

# EFFECTIVE MOMENT OF INERTIA AND DEFLECTIONS OF REINFORCED CONCRETE BEAMS UNDER LONG-TERM LOADING

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## ABSTRACT

The paper presents a method for estimating long-term deflections of reinforced concrete beams by considering creep and shrinkage effects separately. Based on equilibrium and compatibility conditions a method is developed for investigating the properties of a cracked transformed section under sustained load. The concept of effective moment of inertia is extended to predict initial-plus-creep deflections. Long-term deflections computed by the proposed method are compared with the experimental results available in the literature and with the values obtained by the ACI Code method. This comparison shows that for beams having span/thickness ratio greater than 25, the ACI Code method underestimates long-term deflections while the proposed method gives a better estimate with computed to measured deflections ratio equal to 1.0.

## NOMENCLATURE

$A_s$	=	Area of tension steel
$A_s'$	=	Area of compression steel
$b$	=	Width of beam section
$C$	=	Creep coefficient (multiplier) for concrete strain at an uncracked section at time $t$
$C'$	=	Creep coefficient (multiplier) for concrete strain at a cracked section at time $t$
$C_t$	=	Creep coefficient for plain concrete at time $t$
$C_u$	=	Ultimate creep coefficient for plain concrete
$d$	=	Effective depth of beam section
$d'$	=	Compression steel depth factor
$E$	=	Modulus of elasticity
$E_c$	=	Modulus of elasticity of concrete under short-term loading

$E_{ct}$	=	Reduced or sustained modulus of elasticity of concrete at time t
$E_s$	=	Modulus of elasticity of steel
$f_c'$	=	Compressive strength of concrete under short-term loading
$f_r$	=	Modulus of rupture
$f_{c, fct}$	=	Extreme fibre compressive stresses in concrete at a cracked section, immediately after loading and at time t, respectively
$f_s, f_{st}$	=	Stresses in tension steel at a cracked section, immediately after loading and at time t respectively
$f_s', f_{st}'$	=	Stresses in compression steel at a cracked section, immediately after loading and at time t, respectively
$f_u, f_{ut}$	=	Extreme fibre compressive stresses in concrete at an uncracked section, immediately after loading and at time t, respectively
$h$	=	Total thickness of beam section
$I$	=	Moment of inertia
$I_c, I_{ct}$	=	Moments of inertia of cracked transformed sections immediately after loading and at time t, respectively
$I_e, I_{et}$	=	Effective moments of inertia for calculating beam deflections immediately after loading and at time t, respectively
$I_g$	=	Moment of inertia of gross concrete section ignoring reinforcement
$I_u, I_{ut}$	=	Moments of inertia of uncracked transformed sections immediately after loading and at time t, respectively
$k, k_t$	=	Neutral axis depth factors for cracked transformed section immediately after loading and at time t, respectively
$L$	=	Span length
$M$	=	Bending moment
$M_a$	=	Maximum span moment
$M_{cr}$	=	Cracking moment of a section
$m$	=	Beam constant (Sec Eq. 17)
$n$	=	Modular ratio for short-term/immediate loading
$n_t$	=	Increased modular ratio at time t
$n_{tt}$	=	Transformed area multiplier for tension steel at a cracked transformed section at time t
$p$	=	Tension steel ratio = $A_s/bd$
$p'$	=	Compression steel ratio = $A_s'/bd$
$S_c, S_{ct}$	=	Section moduli for cracked transformed sections with respect to extreme compression fibre, immediately after loading and at time t respectively
$S_u, S_{ut}$	=	Section moduli for uncracked transformed sections with respect to extreme compression fibre, immediately after loading and at time t respectively
$t$	=	time under sustained loading
$v_0$	=	Short-term/immediate (initial) deflection after loading

- $v_{cp}$  = Initial-plus-creep deflection at time  $t$   
 $v_{sh}$  = Shrinkage deflection at time  $t$   
 $v_t$  = Total long-term deflection at time  $t$   
 (Second subscripts M, A and P with  $v_0$  and  $v_t$  stand for measured deflection, deflection computed by the ACI Code method and deflection computed by the proposed method respectively)  
 $y, y_t$  = Ratios of neutral axis depth to total depth (thickness) of uncracked transformed sections immediately after loading and at time  $t$ , respectively  
 $\epsilon_c, \epsilon_{ct}$  = Extreme fibre compressive strains at cracked sections immediately after loading and at time  $t$  respectively  
 $\epsilon_{sh}$  = Shrinkage strain for plain concrete at time  $t$   
 $\epsilon_{sh}, \epsilon_u$  = Ultimate shrinkage strain for concrete  
 $\epsilon_u, \epsilon_{ut}$  = Extreme fibre compressive strains at uncracked sections immediately after loading and at time  $t$  respectively  
 $\lambda$  = Long-term deflection multiplier (ACI Code method)

## INTRODUCTION

The use of higher strength materials and the strength method of design, resulting in shallower members, has increased the importance of serviceability checks in the design of reinforced concrete structures. For flexural members immediate or short-term deflections as well as time-dependant or long-term deflections, under service loads, need to be checked.

Immediate deflections are influenced by the magnitude and distribution of loads, span and conditions of restraint, section properties, material properties and the amount and extent of flexural cracks [1]. Taking into account most of these factors the ACI Code 318-89 [2] recommends that immediate deflection of a reinforced concrete flexural member under service loads be computed by using the formulas of elastic deflection with flexural rigidity equal to  $E_c I_e$ , where  $E_c$  is the short-term modulus of elasticity of concrete and  $I_e$  is the effective moment of inertia for the member. A comparison of deflections calculated by the ACI Code method with experimental results shows that the method gives excellent prediction of immediate deflections [3,4].

Long-term deflections are mainly due to the effect of creep and shrinkage and are influenced by a number of factors including stresses in the concrete, amount of tension and compression reinforcement, size of member, curing conditions, temperature, relative humidity, age of concrete at time of loading, and duration of

loading [1]. The ACI Code [2] suggest that additional long-term deflection resulting from the combined effect of creep and shrinkage under sustained load be obtained by multiplying the corresponding immediate deflection by a factor  $\lambda$  that depends only on the duration of loading and compression reinforcement ratio. The application of this method to a large set of experimental results indicates a variability of  $\pm 62$  percent in the estimation of long-term deflections [4]. There may be, therefore, instances when the simple method suggested by the ACI Code may not be applicable and it may become essential to estimate long-term deflection by considering creep and shrinkage effects separately.

This paper proposes a method for calculating long-term deflections of reinforced concrete beams by considering creep and shrinkage effects separately. Deflections due to creep are generally much greater than deflections resulting from all other time-dependent effects combined and are of primary interest [1]. The aim of this paper is to develop a method for predicting flexural rigidity and deflection due to creep under sustained loading. Giving due consideration to the effect of creep, compatibility and equilibrium, the paper develops expressions for some basic properties of a cracked transformed section which will facilitate the calculation of its moments of inertia at any time  $t$ . The concept of effective moment of inertia is extended to estimate member flexural rigidity and initial-plus-creep deflection under sustained load. Separate calculations for shrinkage deflection are proposed to be carried out by Branson's method [5]. To establish the suitability of the method, deflection predictions using the proposed method are compared with experimentally measured long-term deflections as well as with the values obtained by the ACI Code [2] procedure.

## LONG-TERM DEFLECTIONS

Total long-term deflection  $v_t$  at any time  $t$  under sustained load can be expressed as a sum of initial-plus-creep deflection  $v_{cp}$  and shrinkage deflection  $v_{sh}$ , i.e.,

$$v_t = v_{cp} + v_{sh} \quad (1)$$

Considering service loads,  $v_{cp}$  can be computed by using the formulas of elastic deflection employed for calculating immediate deflection with  $E_c I_e$ , the flexural rigidity of the member immediately after loading, replaced by  $E_{ct} I_{et}$ , the flexural rigidity at time  $t$  under sustained loading. Here  $E_{ct}$  is the reduced modulus of elasticity of concrete and  $I_{et}$  is the effective moment of inertia of the member at time  $t$ . A method for calculating  $I_{et}$  is developed later in the paper.

Deflections due to shrinkage are independent of the applied loading but are in the same direction as those caused by gravity loading. A number of methods have been suggested for estimating shrinkage deflections. In the method proposed in this paper, Branson's method [5] is used for estimating shrinkage deflections.

## EFFECTIVE MOMENT OF INERTIA

For calculating immediate deflections under service loads the ACI Code [2] recommends formulas for  $E_c$  and  $I_e$ . The formula for the effective moment of inertia  $I_e$  is:

$$I_e = \left( \frac{M_{cr}}{M_a} \right)^3 I_g + \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right] I_c \quad (2)$$

where  $M_{cr}$  is the cracking moment,  $M_a$  is the maximum span moment,  $I_g$  is the moment of inertia of the gross concrete section ignoring reinforcement, and  $I_c$  is the moment of inertia of the cracked transformed section.

In order to include the effect of reinforcement on the properties of uncracked section,  $I_g$  in Eq. (2) can be replaced by  $I_u$ , the moment of inertia of the uncracked transformed section to yield the following equation for  $I_e$ :

$$I_e = \left( \frac{M_{cr}}{M_a} \right)^3 I_u + \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right] I_c \quad (3)$$

The use of  $I_u$  in place of  $I_g$  has already been recommended by Branson [5].  $I_u$  may be more accurately used for estimating  $I_e$  for heavily reinforced members and lightweight concrete members. Obviously, the cracking moment  $M_{cr}$  for use in Eq. (3) should also be determined by using the uncracked transformed section. If the modulus of rupture,  $f_r$  is evaluated correctly, the accuracy in determining the first cracking point will be improved using  $I_u$  [5].

Using Eq. (3), the flexural rigidity  $E_c I_e$  for a member immediately on loading can be expressed as follows:

$$E_c I_e = \left[ \left( \frac{M_{cr}}{M_a} \right)^3 + \left( 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right) I_c / I_u \right] I_u E_c \quad (4)$$

As already indicated, the flexural rigidity for predicting initial-plus-creep deflection can be expressed as  $E_{ct} I_{et}$ . The reduced or sustained modulus of elasticity for concrete  $E_{ct}$  can be taken equal to  $E_c/(1 + C_t)$ , where  $C_t$  is the creep coefficient and is a function of relative humidity, concrete quality, duration of applied loading, and age of concrete when loaded [1]. The following relationship is suggested [5] between  $C_t$ , ultimate creep coefficient  $C_u$  and duration of loading  $t$  (days) for concrete having "standard conditions", i.e., 4 in (100 mm) slump or less, 40 percent relative humidity, moist cured and loaded at an age of 7 days:

$$C_t = \frac{t^{0.60}}{10 + t^{0.60}} C_u \quad (5)$$

Correction factors for  $C_t$  have also been suggested [5] when the conditions are different from the above-mentioned standard conditions.

For initially cracked simply supported beams (being considered in this study), the values of  $M_a$  and  $M_{cr}/M_a$  will remain constant under sustained load. Effective moment of inertia  $I_{et}$ , and flexural rigidity  $E_{ct} I_{et}$  at time  $t$  can then be expressed by the following equations which are similar to Eqs. (3) and (4) respectively:

$$I_{et} = \left( \frac{M_{cr}}{M_a} \right)^3 I_{ut} + \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right] I_{ct} \quad (6)$$

$$E_{et} I_{et} = \left[ \left( \frac{M_{cr}}{M_a} \right)^3 + \left( 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right) I_{ct} / I_{ut} \right] I_{ut} E_{ct} \quad (7)$$

Here  $I_{ut}$  and  $I_{ct}$  are the moments of inertia of uncracked and cracked transformed sections respectively at time  $t$ .

While discussing the development of Eq. (2) it has been stated [5] that the equation is not sensitive to the power of  $M_{cr}/M_a$  with values of 2.8 or 3.2. Noting this discussion and the limited data presently available on time-dependent deflections the power of 3 has been retained in Eqs. (6) and (7).

In continuous members some redistribution of moments may occur under sustained load. Ignoring this redistribution the values of  $I_{et}$  for the critical positive and negative moment sections of a continuous member can be calculated with the respective values of  $M_a$  and  $M_{cr}/M_a$  used in computing  $I_e$  for those sections.

## TRANSFORMED SECTIONS

The properties of uncracked and cracked transformed sections immediately on loading are obtained by applying the concept of the modular ratio,  $n = E_s/E_c$ , where  $E_s$  is the modulus of elasticity of steel [6]. The modular ratio concept assumes equal strains in the reinforcing steel and the surrounding concrete, i.e., perfect bond.

For reinforced concrete beams under sustained load an "increased" modular ratio  $n_t = E_s/E_{ct}$  can be defined. Noting the basic assumption of modular ratio concept,  $n_t$  may be used for calculating the transformed areas of tension and compression reinforcement, and hence  $I_{ut}$ , for an uncracked section. Similarly,  $n_t$  may be used for calculating the transformed area of compression reinforcement at a cracked section. However, this simple increased modular ratio approach does not seem to be valid for computing the transformed area of tension reinforcement at a cracked section under sustained load. The creep of concrete will increase bond deterioration and slip between steel and concrete which occur adjacent to cracks. The quasi-elastic method based on increased modular ratio will thus overestimate the tension steel area transformed to equivalent concrete area. The transformed area of tension reinforcement and other properties of a cracked transformed section under sustained load need to be obtained by considering strain compatibility and equilibrium conditions. Simultaneously, it is important to consider any relationship that may exist between the properties of uncracked and cracked transformed sections immediately after loading and under sustained load. The following equations for the properties of the two types of sections have been developed by assuming that the stresses are within linear elastic range. This assumption generally implies that the steel stress is below yield point and the concrete stress is not greater than  $0.5 f_c'$ , where  $f_c'$  is the cylinder compressive strength [6].

### Uncracked Section

For an uncracked section subjected to a bending moment  $M$ , Fig. 1 shows transformed sections, strain distributions and stress distributions immediately after loading ( $t = 0$ ) and at time  $t$  under sustained load. Since the stresses and strains are within the elastic range, the extreme fibre strains,  $\epsilon_u$  immediately after loading and  $\epsilon_{ut} = (1 + C) \epsilon_u$ , at time  $t$ , can be expressed as follows:

$$\epsilon_u = \frac{f_u}{E_c} = \frac{M y h}{E_c I_u} \quad (8)$$

and

$$\epsilon_{ut} = \frac{f_{ut}}{E_{ct}} \quad \text{or} \quad (1+C)\epsilon_u = \frac{M y_t h}{E_{ct} I_{ut}} \quad (9)$$

where  $h$  is the total depth (thickness) of the section,  $f_u$  and  $f_{ut}$  are the extreme fibre compression stresses in concrete,  $y$  and  $y_t$  are the ratios of neutral axis depth to the total depth (thickness), and  $I_u$  and  $I_{ut}$  are the moments of inertia of uncracked transformed sections immediately after loading and at time  $t$ , respectively.  $C$  is the creep coefficient (multiplier) for concrete strain at the uncracked section under consideration. Because of the influence of section properties, the value of  $C$  may be less than  $C_t$ , the creep coefficient obtained from tests on plain concrete.

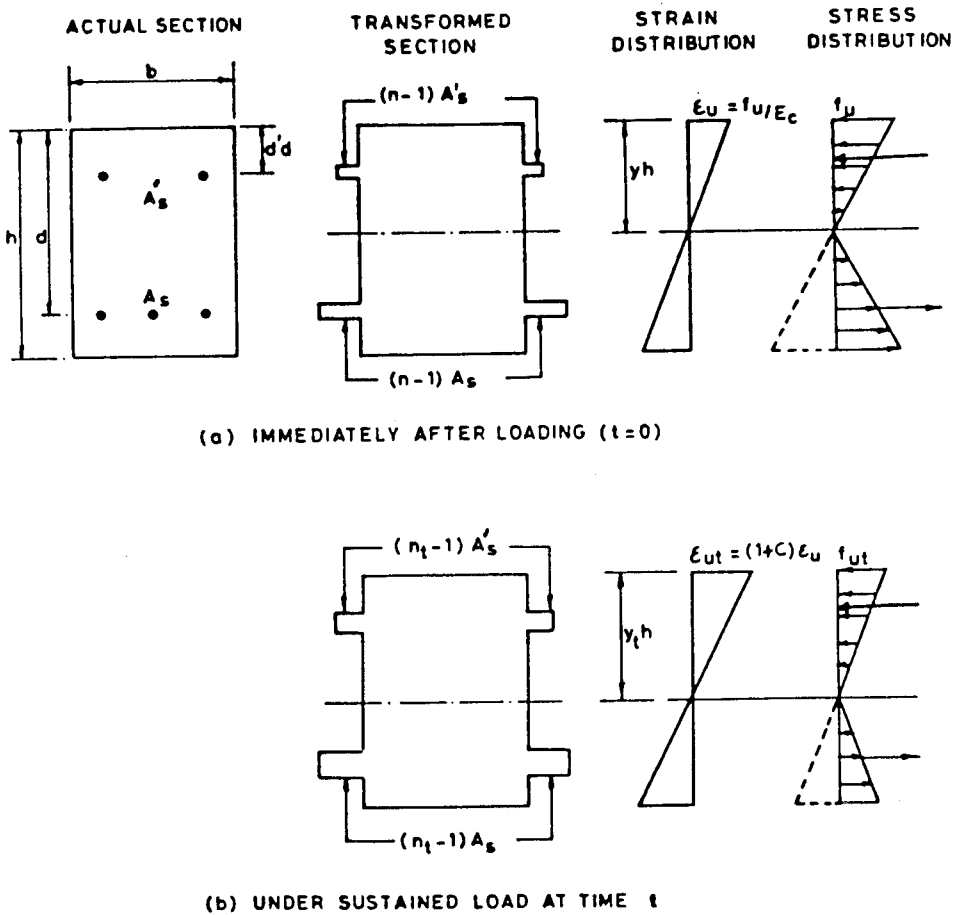


Fig. 1: Uncracked section



From Eqs. (8) and (9) the following relationship between  $C$ ,  $C_t$  and the properties of uncracked transformed sections is obtained:

$$1 + C = \frac{E_c I_u}{E_{ct} I_{ut}} \frac{y_t}{y} \quad (10)$$

Substituting  $E_{ct} = \frac{E_c}{(1 + C_t)}$  in Eq. (10),

$$1 + C = (1 + C_t) \frac{I_u}{I_{ut}} \frac{y_t}{y} = (1 + C_t) \frac{S_u}{S_{ut}} \quad (11)$$

where  $S_u = I_u/y$  and  $S_{ut} = I_{ut}/y_t$  are the section moduli for the uncracked transformed sections, with respect to the extreme compression fibre, immediately after loading and at time  $t$  under sustained load.

### Cracked Section

Figure 2 shows transformed sections, strain distributions and stress distributions for a cracked section immediately after loading and at time  $t$  under sustained load with bending moment  $M$ . For the transformed section immediately after loading, the modular ratio  $n$  has been used to define the transformed areas of tension and compression steels. For the transformed section at  $t$ ,  $n_t$  has been used to define the transformed area of compression steel only. In keeping with the previous discussion, the transformed area for tension steel at time  $t$  has been taken equal to  $n_{tt} A_s$ , where  $n_{tt}$  is defined as the transformed area multiplier for tension steel at time  $t$ .  $C'$  is the creep coefficient for concrete strain at the cracked section under consideration. At this stage it may be assumed that  $C'$  is not equal to  $C$ . It is to be noted that with the cracked transformed section modeled by using  $n_{tt}$ , the value at the level of tension steel in the strain distribution (Fig. 2b) does not represent the steel strain at time  $t$ . However, the strain in tension steel can be obtained from the corresponding stress once its value has been determined.

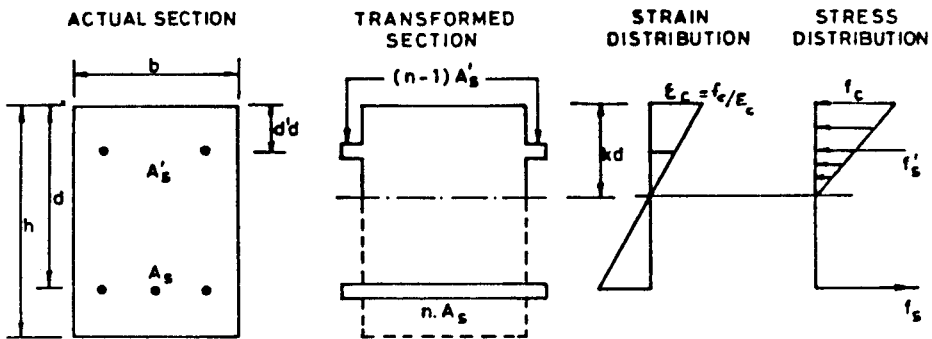
Following the procedure used for uncracked section, a relationship between  $C'$ ,  $C_t$  and the properties of cracked transformed section can be obtained:

$$1 + C' = \frac{E_c I_c}{E_{ct} I_{ct}} \frac{k_t}{k} \quad (12)$$

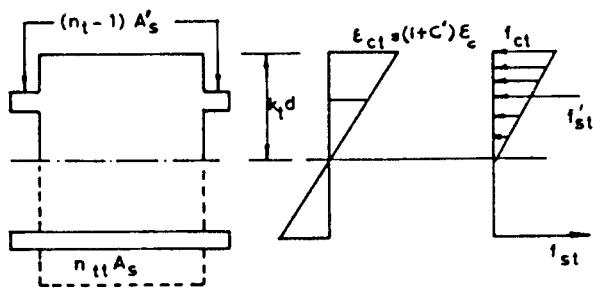
or

$$1 + C' = (1 + C_t) \frac{I_c}{I_{ct}} \frac{k_t}{k} = (1 + C_t) \frac{S_c}{S_{ct}} \quad (13)$$

where  $k$  and  $k_t$  are the neutral axis depth factors and  $I_c$  and  $I_{ct}$  are the moments of inertia of the cracked transformed sections immediately after loading and at time  $t$  under sustained load, respectively.  $S_c$  taken as  $I_c/kd$  and  $S_{ct}$  taken as  $I_{ct}/k_t d$  are the section moduli for cracked transformed sections immediately after loading and at time  $t$ .  $d$  is the effective depth of the section.



(a) IMMEDIATELY AFTER LOADING ( $t=0$ )



(b) UNDER SUSTAINED LOAD AT TIME  $t$

Fig. 2: Cracked section

From Eqs.(11) and (13) the following expressions are obtained:

$$\frac{S_{ct}}{S_{ut}} = \frac{S_c (1+C)}{S_u (1+C')} \quad (14)$$

or

$$\frac{I_u I_{ct} y_t k (1+C')}{I_{ut} I_c y k_t (1+C)} = 1.0 \quad (15)$$

Eqs. (14) and (15) represent relationships between the uncracked and cracked sections. These equations can be used to check the validity of any procedure used for calculating the properties of the two types of sections.

For the cracked transformed section at time t, the value of  $k_t$  can be determined by considering equilibrium conditions immediately after loading and under sustained load.

Taking moments about the tension steel, for the stresses immediately after loading (Fig. 2a):

$$M = \frac{1}{6} f_c k b d^2 (3-k) + p' b d^2 (n-1) f_c \frac{k-d'}{k} (1-d') \quad (16)$$

or

$$\frac{M}{f_c b d^2} = \frac{3k - k^2}{6} + p'(n-1) \frac{k-d'}{k} (1-d') = m \text{ (constant)} \quad (17)$$

Similarly, from the stress distribution at time t (Fig. 2b):

$$f_{ct} = \left( \frac{1+C'}{1+C_t} \right) f_c \quad (18)$$

and

$$\frac{M}{f_c b d^2} = \left[ \frac{3k_t - k_t^2}{6} + p'(n-1) \frac{k_t - d'}{k_t} (1-d') \right] \left( \frac{1+C'}{1+C_t} \right) \quad (19)$$

Here  $f_c$  and  $f_{ct}$  are the stresses at the extreme fibre in compression, immediately after loading and at time t,  $p'$  is the compression steel ratio,  $b$  is the width, and  $d'$  is

a factor which defines the depth of the compression steel  $A_s'$  with respect to the effective depth  $d$ .

Equating the right hand sides of Eqs. (17) and (19), and rearranging:

$$\frac{1+C'}{1+C_t} = \left[ \frac{3k_t - k_t^2}{6} + p'(n_t - 1)(1 - d') \frac{k_t - d'}{k_t} \right] = m \quad (20)$$

Trial and error solution of Eq. (20) will provide the value of  $k_t$ . In the case of a singly reinforced rectangular section, Eq. (20) reduces to a quadratic equation, whose solution yields the following expression for  $k_t$ :

$$k_t = 1.5 \pm \sqrt{2.25 - \left( \frac{1+C_t}{1+C'} \right) (3k - k^2)} \quad (21)$$

Once the value of  $k_t$  has been determined,  $n_{tt}$  is obtained by taking moments of areas of the transformed section at time  $t$  (Fig. 2), about the neutral axis. The final expression for  $n_{tt}$  is:

$$n_{tt} = \frac{k_t^2 + 2(n_t - 1)p'(k_t - d')}{2p(1 - k_t)} \quad (22)$$

In the case of a singly reinforced rectangular section, Eq. (22) reduces to the following form:

$$n_{tt} = \frac{k_t^2}{2p(1 - k_t)} \quad (23)$$

Having calculated the values of  $k_t$  and  $n_{tt}$ , the moment of inertia  $I_{ct}$  for the cracked transformed section at time  $t$  (Fig. 2b) can be worked out.

If all the properties for uncracked and cracked sections are correctly calculated, Eqs. (14) and (15) will be satisfied.

## Stresses in Concrete and Steel

Stresses in concrete at different fibres of uncracked or cracked section can be calculated directly by using the properties of corresponding transformed section and

the bending equation of elastic homogeneous beams. Stress in reinforcing steel will be a product of the appropriate value of modular ratio/transformed area multiplier and the stress in the transformed section at the level of the reinforcement. The value of  $n$  will, therefore, be used for calculating stresses in tension and compression steels at uncracked and cracked sections immediately after loading. Under sustained load,  $n_t$  will be used for calculating stresses in tension and compression steels at uncracked sections as well as in compression steel at a cracked section, while  $n_{tt}$  will be used for calculating stress in tension steel at a cracked section.

## COMPUTATION PROCEDURE AND SIMPLIFIED EQUATIONS

For a given section and material parameters, the properties of uncracked transformed sections ( $y$ ,  $I_u$ ,  $y_t$  and  $I_{ut}$ ) immediately after loading and under sustained loading (Fig. 1) and those of a cracked transformed section ( $k$  and  $I_c$ ) immediately after loading (Fig. 2) can be readily computed. The value of  $C$  can then be calculated from Eq. (11). For the determination of  $C'$  the following procedure can be adopted:

- a) Assume a value of  $k_t$
- b) Calculate  $n_{tt}$  from Eq. (22)
- c) Calculate  $I_{ct}$  for the cracked transformed section using  $n_t$  for compression steel and  $n_{tt}$  for tension steel (Fig. 2)
- d) Calculate  $(1 + C')$  from Eq. (13)
- e) Substitute all the relevant values in Eqs. (20) and (15) to see whether these equations are satisfied
- f) Repeat the above steps to obtain a value of  $k_t$  and the corresponding values of  $n_{tt}$ ,  $I_{ct}$  and  $C'$  which satisfy Eqs. (20) and (15).

The above procedure was used to develop a computer program to analyze a large number of rectangular beam sections with different combinations of practically possible section and material parameters. The results of this parametric study indicate that Eqs. (20) and (15) are satisfied only when  $C' = C$ , i.e., the creep coefficients (multipliers) for concrete strains in uncracked and cracked transformed sections at any time under sustained loading, are equal. This is a logical finding because the same value of the creep coefficient,  $C_t$  for plain concrete has been used

in the derivation of equations for the two types of transformed sections and that Eqs. (11) and (13), for  $C$  and  $C'$ , respectively, are independent of the bending moment and the stresses being induced.

By substituting  $C' = C$ , Eqs. (14) and (15) take the following simplified form:

$$\frac{S_{ct}}{S_{ut}} = \frac{S_c}{S_u} \quad (24)$$

and

$$\frac{I_u}{I_{ut}} \frac{I_{ct}}{I_c} \frac{y_t}{y} \frac{k}{k_t} = 1.0 \quad (25)$$

Eq. (24) shows that if all the values are correctly substituted, the ratio of section moduli with respect to extreme compression fibre for the two types of sections under sustained load, with stresses within elastic range, will remain constant and will be equal to the ratio for the sections immediately after loading.

The substitution of  $C' = C$  simplifies the above-mentioned iterative procedure for determining the properties ( $k_t$ ,  $n_{tt}$  and  $I_{ct}$ ) of the cracked transformed section under sustained load. In the case of singly reinforced beams,  $k_t$  can be computed directly from Eq. (21).

The results of this study also indicate that Eqs. (15) and (25) are not satisfied when the properties ( $k_t$ ,  $I_{ct}$  and  $C'$ ) of the cracked transformed section, based on a simple increased modular ratio approach ( $n_{tt} = n_t$ ), are substituted in these equations with all other terms remaining unchanged.

Figures. 3-6 present typical results to show the influence of different parameters on flexural rigidities of uncracked and cracked transformed sections computed by the proposed method. For two different sets of section and material parameters, Fig. 3 shows the influence of tension steel ratio on the flexural rigidities of singly reinforced uncracked and cracked transformed sections, immediately after loading and under sustained load. Keeping all other section and material parameters constant, Fig. 4 shows the influence of effective depth  $d$  on  $n_{tt}$  and flexural rigidities. Fig. 5 shows the beneficial effect of compression reinforcement on increasing flexural rigidities and therefore reducing deflections under sustained loads. The effect of compressive strength of concrete on flexural rigidities is shown in Fig. 6. Compressive strength of concrete influences the modulus of elasticity  $E_c$ , modular ratio  $n$  and the ultimate creep coefficient  $C_u$ . The graphs in Fig. 6 have

been drawn for different values of  $f_c'$  by using the appropriate values of  $E_c$  and  $C_u$  based on the study at Cornell University [6]. The results in Figs. 3-6 show that besides compression steel ratio  $p'$ , effective depth  $d$ , tension steel ratio  $p$  and compressive strength of concrete  $f_c'$  may have appreciable effect on flexural rigidities. These factors should, therefore, be considered in predicting deflections of reinforced concrete beams under sustained loads.

The results of parametric study show that in some cases a difference as high as 15 percent in the values of  $E_{ct} I_{ct}/I_c I_c$  and  $E_{ct} I_{ut}/E_c I_u$  may occur. One typical set of results showing the difference in flexural rigidities of uncracked and cracked sections is presented in Fig. 4. In Figs. 5 and 6 only the graphs for  $E_{ct} I_{ct}/E_c I_c$  have been shown as the values for  $E_{ct} I_{ut}/E_c I_u$  were very close to these values. The study, therefore, indicates that for most sections:

$$\frac{E_{ct} I_{ct}}{E_c I_c} = \frac{E_{ct} I_{ut}}{E_c I_u}$$

or

$$\frac{I_{ct}}{I_{ut}} = \frac{I_c}{I_u} \tag{26}$$

Replacing  $I_{ct}/I_{ut}$  by  $I_c/I_u$  and  $E_{ct}$  by  $E_c(1 + C_t)$  in Eq.(7), therefore, yields:

$$E_{ct} I_{ct} = E_c I_c \left[ \frac{I_{ut}}{(1 + C_t) I_u} \right] \tag{27}$$

In other words, the initial-plus-creep deflection  $v_{cp}$  at any time  $t$  under sustained loading can be predicted for most cases by multiplying the corresponding immediate deflection with  $(1+C_t) I_u/I_{ut}$ .

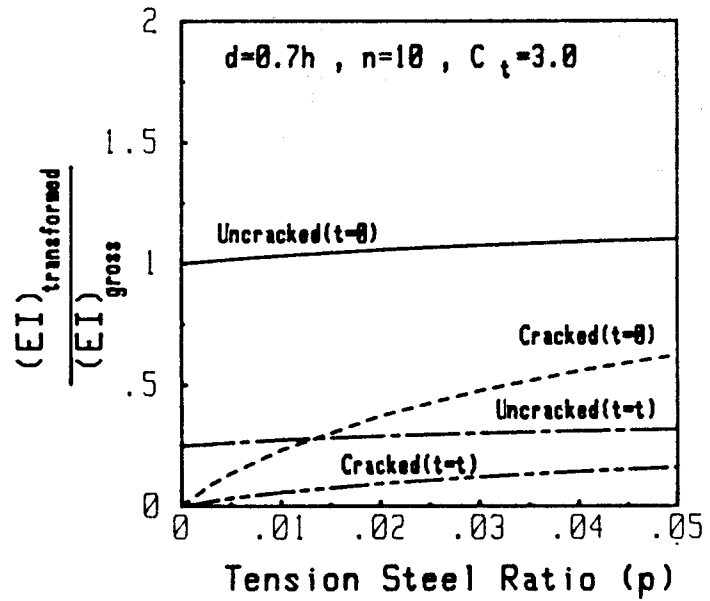
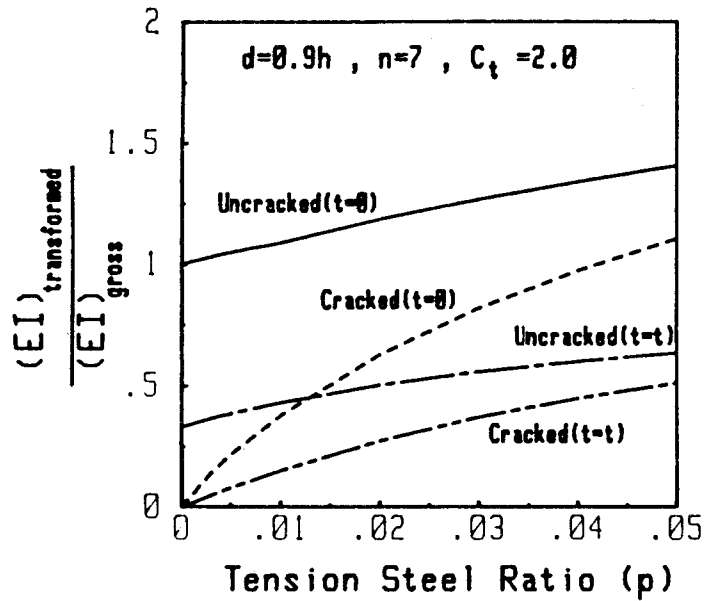


Fig. 3 Influence of tension steel ratio on immediate and long-term flexural rigidities of singly reinforced rectangular sections



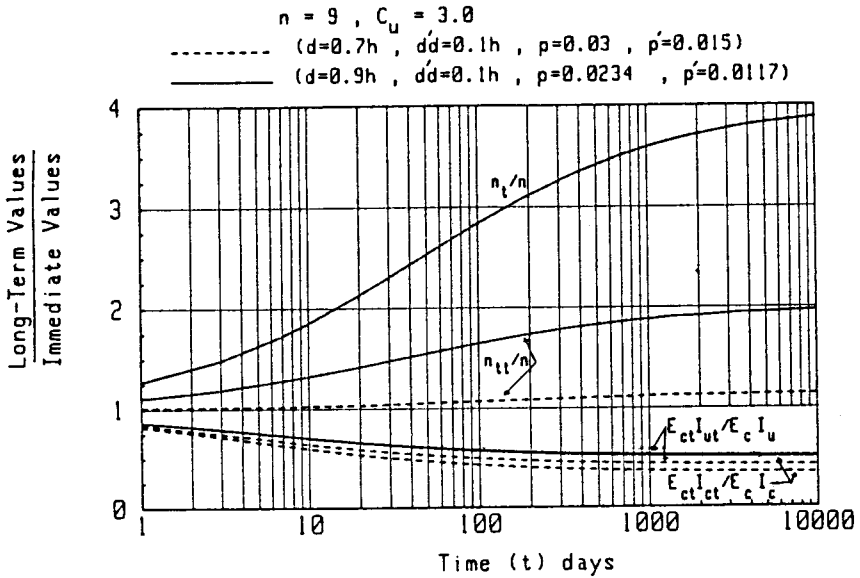


Fig. 4: Influence of effective depth on flexural rigidities and  $n_{tt}$

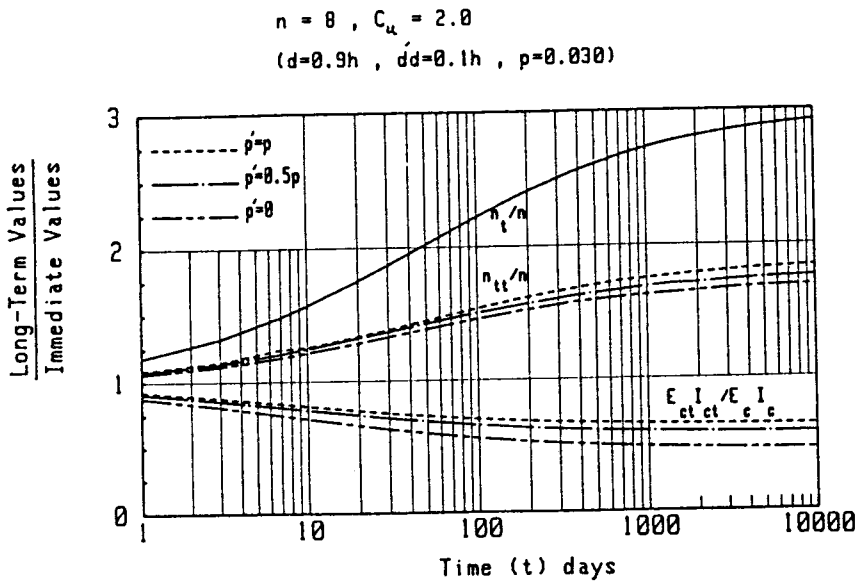
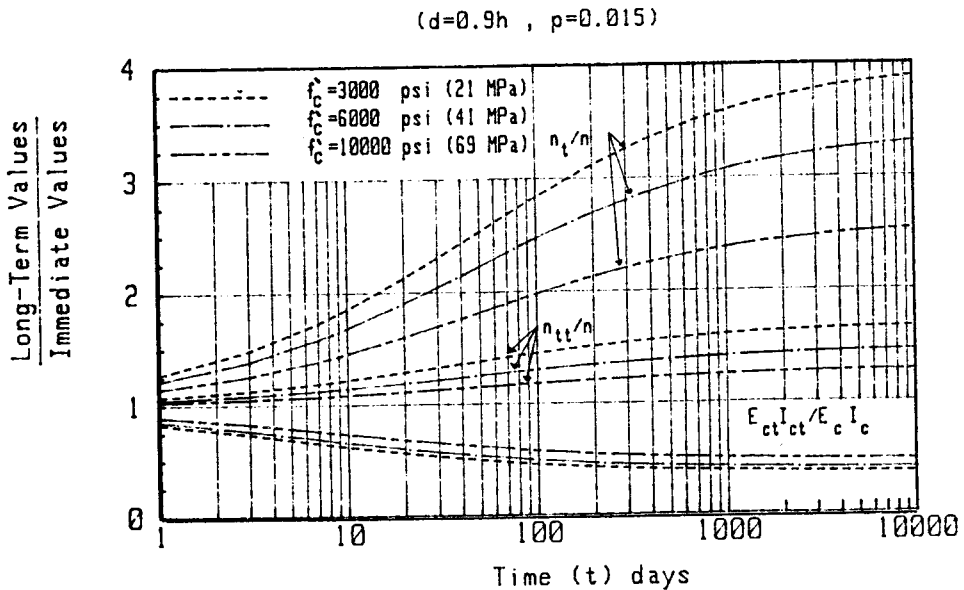


Fig. 5: Influence of compression steel ratio on flexural rigidities and  $n_{tt}$



**Fig. 6: Influence of compressive strength of concrete on flexural rigidities and  $n_{tt}$  for singly reinforced rectangular sections**

### COMPARISON OF MEASURED AND COMPUTED LONG-TERM DEFLECTIONS

Using the data reported in different laboratory studies [7-11], immediate and long-term deflections of a number of simply supported beams were computed by the proposed method and were compared with the measured values and the values estimated by the ACI Code method. Table 1 shows the necessary data, measured and computed deflections and the results of statistical analysis for all the beams. In Table 1,  $v_{0M}$  and  $v_{tM}$  are the measured immediate and long-term deflections respectively,  $v_{0A}$  and  $v_{tA}$  are the immediate and long-term deflections computed by using the ACI Code method, while  $v_{0P}$  and  $v_{tP}$  are the immediate and long-term deflections computed by the proposed method.  $v_{tP}$  is the sum of the initial-plus-creep deflection  $v_{cp}$  and the shrinkage deflection  $v_{sh}$  (Eq. 1).

For rectangular beams in Table 1, initial-plus-creep deflections  $v_{cp}$  were computed using  $E_{ct} I_{ct}$  with  $I_{ct}$  determined by the already stated iterative procedure. For the T-beams of Yu and Winter the values of  $v_{cp}$  were calculated by adopting

**Table 1: Comparison of Measured and Computed Deflections for Simply Supported Beams**

Beam No. +	L/h	p'	Measured Deflections		Computed Deflections++									
			v <sub>OM</sub> (mm)	v <sub>tI</sub> (mm)	ACI Code Method				Proposed Method					
					v <sub>OA</sub> (mm)	$\frac{v_{OA}}{v_{OM}}$	v <sub>tA</sub> (mm)	$\frac{v_{tA}}{v_{tM}}$	v <sub>OP</sub> (mm)	$\frac{v_{OP}}{v_{OM}}$	v <sub>cp</sub> (mm)	v <sub>sh</sub> (mm)	v <sub>tP</sub> (mm)	$\frac{v_{tP}}{v_{tM}}$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Rectangular Beams -- Washa and Fluck <sup>7</sup> : t=30 months, C <sub>u</sub> =2.5, ε <sub>sh,u</sub> =0.00075														
A1, A4	20	0.0163	13.46	23.62	15.34	1.14	29.88	1.26	13.80	1.02	28.12	0	28.12	1.19
A2, A5	20	0.0077	15.75	32.26	15.77	1.00	35.38	1.10	14.52	0.92	32.84	5.35	38.19	1.18
A3, A6	20	0	17.02	44.70	16.26	0.96	44.23	0.99	15.30	0.90	39.68	9.07	48.75	1.09
B1, B4	30	0.0167	23.37	51.05	25.64	1.10	49.69	0.97	23.22	0.99	51.47	0	51.47	1.00
B2, B5	30	0.0083	24.89	65.02	25.88	1.04	57.29	0.88	23.87	0.96	58.03	7.69	65.72	1.01
B3, B6	30	0	26.42	86.36	26.18	0.99	71.21	0.83	24.63	0.93	69.31	13.71	83.02	0.96
C1, C4	50	0.0167	40.13	80.01	43.63	1.09	84.57	1.06	39.76	0.99	84.37	0	84.37	1.05
C2, C5	50	0.0083	43.43	100.58	44.38	1.02	98.26	0.98	41.19	0.95	95.83	13.35	109.18	1.09
C3, C6	50	0	47.75	140.72	45.35	0.95	123.35	0.88	42.92	0.90	116.93	23.79	140.72	1.00
D1, D4	30	0.0167	11.94	27.69	15.83	1.33	30.68	1.11	14.46	1.21	30.64	0	30.64	1.11
D2, D5	30	0.0083	14.22	33.02	16.11	1.13	35.66	1.08	14.97	1.05	34.80	4.81	39.61	1.20
D3, D6	30	0	17.78	48.51	16.46	0.93	44.77	0.92	15.60	0.88	42.49	8.56	51.05	1.05
E1, E4	70	0.0159	59.44	123.95	52.72	0.89	103.28	0.83	47.57	0.80	106.53	0	105.92	0.86
E2, E5	70	0.008	55.88	128.78	53.11	0.95	118.50	0.92	48.82	0.87	120.36	15.45	135.81	1.05
E3, E6	70	0	62.99	184.91	53.59	0.85	145.76	0.79	50.26	0.80	142.34	27.52	169.87	0.92
Tee Beams -- Yu and Winter <sup>8</sup> : t=6 months, C <sub>u</sub> =2.0, ε <sub>sh,u</sub> =0.000625														
A	20	0	34.04	67.31	29.97	0.88	66.29	0.98	28.96	0.85	61.72	4.57	66.29	0.94
B	20	0.0051	31.49	56.64	29.46	0.94	52.83	0.90	28.45	0.90	58.42	2.54	60.96	1.03
C	20	0.01	30.23	56.89	29.72	0.98	49.53	0.87	28.70	0.95	56.90	0	56.90	1.00
D	20	0	32.26	67.06	32.77	1.02	72.14	1.08	32.26	1.00	65.02	4.57	69.60	1.04
E	14	0	12.95	29.21	14.22	1.10	31.24	1.07	13.72	1.06	28.46	2.29	31.75	1.09
F	30	0	55.88	100.30	53.59	0.96	118.90	1.17	52.58	0.94	113.28	7.87	121.20	1.21

Table 1 continued on the next page .....

Table 1 continued

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Rectangular Beams -- Corley and Sozen <sup>9</sup> : t=23 months, $C_u=3.0$ , $\epsilon_{sh,u}=0.0003$														
C1	12	0	3.05	7.37	2.95	0.92	7.37	1.00	2.79	0.92	7.11	0.51	7.62	1.03
C3	17	0	7.87	17.27	7.62	0.97	20.32	1.18	7.37	0.94	20.83	1.02	21.84	1.26
C4	17	0	6.10	15.49	5.84	0.96	15.75	1.02	5.84	0.96	15.75	1.02	16.76	1.08
Rectangular Beams -- Bakoss et al <sup>10</sup> : t=500 days, $C_t=2.4^*$ , $\epsilon_{sh}=0.00065^*$														
1B1	25	0	-	9.08	1.11	-	2.83	0.31	0.94	-	2.60	6.41	9.01	0.99
1B2	25	0	8.94	25.02	9.70	1.09	24.73	0.99	8.25	0.92	22.55	6.41	28.96	1.16
Rectangular Beams** -- Clarke et al <sup>11</sup> : t=6 months, $C_t=2.26^*$														
B1	14	0.012	4.78	8.15	4.91	1.03	8.59	1.05	4.50	0.94	9.64	0	9.64	1.18
B2	14	0.012	4.30	7.93	4.72	1.10	8.26	1.04	4.31	1.00	9.26	0	9.26	1.17
S	For all beams:													
T A	Mean					1.012	0.974	0.946	1.069					
A N	Standard deviation					0.101	0.173	0.083	0.098					
T A	For beams with L/h = 25:													
I L	Mean					1.022	0.915	0.942	1.044					
S Y	Standard deviation					0.122	0.20	0.104	0.098					
T S														
I I														
C S														
A														
L														

+ For beam description, loading and other properties see the original reference.

++ In computing deflections by the ACI Code method and the proposed method,  $E_c$  and  $f_r$  were calculated by the ACI Code formulas and the values of  $f'_c$  at the time of loading.

\* Measured values at time t.

\*\* Data on shrinkage deformations not given in the reference. Therefore, beams having  $p'=p$  (zero shrinkage deflection) have been included from that reference in this table.

the simple approach of multiplying the computed immediate deflections by the corresponding factor  $(1 + C_t)I_u/I_{ut}$ .

Table 1 indicates that the use of  $I_u$  in place of  $I_g$  in the proposed method results in a slight decrease in the estimation of immediate deflections when compared with the ACI Code method. The mean and standard deviation of the ratio of computed to measured immediate deflections  $v_{0P}/v_{0M}$  by the proposed method are 0.946 and 0.083 respectively compared with 1.012 and 0.101 for  $v_{0A}/v_{0M}$  by the ACI Code method. Nevertheless, the mean and standard deviation of the ratio of computed to measured long-term deflections  $v_{tP}/v_{tM}$  by the proposed method are 1.069 and 0.098 respectively compared with the corresponding values of 0.974 and 0.173 for  $v_{tA}/v_{tM}$  by the ACI Code method.

The results in Table 1 show that the ACI Code method generally underestimates long-term deflections of beams having span to thickness ratio,  $L/h$ , greater than 25. The mean and standard deviation of computed to measured long-term deflections for beams with  $L/h > 25$  (Table 1) are 0.915 and 0.20 respectively by the ACI Code method compared with 1.044 and 0.098 by the proposed method. For these beams the proposed method therefore gives a better estimate by considering creep and shrinkage components of deflections separately as well as by employing a detailed procedure for the computation of flexural rigidity under sustained load. The same is further true for beam 1B1 of Bakoss et al [10], in which loading produces  $M_a < M_{Cr}$ . For this beam  $v_{tA}/v_{tM}$  is 0.33 and  $v_{tP}/v_{tM}$  is 0.99.

Since the main objective of this study was to develop a method for estimating effective moment of inertia and flexural rigidity of beams under sustained loads, a comparison of measured and computed initial-plus-creep deflections was made, as shown in Table 2. This Table includes those beams from Table 1 for which compression and tension steel ratios are equal ( $p' = p$ ) and therefore theoretical values of shrinkage deflection  $v_{sh}$  are zero. Table 2 also includes the beams from Table 1 for which it was possible to deduct the shrinkage deflection component from the measured long-term deflection  $v_{tM}$  at time  $t$  by using the data of companion specimens made and tested to study the shrinkage behaviour. The mean value and standard deviation of the ratios of computed to measured deflections for the sample of 16 beams in Table 2 are 1.09 and 0.11, respectively. This comparison in Table 2 shows that the proposed method, based on the calculation of flexural rigidity under sustained loads, gives a good estimate of initial-plus-creep deflection.

**Table 2: Comparison of Measured and Computed Initial-Plus-Creep Deflections**

Beam No.+	Measured (mm)	Computed (mm)	Ratio*
A1. A4	23.62	28.12	1.19
B1. B4	51.05	51.48	1.01
C1. C4	80.00	84.37	1.05
C2. C5	87.88	95.83	1.09
C3. C6	110.22	116.93	1.06
D1. D4	27.69	30.64	1.11
D2. D5	27.92	34.80	1.25
D3. D6	39.62	42.49	1.07
E1. E4	123.95	106.53	0.86
E2. E5	117.37	120.36	1.03
E3. E6	154.42	142.34	0.92
C	56.89	56.90	1.00
C1	6.00	7.11	1.18
1B1	18.12	22.55	1.24
B1	8.15	9.64	1.18
B2	7.93	9.26	1.17
Mean			1.09
Standard Deviation			0.11

+ Beam Nos. are same as given in Table 1 from different references.

\* Ratio of computed to measured initial-plus-creep deflections.

Table 3 shows the computed values, immediately after loading and at  $t = 30$  months under sustained load, of the neutral axis depth factor, concrete stress at the extreme compression fibre and stresses in compression as well as tension steels for the maximum moment (mid-span) cracked sections of the beams tested by Washa and Fluck [7]. These values were computed using the elastic bending equation and the properties of cracked transformed sections determined by the proposed method. The results in Table 3 show that due to the effect of creep, the neutral axis depth increases, concrete stress decreases and stresses in compression and tension steels

**Table 3: Computed Neutral Axis Depth Factors and Stresses in Concrete and Steel at the Maximum Moment Cracked Section of Beams Tested by Washa and Fluck**

Beam No.	k	$k_t$	$f_c$ (MPa)	$f_{ct}$ (MPa)	$f_s$ (MPa)	$f_{st}$ (MPa)	$f_s$ (MPa)	$f_{st}$ (MPa)
A1, A4	0.364	0.361	9.17	5.86	37.92	76.05	135.69	138.52
A2, A5	0.384	0.396	9.96	7.45	43.51	104.94	135.42	137.83
A3, A6	0.405	0.450	10.89	10.00	-	-	135.35	137.69
B1, B4	0.397	0.411	9.72	7.03	23.72	59.16	136.87	143.97
B2, B5	0.408	0.427	10.14	8.21	26.61	75.02	135.69	140.80
B3, B6	0.422	0.447	10.62	10.14	-	-	134.25	153.56
C1, C4	0.390	0.396	9.24	6.34	31.17	68.67	136.04	141.55
C2, C5	0.405	0.422	9.80	7.59	35.30	90.67	135.07	139.42
C3, C6	0.424	0.458	10.52	9.86	-	-	133.97	135.76
D1, D4	0.390	0.396	9.31	6.34	31.44	69.23	136.80	142.31
D2, D5	0.405	0.422	9.86	7.57	35.58	91.43	135.83	140.18
D3, D6	0.424	0.458	10.55	9.93	-	-	134.73	136.52
E1, E4	0.393	0.410	9.72	7.17	21.79	56.95	140.04	147.48
E2, E5	0.403	0.422	10.07	8.27	24.34	70.74	138.88	144.11
E3, E6	0.416	0.438	10.50	10.07	-	-	137.49	138.66

increase with time. All these changes are in line with the usual description of reinforced concrete beam behaviour under creep effect [1,6].

## CONCLUSIONS

1. The method proposed in this paper considers separately the effects of creep and shrinkage in estimating long-term deflections of reinforced concrete beams. For estimating initial-plus-creep deflection the concept of effective moment of inertia has been extended by considering the properties of uncracked and cracked transformed sections under sustained loads. For calculating shrinkage deflections Branson's method has been used. A comparison of the computed deflections with the test results from different sources confirms the suitability of the proposed method.
2. The quasi-elastic approach using increased modular ratio may not give correct properties of cracked transformed section for members subjected to sustained load. A better estimate of the properties of cracked transformed section can be obtained by giving due consideration to compatibility and equilibrium conditions. The properties of cracked section thus determined can be used for predicting not only initial-plus-creep deflections but also stresses (and strains) in concrete and steel reinforcement.
3. A comparison of computed and measured long-term deflections in the paper indicates that the ACI Code method underestimates these deflections for members with  $L/h$  greater than 25. The proposed method gives a better estimate of long-term deflections for beams with  $L/h$  greater than 25.
4. The results of parametric study show that for any type/grade of concrete the material properties ( $E_c$ ,  $C_t$ ) have appreciable affect on flexural rigidities of reinforced concrete members under sustained loads. An improvement in the determination of these properties will, therefore, lead to a better estimation of long-term deformations.

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