{(rigid wall)}$$

$$\frac{du^+}{dy} = v^+ = \frac{d^2 v^+}{dy} = \frac{d\theta^+}{dy} + H\theta^+ = 0 \text{ at } y = 1 \text{ (free surface)} \qquad (24)$$

where:

$$u^+ = u^*/\sin \delta, \; x^+ = x/\sin \delta, \; v^+ = v \;, \; \theta^+ = \theta^*$$

<u>**Transverse Rolls:**</u> The equations needed for the analysis of transverse rolls are obtained by setting $a_2 = 0$ in the system (21). Then, if the dependent variables u^* and v^* are replaced by the perturbed stream function ψ^* such that

$$v^* = - i a_1 \psi^*, \quad u^* = \frac{d\psi^*}{dy}$$

and p^* is eliminated fromthe resulting equations, we find, after some rearranging,

$$(D^2 - a_2^2)^2 \psi^* - i a_1 \sin \delta \, Re[u_v(D^2 - a_1^2) - D^2 u_v]\psi^*$$

$$= [i a_1 \theta^* \cos \delta + \frac{d\theta^*}{dy} \sin \delta] \qquad (26a)$$

$$(D^2 - a_1^2)\, \theta^* - i a_1 [Pe\, u_v\, \theta^* \sin \delta - Ra \frac{\partial \bar{\theta}}{\partial y} \psi^*]$$

$$- \frac{Ra}{Pe} \frac{\partial \bar{\theta}}{\partial x} \frac{d\psi^*}{dy} = 0 \qquad (26b)$$

These equations must be solved subjec to the following boundary conditions:

$$\psi^* = \frac{d\psi^*}{dy} = \frac{d\theta^*}{dy} = 0 \qquad y = 0 \quad \text{(rigid wall)}$$

$$\psi^* = \frac{d^2\psi^*}{dy^2} = \frac{d\theta^*}{dy} + H\theta^* = 0 \qquad y = 1 \quad \text{(free surface)} \tag{27}$$

SOLUTIONS OF THE EIGENVALUE PROBLEMS

The two eigenvalue systems given by equations (23-24) for the longitudinal rolls and by equations (26-27) for transverse rolls are solved here to a higher order of approximations by applying the Chandrasekhar for the first system and Galerkin method for the second system.

Solution for the Longitudinal Rolls

To solve the stability problems given equations (23) and subject to the boundary conditions (24), the function $\theta_+(y)$ is represented in a series of orthogonal functions that satisfy its boundary conditions, by following Chardrasekhar, as

$$\theta^+(y) = \sum_{m=1}^{\infty} A_m\theta_m^*(y), \quad \theta_m^*(y) = \cos \mu_m y \tag{28}$$

where the eigenvalues μ_m's and the orthogonality condition for the eigenfunctions $\theta_m^*(y)$ are the same as those given previously under the section of the base flow analysis by equations (8b) and (8c), respectively.

When the above solution for $\theta^+(y)$ is introduced into equation (23b) together with the.expression for $V^+(y)$ chosen as

$$v^+(y) = a_2^2 \cos \delta \sum_{m=1}^{\infty} A_m \; v_m^*(y) \tag{29a}$$

one finds

$$(D^2 - a_2^2)^2 \; v_m^* = \cos \mu_m y \tag{29b}$$

The solution of this equation subjected to the boundary conditions (29) for $v_m^*(y)$ gives $v_m^*(y)$. Choosing

$$u^+(y) = \sum_{m=1}^{\infty} A_m \; u_m^*(y), \tag{30a}$$

and introducing this expansion together with those obtained above for $\theta^+(y)$ and $v^+(y)$ into equations (23a), one obtains

$$(D^2 - a_2^2) \; u_m^* = \frac{1}{2} a_2^2 \sin 2\delta \frac{d\bar{u}_v}{dy} v_m^*(y) + \cos \mu_m y \tag{30b}$$

The complete solution of $u_m^*(y)$ is obtained from the above equation together with the boundary conditions (24) for $u_m^*(y)$.

To evaluate the eigenvalues of this system, the above solutions for $\theta^+(y)$, $v^+(y)$ are introduced into equation (23c) and the orthogonality condition (8c) is utilized to yield the following infinite order secular determinant for the evaluation of the Rayleigh number, R_a,

$$\left\| E_{mn} + Ra \; F_{mn} \right\| = 0 \quad \text{for } (n = 1,2,3, \ldots) \tag{31}$$

where

$$E_{mn} = (\mu_m^2 + a_2^2) \; N_m \; \delta_{mn}$$

N_m is given by (8c), and

$$F_{mn} = \int_0^1 \left[\frac{1}{Pe} \frac{\partial \bar{\theta}}{\partial x^+} u_m^*(y) + a_2^2 \cos \delta \frac{\partial \bar{\theta}}{\partial y} v_m^*(y)\right] dy$$

Solution for the Transverse Rolls

The system for the transverse rolls governed by the differential equations (26) and the boundary conditions (27) is complex. Therefore, the solution of this system should be complex. The Galerkin method is applied to construct solutions for the dependent variables $\psi^*(y)$ and $\theta^*(y)$ so that the boundary conditions (27) are satisfied. The proper solution for this problem is taken as:

$$\psi_1^*(y) = \sum_{m=1}^{N} A_m \phi_m(y) \tag{32}$$

$$\theta_1^*(y) = \sum_{m=1}^{N} B_m \theta_m(y) \tag{33}$$

The orthogonal functions $\phi_i(y)$ and $\theta_i(y)$ which satisfy their boundary conditions are given as:

$$\phi_i(y) = \frac{\cosh \alpha_i y - \cos \alpha_i y}{\cosh \alpha_i - \cos \alpha_i} - \frac{\sinh \alpha_i y - \sin \alpha_i y}{\sinh \alpha_i - \sin \alpha_i} \tag{34.a}$$

$$\theta_i(y) = \cos \mu_i y \quad \text{for} \quad (i = 1, 2, \ldots, 2N) \tag{34b}$$

and the eigenvalues α_i and μ_i are obtained respectively from the solution of the following transcendental equations

$$\coth \alpha - \cot \alpha = 0 \tag{35a}$$

$$\mu \sin \mu - H \cos \mu = 0 \tag{35b}$$

The functions $\phi_i(y)$ satisfy the following orthogonality condition:

$$\int_0^1 \phi_i(y)\phi_j(y)dy = \delta_{ij} \, N\phi_i \tag{36}$$

where, $N\Phi_i = \dfrac{2 \tanh^2 \alpha_i \sinh \alpha_i \sin \alpha_i (\cosh \alpha_i \cos \alpha_i - 1)}{(\sinh \alpha_i - \sin \alpha_i)^4}$

The orthogonality condition for $\theta_i(y)$ functions is the same as that given previously by equations (8c). The coefficient A_m , B_m , are complex quatities. By substituting (32) and (33) into equations (26), and utilizing the orthohonality property for each equation, and utilizing the orthogonality property for each equation, we obtain the following equation in the matrix form as:

$$[y_{nm}] \cdot \bar{X}_m = 0 \tag{37}$$

where the elements $(y)_{nm}$ of the $[y_{nm}]$ matrix are 2x2 matrices resulting from orthogonalization of equations (26) and the vector X_m consists of the coefficients of the series (33).

The solution of the homogeneous system given by equation (37) is characterized by the following secular determinant, of order 2N x 2N for the determination of the eigenvalues, R_a.

$$\left\| y_{mn} \right\| = 0 \qquad (n = 1, 2, \ldots, N) \tag{38}$$

RESULTS AND CONCLUSIONS

Before presenting results on the effects of the Reynolds number Re based on the vertical position, the Prandtl number Pr, the Biot number H, the dimensionless axial distance $x^+ = \left[\frac{x}{d}\right] / (Pe \sin \delta)$, the inclination angle δ and the uniform entrance temperature to discuss the effects of these parameters on the base flow temperature distributions which strongly influence stability.

To present the graphical results,it is concenient to redefine new base flow temperatures $\theta_b(x^+,y)$ and θ_{ob} as:

$$\theta_b(x,y) = \frac{\bar{\theta}(x^+,y)}{1 + \frac{1}{H}} \tag{39a}$$

$$\theta_{ob} = \frac{\theta_o}{1 + \frac{1}{H}} \tag{39b}$$

and a new dimensionless distance $\bar{x}$ as

$$\bar{x} = \frac{1}{2} x^+ = \left(\frac{X}{d}\right) / (2 \cdot Pe \cdot \sin \delta) \tag{39c}$$

Figures 2 and 3 shows the results for the base flow temperature distribution $\theta_b(\bar{x},y)$ in the developing region as a function of the axial dimensioless coordinate x and the Biot number H = 0.1 and 100 for various values of the entrance temperature, θ_{ob} = -1, and 1. As shown in these figures, for θ_{ob} = -1 and 0, the fluid temperature $\theta_b(\bar{x},y)$ increase as $\bar{x}$ increases. However, for θ_{ob} = 1, that is, when the entrance temperature becomes larger than the surrounding temperature, the flow temperature increases with increasing x for H = 0.1. As the Biot number increase (i.e., H d 100), the hotter fluid in the developing region losses more heat to the surroundings than it gains from the plate. As a reult, the base flow temperature inthe entrance region decreases as the fluid flowing over the inclined heated plate specially for fluid layers near the outside surface as shown in Figure 3. The variation of the thermal entry length x_d with the Biot number H, is also shown in these figures. For example in comparing the cases $\bar{x}_d$ = 100 for H = 0.1 and z_d = 0.5 for H =100, the thermal entry length is observed to decrease with lincreasing H.

The numerical calculations carried out for the longitudinal and transverse rolls, indicate are less in magnitude than those for the transverse rolls for all the ranges of the lsystem parameters considered in this study. This means that longitudinal rolls have priority of loccurence over transverse rolls. So, the results presented in the next figures of stability are for the longitudinal rolls.

The critical Rayleigh numbers Ra_c at the onset of instability are of primary interest in this study. The results presented in Figures 4 and 5 for Pr =7 (i.e., water, and θ_{ob} = 0 (i.e., $T_o = T_{\infty 2}$), show the effects of the Raynolds number Re, the Biot number N and the inclination angle α on the critical Rayleigh number in the developing region. It is shown in these figures that with increasing H,

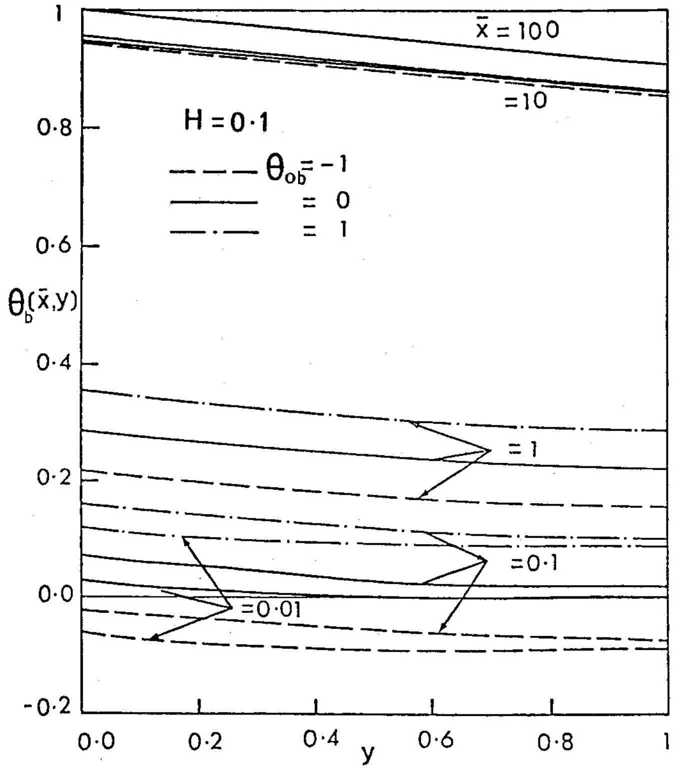

Figure 2: Variation of the Base Flow Temperature Profile θ in the Thermal Entrance Region with the Coordinate x for H = 0.1.

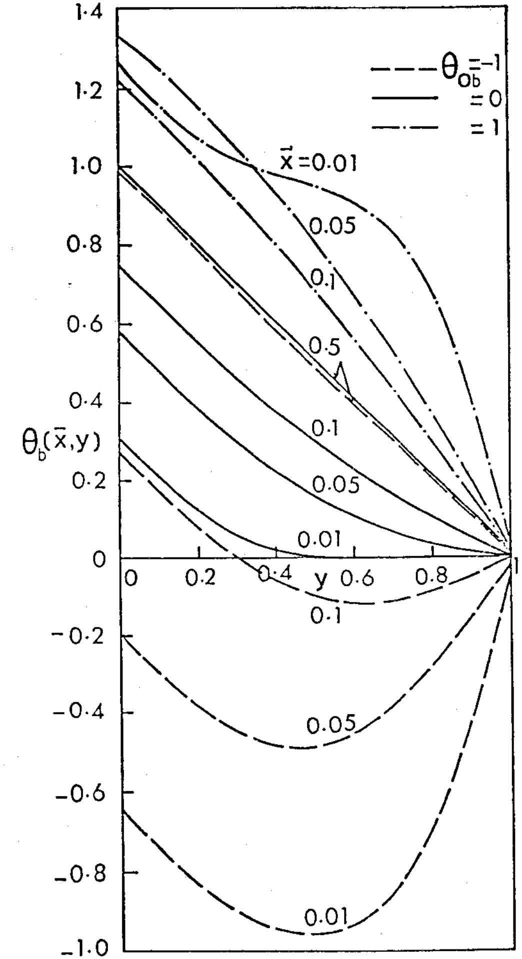

Figure 3: Variation of the Temperature θ with x for H = 100.

stability is improved due to the daming effects of the Biot number on the perturbation of the temperature profile. Another interesting feature of these figures is that in the developing region and with smaller values of Reynolds number (i.e., Re = 50), the critical Rayleigh number for a given x does not change with inclination. The reason for this fluid is caused by the combined effects of the axial base flow temperature gradient $\frac{\partial \theta_b}{\partial x}$ and the transverse flow temperature gradient normal to the fluid layer $\frac{\partial \theta_b}{\partial y}$. In the developing region and for small values of Reynolds number, the effect of the axial temperature on instability is more significant than that for normal transverse gradients. As a result, the stability is controlled only by the axial temperature gradient as apparent from equation (23c). This, in turn, implies that $v^+(y)$ governed by equation (23b) is vanishingly small (i.e., equation (23b) may be omitted). Then the Rayleigh number becomes independent of inclination. For larger values of the Reynolds number Re $\geqslant$ 500, and for Biot numbers H $\geqslant$ 0.1, which is the case in Figure 5, it is found that the critical Rayleigh number increases with inclination. The reason for this can be attributeto the fact that the thermal boundary layer grows very fast as H increases. In this case, the stability is not affected only by $\frac{\partial \theta_b}{\partial x}$, but it is also influenced by $\frac{\partial \theta_b}{\partial y}$. Therefore, the critical Rayleigh number Ra_c is expected to increase with inclination but at a lower, lesser than that for the fully developed region in which $Ra_c \cos \delta = Ra_{ch}$. The graphical results presented in these figures show that the critical Rayleigh number attains a maximum value in the developing region for large values of H. This is to be expected, because in this case, as x increase, the component $\frac{\partial \theta_b}{\partial y}$ increase with a rate higher than the decreasing rate on the component $\frac{\partial \theta_b}{\partial x}$, which in turn tends to bring the fluid to the fully developed state. This explains the peak in the Rayleigh number. A very important conclusion deduced from these figures, is that (i.e., Re = 50) and with H up to ~0.1, the critical Rayleigh number, Ra_c, for given values of H and x is linearly proportional to the Reynolds number, Re, at any inclination, $\delta < 90°$. From this valuable results, the critical Rayleigh number at any Reynolds number within the above range can readily be calculated.

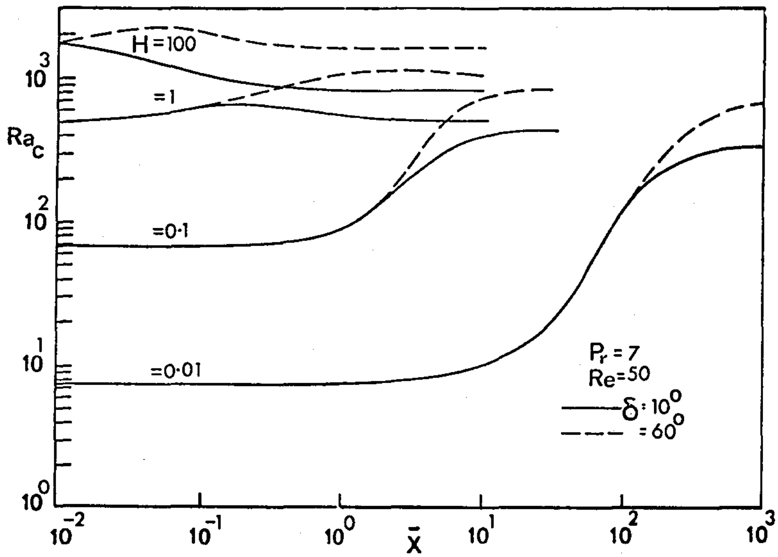

Figure 4: Variation of Ra_c in the Thermal Developing Region with δ and H for Pr = 7.

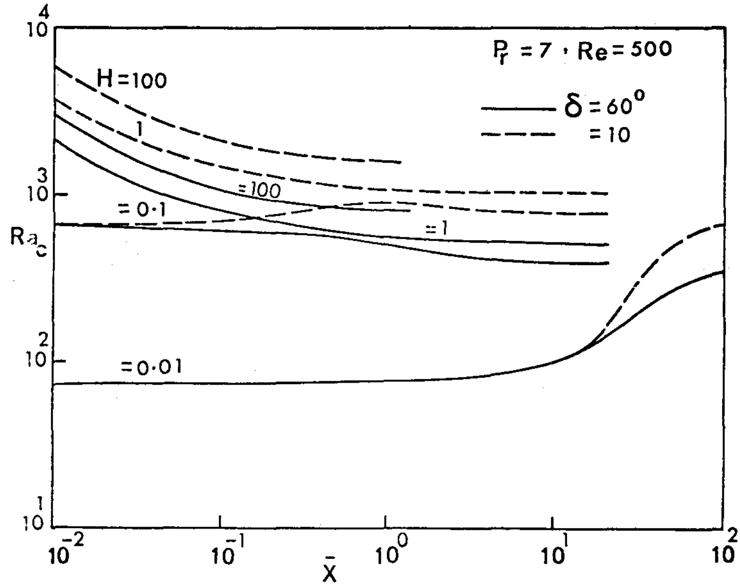

Figure 5: Variation of the Critical Rayleigh Number Ra_c in the Thermal Developing Region with the Inclination Angle δ and Biot Number H for Pr = 7, Re = 500.

To show the variation of the liquids of high Prandtl number, Pr = 100 (i.e., oils). The results obtained for this case are presented inFigures 6 and 7 for Re = 50 and 10 respectively. Both figures show that the critical Rayleigh numbers for the value of Pr considered here, are larger than those plotted for Pr = 7. This means that, for channel flow problems studied here and in Reference [16], increasing the Prandtl number stibilizes the laminar forced flow. While for natural convection flow problems, investigated in the literature, it is known that increasing the Prandtl number destabilizes the laminar natural flow. The reason for the stabilizing effect of the Prandtl number on the fluid for the problem considered here can be attributed to the fact that, for large values of Prandtl number, the contibution of the axial temperature gradient on thermal instability becomes less. So, the destabilization due to $\frac{\partial \theta_b}{\partial \bar{x}}$ is reduced. For the fully developed case, $\frac{\partial \theta_b}{\partial \bar{x}}$ vanishes and the stability problem reduces to the conventional free-surface problem irrespective of the presence of flow. The critical Rayleigh number for all these figures, in the fully developed region becomes independent of Pradtl number, and Ra_c at any inclination is determined from the relation $Ra_c \cdot \cos \delta = Ra_{ch}$, where Ra_{ch} is the critical horizontal value for the case of free surface problem.

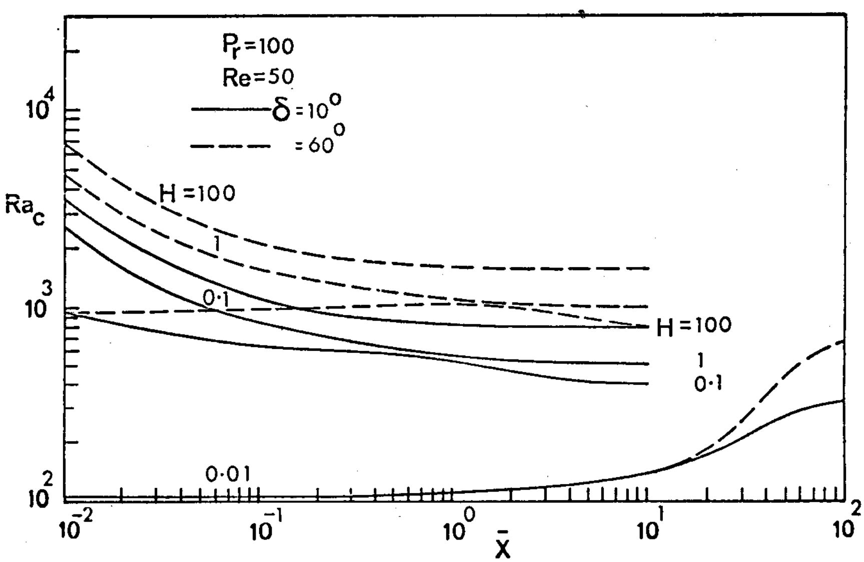

Figure 6: Variation of Ra_c in the Thermal Entrance Region with δ and H for Pr = 100, Re = 50.

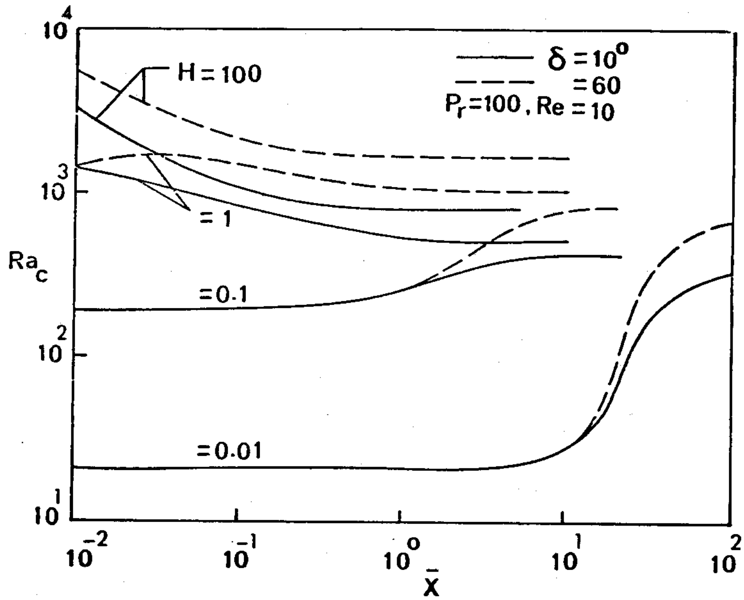

Figure 7: Variation of Ra_C in the Thermal Entrance Region with δ and H for Pr = 100, Re = 10.

The effect of the uniform entrance temperature θ_{ob}, on the initiation of cellular convection has been considered in Figure 8, for Pr = 100 and θ 60. For small Biot numbers H = 0.01 and 0.1, the stability is slightly improved compared to that of θ_{ob}, if θ_{ob} is reduced to θ_{ob} = -1 (i.e., $T_o < T_{\infty 2}$, that is, the fluid enters with temperature less than the surrounding temperature) and the system becomes less stable by a narrow margin. for θ_{ob} = 1 (i.e., $T_o < T_\infty$). For large values of Biot number H = 100, the fluid is most stable for $\theta_{\neq\approx}$ = 0 near the upstream region, and least stable for θ_{ob} = 1 in the other portion of the developing region. These results can be explained as follows: From the base flow temperature profiles in the entrance region for H = 100, as shown in Figure 3, it can be concluded that, for the case when θ_{ob} = 1, the entrance region is very short and heat losses to the surrounding are doubled. As a result, instability sets in earlier than that with θ_{ob} = 0. When θ_{ob} -1, the liquid film flowing along the plate is heated at much faster rate than the other cases, as it receives heat from both the lower plate and the atmosphere. This makes the fluid less stable near the upstream region, because $\frac{\partial \theta_b}{\partial \bar{x}}$ is very large, and most stable for the other portion of the entrance region due to the slow change in the temperature profile with the axial distance x, or the transverse coordinate -y.

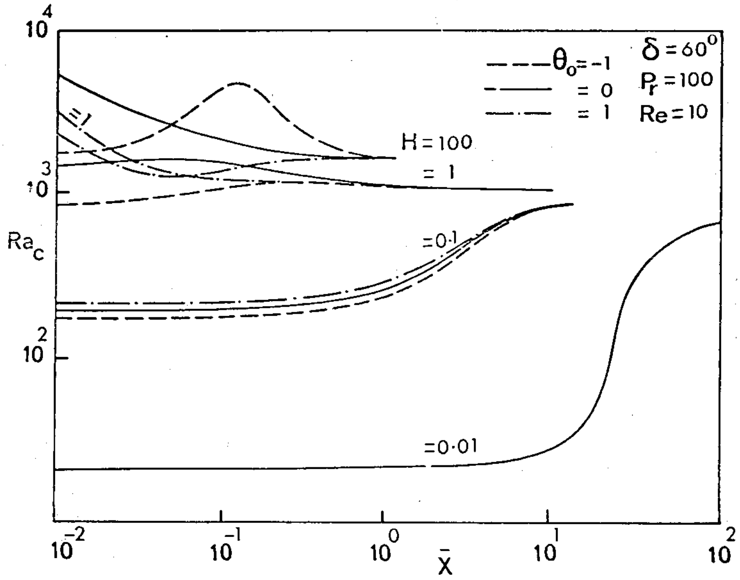

Figure 8: Effect of the Entrance Temperature θ on the Critical Rayleigh Number Ra_c in the Thermal Entrance Region for Pr = 100, Re = 10.

Convergence of Solutions

The eigenvalues β_m of the eigenvalue problem (9) related to the base flow temperature are determined in the present analysis using the Galerkin method. The results obtained in this manner are compaired with those numerically calculated in references [17-18]. It is found that the analytical results are very accurate as listed in Table 1. It is to be noted that, the β_m's (Galerkin) listed in this table are computed from the determinant (10) of order 10 x 10.

Table 1: Comparison Between Galerkin and Numerical [17-18] Solutions for β_n.

n	H =0		H = ∞	
	$\beta_{n(Galerkin)}$	$\beta_{n(Num.)}$	$\beta_{n(Galerkin)}$	$\beta_{n(Num.)}$
1	4.28723	4.28724	2.263078	2.263106
2	8.30375	8.30372	6.297614	6.29768
3	12.3107	12.3144	10.30764	10.3077
4	16.3147	16.333	14.30764	14.3077

The covergence of the solution constructed for the longitudinal rolls solved the Chandrasekhar method are checked and found to converge very fast. Some of these numerical results are presented in Table 2. The results shown in this table indicate that the difference between the third approximate and the second approximate Rayleigh numbers is always less than 1 percent.

Table 2: Illustrates the Convergence of the Critical Rayleigh Number for (Pr = 7, Re = 500, H = 0.1 and δ = 10°).

$\bar{x}$	Ra_c		
	n = 1	n = 2	n = 3
0.005	678.0344	678.2218	678.1258
0.1	608.6651	608.8592	607.0391
0.05	544.8353	544.6155	542.416
0.1	530.4766	530.1381	528.0033
1	417.5945	415.9223	412.8229
10	392.3677	390.4931	387.638

NOMENCLATURE

a	:	magnitude of wave vector, $a = a1^2 + a2^2$
a_1	:	wave number in x-direction
a_2	:	wave in z-direction
c	:	a complex number
C^*	:	amplication or damping factor
d	:	thickness of the film
g	:	gravity
h	:	heat transfer coefficient at the surface
H	:	Biot number at the upper surface, hd/k
k	:	thermal concuctivity of the liquid
p'	:	perturbed pressure
p	:	pressure
Pe	:	Peclet number, $U_o d/\alpha$
Pr	:	Prandtl number, v/α
q_o	:	uniform conductive heat flux at the lower surface
Ra	:	Rayleigh number, $\gamma g q_o d^4/k\alpha v$
Re	:	Reynold number, $U_o d/v$
t	:	time
T	:	fluid temperature
$\bar{u}$	:	dimensionless bass flow velocity
u',v',w'	:	perturbed velocity components
U,V,W	:	dimensionless perturbed velocity components
U_o	:	characteristics velocity
$\bar{U}$	:	dimensional base flow velocity at the vertical position
x,y,z	:	dimensionless catesian coordinates
$\bar{X},Y,Z$	:	cartesian coordinates with Y measured normal to the fluid layer
X	:	$(\frac{x}{d})/(2Pe \sin\delta)$

Greek Letters

α	:	thermal diffusivity
γ	:	coefficient of thermal expansion for fluid
δ	:	angle measured from horizontal
$\bar{\theta}$	:	dimensionless base flow temperature
θ_o	:	dimensionless entrance temperature
$\hat{\theta}$	:	dimensionless perturbed temperature
v	:	kinematic vilocity
p	:	density of fluid
t	:	dimensionless time
w	:	dimensionless perturved stream functions
∇^2	:	Laplacian operator

Superscripts

–	:	refer to mean quantities
/ ′ ^ ′ * ′ +	:	refer to perturbed quatities

Subscripts

∞	:	refer to outside environment
o	:	fixed quantities

REFERENCES

1. **T.A. Benjamin,** : "Wave Formation in Laminar Flow Down an Inclined Plane." J. Fluid Mech., 2, 554-574 (1957).

2. **Chi-Shun Yih,** : "Stability of Liquid Flow Down an Inclined Plane." Phys. Fluid, 6, (3), 321-334 (1963).

3. **S.G. Bankoff,** : "Stability of Liquid Flow Down a Heated Inclined Plane." Int. J. Heat Mass Transfer, 14, 377-385 (1971).

4. **A.R. Wazzan, T. Okamura,** and **A.M.O. Smith,** : "The Stability of Water Flow Over Heated and Cooled Flat Plates." J. Heat Transfer, 90c, 109-114, (1968).

5. **W. Nakayama, G.J. Hwang** and**K.C. Cheng,** : "Thermal Instability in Plane Poiseille Flow." J. Heat Transfer, Trans. ASME, Series C, 92, (1), 61-68 (1970).

6. **M. Akiyama, G.J. Hwang,** and **K.C. Cheng,** : "Experiments on the Onset of Longitudinal Vortices in Laminar Forced Convection Between Horizontal Plates." J. Heat Transfer, 93c, No. 4, 335-341 (1971).

7. **G.J. Hwang** and **K.C. Cheng,** : "Convective Instability in the Thermal Entrance Region of a Horizontal Parallel-Plate Heated from Below." J. Heat Transfer, 95c, 72-77 (1973).

8. **Y. Kamotani** and **S. Ostrach,** : "Effect of Thermal Instability on Thermally Developing Laminar Channel Flow." J. Heat Transfer, 98c, 62-66 (1976).

9. **J. E. Lloyd** and **E.M. Sparrow,** : "On the Instability of Natural Convection Flow on Inclined Ssurfaces." J. Fluid Mech., 42, 465-470 (1970).

10. **S.E. Haaland** and **E.M. Sparrow,** :"Vortex Instability of Natural Convection Flow on Inclined Surfaces." Int. J. Heat Mass Transfer, 51 659-666 (1973).

11. **G.J. Hwang** and **K.C. Cheng,** : "NewThermal Instability of Laminar Natural Convection Flow on Inclined Isothermal Plates." Canadian J. Chemical Eng., 51, 659-666 (1973).

12. **J.B. Lee** and **G.S.H. Lock,** : "Instability in Boundary-Layer Free Convection along an Inclined Plate." Transactions Candian Society for Mechanical Engineering, 1, 197-203 (1972).

13. **R.A. Kahawita** and **R.N. Meroney,** : "The Vortex Mode of Instabilty in Natural Convection Flow Along Inclined Plates." Int. J. Heat Mass Transfer, 17 541-548 (1974).

14. **Chia-Jung Hsu,** : "An Exact Analysis of Low Paclet Number thermal entry Region Heat transfer in Transversely Non-uniform Velocity Fields." AIChe, 17, (3), 732-740 (1971).

15. **R.S. Wu,** and **K.C. Cheng,** : "Low Peclet Number Heat Transfer in the Thermal Entrance Region of Parallel-Plate Channels with Unequal Wall Temperatures." Canadian J of Chemical Eng., 54, 526-531 (1976).

16. **K.C. Cheng,** and **R.S. Wu,** : "Axial Heat Conduction Effects on Thermal Instability of Horizontal Plane Poiseuille Flows Heated from Below." J. Heat Transfer, 98c, 564-568 (1976).

17. **R.D. Cess,** and **E.C. Shaffer,** : "Heat Transfer to Laminar Flow Between Two Parallel Plates with a Prescribed Heat Flux at the Walls." Appl. Sci. Res., A9 64-70 (1959).

18. **R.D. Cess** and **E.C. Shaffer,** : "Laminar Heat Transfer Between Parallel Plates with Unsymmetrically Prescribed Heat Flux at the Wall." Appl. Sci. Res., A9, 64-70 (1959).

19. **A.H. Eraslan** and **N.F. Eraslan,** : "Heat transfer in Magnetohydrodynamic Channel Flow." Phys. Fluids, 12, (1), 120-128 (1969).