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ON THE DIRECT PROBLEM AND SCATTERING DATA FOR A SINGULAR SYSTEM OF DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

By

A. A. DARWISH
Department of Mathematics, Faculty of Science
Qatar University, Qatar

عن المسألة الأمامية وبيانات التشتت لنظام من المعادلات التفاضلية الشاذة ذات معاملات غير متصلة .

عادل عبد الفتاح درويش

تم في هذا البحث دراسة المسألة الأمامية لنظام من المعادلات التفاضلية من نوع شتورم ليوفيل ذات دالة كثافة غير متصلة كما تم استنتاج بيانات التشتت للمسألة المدروسة .

Key Words: Direct problem, Scattering data, Parseval's equation, Discontinuous coefficients

ABSTRACT:

A system of Sturm-Liouiville differential equations of order n with a density matrix function is considered. The direct problem of the considered system is studied and hence the scattering data of the problem is obtained.

INTRODUCTION

We consider the system of Sturm-Liouivlle differential equations of order n

$$-y''+Q(x)y=\lambda\rho(x)y, \qquad (0\leq x,\infty) \tag{1}$$

and the boundary condition

$$y'(0)=0, \tag{2}$$

where Q(x) is a self adjoint matrix function of order n with real elements

The condition

$$\int_0^\infty x\|Q(x)\|dx < \infty \tag{3}$$

is assumed to hold throughout in this paper. Also, the matrix function $\rho(x)$ has the form

$$\rho(x)=\begin{cases}\alpha_n^2, & a_n\leq x<a_n\\ E_n, & a_n\leq x<\infty,\end{cases} \tag{4}$$

where

$\alpha_n>0, a_0=0,\ a_n\neq a_{n+1}, n=\overline{1,m-1}, a_m\neq 1$ and α_n are the diagonal elements of a matrix of order nxn such that they do not coincide with the identity matrix E_n.

Denote by W_n an $L_2(0,\infty;\rho(x))$ the set of the matrix functions $\rho(x)$ and the set of all vector functions

$f(x)=\{f_1(x),f_2(x),\dots f_n(x)\}$ with elements in $L_2(0,\infty)$ respectively.

In $L_2(0,\infty;\rho(x))$ we introduce the scalar product

$$(f,g)=\int_0^\infty\sum_{j=1}^n f_j(x)\overline{g}_j(x)dx$$

and consider that (1)-(2) arise in $L_2(0,\infty;\rho(x))$.

The problem (1)-(2) was investigated earlier in the scalar form in the papers [1.8] when $\rho(x)=E_n$ and for the case n=1 this problem has been discussed in the works [2,4,5,7]. So, this paper is aimed to extend those previous results.

It is well known [4] that the collection of quantities $\left\{S(k),-t_n^2,M_n,n=\overline{1,m}\right\}$ is called the scattering data of the system (1)-(2), where S(k) is the scattering matrix function and M_n are nonnegative matrices of order n whose ranks coincide with the multiplicity of the eigenvalues $-\tau_n^2$ of the problem (1)-(2). This article is aimed to study the direct problem and hence to obtain the scattering data of the problem (1)-(2).

NOTATIONS

Throughout this paper we use the following notations:

- E_n is the unit matrix in n-dimensional Euclidean space.
- $\tilde{F}$ denotes the transposed matrix of F.
- F* is the adjoint matrix of F.
- F' denotes the differentiation with respect to k.
- $\|Q(x)\|$ is the Euclidean norm of Q.

1. SOLUTION OF SYSTEM (1) AND ITS SCATTERING FUNCTION

We shall mainly use the basic solutions that have been in [8,9].

Every n vector solution $Y(x,\lambda)$ of (1) can be written in the form of a quadratic matrix of order n which satisfies the equation

$$-Y''+Q(x)Y=\lambda\rho(x)Y,\ 0\le x<\infty \qquad (5)$$

It is evident that the columns of any matrix solution of equation (5) are solutions of equation (1). Thus, we consider the matrix differential equation (5) with the boundary condition

$$Y'(0)=0 \qquad (6)$$

instead of (1)-(2).

Denote by

$$k=\lambda^{1/2}=\mu+i\tau \text{ and } 0\le \arg k<\pi$$

and $\sigma(x)=\int_x^\infty \|Q(t)\|dt\,;\ \sigma_1(x)=\int_x^\infty t\|Q(t)\|dt.$

Let us denote by $\varphi_n(x,k)$ the matrix solutions of the canonical equation (5) as $x\in[a_{n-1},a_n)$

These solutions satisfy the following conditions

$$\begin{cases}\varphi_n(a_{n-1},k)=E_n, & \psi_n(a_{n-1},k)=0\\ \varphi'_n(a_{n-1},k)=0, & \psi'_n(a_{n-1},k)=E_n.\end{cases} \qquad (7)$$

As already known [1,4], these solutions can be represented in the form

$$\varphi_n(x,k)=\cos k a_n(x-a_{n-1})+\int_{a_{n-1}}^{x} A_n(x,t)\cos k a_n(t-a_{n-1})dt \qquad (8)$$

$$\psi_n(x,k)=\frac{\sin k a_n(x-a_{n-1})}{k}\alpha_n^{-1}+\int_{a_{n-1}}^{x} B_n(x,t)\frac{\sin k a_n(t-a_{n-1})}{k}\alpha_n^{-1}dt;$$

where

$$A_n(x,x)=\frac{1}{2}\int_{a_{n-1}}^{2x-a_{n-1}} q(t)dt\ ,\ \frac{\partial}{\partial t}A_n(x,t)\Big|_{t=a_{n-1}}=0,$$

and

$$B_n(x,x)=\frac{1}{2}\int_{a_{n-1}}^{2x-a_{n-1}} q(t)dt\ ,\ B_n(2x-a_{n-1},a_{n-1})=0;$$

Lemma 1: If the condition (3) is satisfied, then for $x\ge a_n$ and $\tau\ge 0$ equation (1) has a solution F(x,k) that can be represented in the form

$$F(x,k)=\exp(ikx)E_n+\int_x^\infty K(x,t)\exp(ikt)dt, \qquad (9)$$

where the kernel K(x,t) satisfies the inequality

$$\|K(x,t)\|\le\frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp(\sigma_1(x))$$

and the condition $K(x,x)=\frac{1}{2}\int_x^\infty q(t)dt.$

Moreover, if q(x) is differentiable, then K(x,t)is twice differentiable and satisfies both the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2}-\frac{\partial^2 K(x,t)}{\partial t^2}=q(x)K(x,t)$$

and the condition $\lim\limits_{x+t\to\infty}\frac{\partial K(x,t)}{\partial x}=\lim\limits_{x+t\to\infty}\frac{\partial K(x,t)}{\partial t}=0.$

The solution F(x,k) is an analytic function of k in the upper half plane $\tau>0$ and is continuous on the real line. This solution has the following asymptotic behaviour

$F(x,k)=\exp(ikx)[E_n+o(1)],\ F'_x(x,k)=(ik)\exp(ixk)[E_n+o(1)]$ as $x\to\infty$ for all $\tau\ge 0,\ k\ne 0$.

Also,

$F(x,k)=\exp(ikx)[E_n+O(\frac{1}{k})],\ F'_x(x,k)=(ik)\exp(ixk)[E_n+O(\frac{1}{k})]$ as $|k|\to\infty$ and for all $\tau\ge 0$.

Next, if we continue the solution F(x;k) of the equation (1) to the interval $[a_{n-1},a_n)$ thus we find the following asymptotic form

$$F(x,k)=\begin{cases}\exp(ika_n)\left[\cos ka_n(x-a_{n-1})+\frac{1}{a_n}\sin ka_n(x-a_{n-1})\right]\left[E_n+O(\frac{1}{k})\right], & a_{n-1}\le x<a_n\\ \exp(ikx)\left[E_n+O(\frac{1}{k})\right] & ,a_n\le x<\infty\end{cases} \quad (10)$$

First we introduce the concept of the Wronskian of a pair of solutions of system (1). Denote by $W[\varphi_n,\psi_n]$ the Wronskian of two matrix solutions $\varphi_n(x,k)$ and $\psi_n(x,k)$ such that

$$W[\varphi_n,\psi_n]=\tilde{\varphi}_n(x,k)\psi'_n(x,k)-\tilde{\varphi}'_n(x,k)\psi_n(x,k).$$

The following property can be easily shown:

Lemma 2: The Wronskian of two matrix solutions of (1) does depend on x.

Next, since the matrix solutions F(x,k) and F(x,-k) are linearly independent as $\tau=0$, thus, we have

$$\varphi_n(x,k)=F(x,k)\,C_1(k)+F(x,-k)C_2(k),$$

where $C_1(k)$ and $C_2(k)$ are matrices of order n, which we have to find. For this purpose, we have

$$F(a_{n-1},k)\,C_1(k)+F(a_{n-1},-k)C_2(k)=E_n,$$

and

$$F'(a_{n-1},k)\,C_1(k)+F'(a_{n-1},-k)C_2(k)=0.$$

Multiplying the first equality from the left by $\tilde{F}'(a_{n-1},-k)$ and the second equality by $\tilde{F}(a_{n-1},-k)$ we obtain

$$C_1(k)=-\frac{1}{2ik}\tilde{F}'(a_{n-1},-k).\ \text{and}\ C_2(k)=\frac{1}{2ik}\tilde{F}(a_{n-1},k).$$

Thus

$$\varphi_n(x,k)=\frac{1}{2ik}\left[F(x,-k)\tilde{F}'(a_{n-1},k)-F(x,k)\tilde{F}'(a_{n-1},-k)\right]$$

Since

$\varphi'_n(a_{n-1},k)=0$, it yields

$$\tilde{F}'(a_{n-1},-k)\tilde{F}(a_{n-1},k)=\tilde{F}'(a_{n-1},k)\tilde{F}(a_{n-1},-k).$$

Let $\det F'(a_{n-1},k)=0$ as $\tau=0,\ k\neq 0$

Thus, we find a vector $\vec{v}\neq 0$ such that

$$F'(a_{n-1},k)\,\vec{v}\neq 0\ \text{ and }\ \overset{*}{v}\overset{*}{F'}(a_{n-1},k)=0$$

Evidently,

$$F^*(x,k)F(x,k)-F^{*\prime}(x,k)F(x,k)=2ik\,E_n.$$

Multiplying this equality from the left by $\vec{v}^{\,*}$ and from the right by $\vec{v}$, we have

$$\overset{*}{v}\left[\overset{*}{F}(x,k)F'(x,k)-\overset{*}{F'}(x,k)F(x,k)\right]\vec{v}=\vec{v}^{\,*}\,2ikE_n\,\vec{v}.$$

Setting $x=a_{n-1}$, we get

$\vec{v}^{\,*}\vec{v}=0$. Then, if $F'(a_{n-1},k)\vec{v}=0$ we find $\vec{v}=0$

which leads to a contradiction.

Thus for all $x\geq a_{n-1}$ and $\tau=0$, $k\neq 0$ the matrix function $F'(x,k)$ is non singular.

Hence

$$\varphi_n(x,k)=\frac{1}{2ik}[F(x,-k)-F(x,k)S(k)]\tilde{F}'(a_{n-1},k),$$

where (11)

$$S(k)=\tilde{F}'(a_{n-1},-k)[\tilde{F}'(a_{n-1},k)]^{-1}$$

is called the scattering matrix of equation (5) with the initial conditions (7).

Hence:

Theorem 1: The identity

$$\frac{2ik\varphi_n(x,k)}{\tilde{F}'(a_{n-1},k)}=F(x,-k)-S(k)F(x,k), \qquad (12)$$

where

$$S(k)=\tilde{F}(a_{n-1},-k)[\tilde{F}'(a_{n-1},-k]^{-1}. \qquad (13)$$

is valid for all real $k\neq 0$.

The scattering matrix S(k) satisfies the following properties:

(i) $S(k)\overset{*}{S}(k)=\overset{*}{S}(k)S(k)=E_n$,

(ii) $S(-k)=\overset{*}{S}(k)$.

Here, taking into account formulas (8), (9) (10) and (11) we can prove that:

Theorem 2: For large real k, $|k|\to\infty$ the following asymptotic form holds

$$S(k)=S_0(k)+O(\frac{1}{k}), \qquad (14)$$

where

$$\begin{aligned}S_o(k)=\exp\ &(-2ik\,a_n)[\sin\,k\,\alpha_n(a_n-a_{n-1})\\ &+i\alpha_n^{-1}\cos\,k\,\alpha_n(a_n-a_{n-1})][\sin\,k\,\alpha_n(a_n-a_{n-1}) \qquad (15)\\ &-i\alpha_n^{-1}\cos\,k\,\alpha_n(a_n-a_{n-1})]+O(\frac{1}{k})\end{aligned}$$

2. THE DISCRETE SPECTRUM AND PARSEVALS EQUATION

We consider the signal boundary value problem arising from the canonical equation (5) with the conditions (3), (4) and (6).

Theorem 3: The necessary and sufficient conditions that $\lambda \neq 0$ be an eigenvalue of the problem (5)-(6) are
$\lambda = k^2, \tau > 0$, $\det F'(a_{n-1}, k) = 0$.

They are countable in number and its limit points can be on the real axis.

This theorem can be proved via [1,5].

Theorem 4: All the singular points of the matrix $[F'(_{n-1},k)]^{-1}$ are all simple.

Proof: By differentiating the equation

$$-F''(x,k)+q(x)F(x,k)=k'\rho(x)F(x,k) \qquad (16)$$

with respect to k, and taking the Hermitian conjugates of the matrices, we have

$$-[F^*(x,k)]''+q(x)F^*(x,k) = 2\bar{k}\rho(x)F^*(x,k) + \bar{k}^2 F(x,k) \qquad (17)$$

Multiplying (16) from the left by $F^*(x,k)$ and (17) from the right by F(x,k) and subtracting, we obtain

$$F'^*(x,k)F''(x,k) - \left[F^*(x,k)\right]'' F(x,k) = -2k\,F^*(x,k)F(x,k).$$

Since the elements of F'(x,k) and F(x,k) lie in $L_2(0,\infty,\rho(x))$ thus it yields

$$F^*(x,k)F(x,k) - \left[F^*(x,k)\right]' F(x,k) = 2k\int_x^\infty F^*(t,k)\rho(t)F(t,k)dt \qquad (18)$$

Suppose that the point $k_0 = i\tau_0, \tau_0 > 0$ be a zero of $\det F(a_{n-1}, k_0)$. Then there exists a non zero vector $\vec{v}$ such that $F'(a_{n-1},k)\vec{v} = 0$. (19)

Multiplying (18) on the right by $\vec{v}^*$ and letting x goes to a_{n-1}, we get

$$\vec{v}^* F^*(a_{n-1},k)F(a_{n-1},k)\vec{v} - \vec{v}^*\left[F^*(a_{n-1},k)\right]' F(a_{n-1},k)\vec{v} = 2k\int_{a_{n-1}}^\infty \vec{v}^* F^*(t,k)\rho^*(t)F(t,k)\vec{v}\,dt. \qquad (20)$$

From the behaviour of F(x,k); F*(x,k) and using the meanvalue theorem, we have

$$\vec{v}^*\left[F'^*(a_{n-1},k)\right]' \mathrm{I}F'(a_{n-1},k)\vec{v} = -2k\int_{a_{n-1}}^\infty F^*(t,k)\vec{v}^*\mathrm{I}\rho^*(t)F(t,k)\vec{v}\,dt \neq 0 \qquad (21)$$

Suppose that $\vec{v}$ not only satisfies (19) but also the relation

$$F(a_{n-1},k_0)\vec{w} + F(a_{n-1},k_0)\vec{v} = 0. \qquad (22)$$

Here, along the Hermitian conjugate and multiplying on the right by $\mathrm{I\!F}'(a_{n-1},k_0)\vec{v}$, we have

$$\vec{w}^*[F'^*(a_{n-1},k)]\mathrm{I\!F}(a_{n-1},k)\vec{v} + \vec{v}^*\left[F''^*(a_{n-1},k)\right]' \mathrm{I\!F}(a_{n-1},k)\vec{v} = 0$$
(23)

In view of [1] and the Wronskian of F(x,k);F*(x,k) we have

$$\left[F'^*(a_{n-1},k)\right]\mathrm{I\!F}(a_{n-1},k) = \left[F''^*(a_{n-1},k)\right]'\mathrm{I\!F}'(a_{n-1},k) = 0$$

Then by (19), we have

$$\vec{w}^*\left[F_1^*(a_{n-1},k)\right]' \mathrm{I\!F}'(a_{n-1},k)\vec{v} = 0$$

Therefore (23) takes the form

$$\vec{v}^*\left[F^*(a_{n-1},k)\right]' \mathrm{I\!F}'(a_{n-1},k)\vec{v} = 0$$

which contradicts (21). Hence it follows from equations (19) and (22) that $\vec{v} = 0$ and this completes the proof of the theorem.

Lemma 3: When $\tau > 0$, the matrix function

$$R_k(x,t) = \begin{cases} F(x,k)F^{-1}(a_{n-1},k)\mathrm{I}\tilde{\varphi}_n(t,k), & t \le x \\ \varphi_n(x,k)\mathrm{I\!F}^{-1}(a_{n-1},k)\tilde{F}(t,k), & t \ge x \end{cases} \qquad (24)$$

is the kernel resolvent of the problem (5)-(6).

Proof: We can find Greens function of the problem (5)-(6) by using the method of variation of parameters and thus the resolvent is in the form (24).

Lemma 4: Suppose that the vestor function f(t) is finite and has a continuous derivative in $L_2(0,\infty;\rho(x))$ and satisfies the boundary condition (6). Then

$$\int_0^\infty R_k(x,t)\rho(t)f(t)dt = \frac{-f(x)}{k^2} + \frac{1}{k^2}\int_0^\infty R_k(x,t)g(t)dt,$$
where $g(t) = -f''(t) + Q(t)f(t)$.

Moreover, if $\tau > 0$ and $|k| \to \infty$, then

$$\int_0^\infty R_k(x,t)\rho(t)f(t)dt = \frac{-f(x)}{k^2} + O\left(\frac{1}{k^2}\right) \qquad (25)$$

Proof: Using formula (24) we get

$$\int_{a_{n-1}}^{\infty} R_k(x,t)\rho(t)f(t)dt = F^{-1}(a_{n-1},k)\Big\{F(x,k)\left[\int_{a_{n-1}}^{x}[-\frac{1}{k^2}\tilde{\varphi}''(t,k)+\frac{1}{k^2}Q(t)\tilde{\varphi}(t,k)]f(t)dt\right] + \varphi(x,k)\int_x^{\infty}\left\{-\frac{1}{k^2}\tilde{F}''(x,k)+\frac{1}{k^2}Q(t)\tilde{F}(t,k)\right\}f(t)dt\Big\}.$$

Integrating this identity by parts, and taking into account Riemman-Lebsegue theorem, it yields that

$\int_{a_{n-1}}^{\infty} R_k(x,t)g(t)dt = o(1)$. Hence (25) follows directly.

The following lemma is well-known:

Lemma 5: $\bar{R}_{k^2} = R_{\bar{k}^2}$.

With the help of these lemmas we can prove the following theorem:

Theorem 5: The following Parsevals equation is valid:

$$\frac{1}{2\pi}\int_{a_{n-1}}^{\infty} u(x,k)u^*(t,k)dk + \sum_{n=1}^{m} u(x.i\tau_n)u^*(t,i\tau_n) = \delta(x-t)\rho^{-1}(x)E_n, \tag{26}$$

where

$$u(x,k) = F(x,-k) - S(k)F(x,k)$$

and

$$u(x,i\tau_n) = M_n F(x,i\tau_n), N = \overline{1,m}.$$

such that $M_1, M_2, \ldots, M_n$ are non negative matrices.

Proof: Suppose that f(x) satisfies the conditions of lemma 4. Thus (25) holds. Integrating both sides of (25) with respect to k over the semi-circle $\{k: |k| = r, r > 0\}$ in the upper-half plane k>0. It is evident that the integral $\int_{a_{n-1}}^{\infty} R_k(x,t)\rho(t)f(t)dt$ is an analytical function except the zeros of det $F'(a_{n-1},k)$.

Hence, upon using [3], we find that

$$f(x) = \frac{1}{\pi i}\int_{a_{n-1}}^{\infty} k \int_{a_{n-1}}^{\infty}[R_{k+i0}(x,k) - R_{k-i0}(x,k)]\rho(t)f(t)dtdk + \sum_{n=1}^{m}\operatorname{Res}\left[2k\int_{a_{n-1}}^{\infty} R_k(x,t)\rho(t)f(t)dt\right]_{k=i\tau_n} \tag{27}$$

Next, let us compute the first quantity in the right-hand side of equation (27). By lemma 5 it follows that $R_{k-i0} = \bar{R}_{k+i0}$. Then, we can compute R_{k+i0} and thus R_{k-i0} at once. Therefore, using (24) we obtain

$$R_{k+i0}(x,t) - R_{k-i0}(x,t) = \frac{\varphi_n(x,k)2ik\varphi_n^*(t,-k)}{W(k)W^*(-k)},$$

where $W(k) = \det F(a_{n-1},k)$

Thus, taking into account (12) we have

$$R_{k+i0}(x,t) - R_{k-i0}(x,t) = \frac{u(x,k)u^*(t,-k)}{-2ik},$$

where $u(x,k) = \dfrac{2ik\varphi_n(x,k)}{W^*(-k)} = F(x,-k) - S(k)F(x,k).$

Hence

$$\frac{1}{\pi i}\int_{a_{n-1}}^{\infty} k \int_{a_{n-1}}^{\infty}[R_{k+i0}(x,k) - R_{k-i0}(x,k)]\rho(t)f(t)dtdk = \frac{1}{2\pi}\int_{a_{n-1}}^{\infty} dk \int_{a_{n-1}}^{\infty} u(x,k)u^*(t.-k)\rho(t)f(t)dt. \tag{28}$$

Now we compute the second quantity on the right-hand side side of (27). From (24) we have

$$\operatorname*{Res}_{k=i\tau_n}\left[k\int_{a_{n-1}}^{\infty} R_k(x,t)\rho(t)f(t)dt\right] = 2i\tau_n\left[F(x,i\tau_n)P_n I\int_{a_{n-1}}^{\infty}\tilde{\varphi}(t,i\tau_n)\rho(t)f(t)dt + \varphi(x,i\tau_n)\int_x^{\infty}\tilde{I}\tilde{P}_n\tilde{F}(t,i\tau_n)\rho(t)f(t)dt\right] \tag{29}$$

where P_n is the residue of $F^{-1}(a_{n-1},k)$ at $k = i\tau_n$.

Since $F'(a_{n-1},k)$ is an analytical function for $\tau > 0$ and $F^{-1}(a_{n-1},k)$ has a simple poles at $i\tau_n$, then the following relation is valid

$$F(a_{n-1},k) = F(a_{n-1}, i\tau_n) + F(a_{n-1}, i\tau_n)(k - i\tau_n) + \ldots$$

and

$$F^{-1}(a_{n-1},k) = \frac{P_n}{(k-i\tau_n)} + P_n^{(0)} + \ldots \tag{30}$$

From (30) and the relation

$$F(a_{n-1},k)F^{-1}(a_{n-1},k) = F^{-1}(a_{n-1},k)F(a_{n-1},k) = E_n$$

it yields that

$$E_n = \frac{F'(a_{n-1},k)P_n}{k-i\tau_n} + F(a_{n-1},k)P_n + F(a_{n-1},k)P_n^{(0)} + \ldots$$

Hence

$$F(a_{n-1},k)P_n = P_n F(a_{n-1},k) = 0 \text{ and}$$

$$F(a_{n-1},k)P_n + F(a_{n-1},k)P_n^{(0)} = P_n F(a_{n-1},k) + P_n^{(0)} F(a_{n-1},k) = E_n. \quad (31)$$

Let H_n be the operator of orthogonal projection onto P_n. It is easy to show that [3] the ranks of H_n and P_n are the same and that $H_nP_k=P_n$. From (18), we have

$$F^*(x,i\tau_n)F(x,i\tau_n) - \left[F^*(x,i\tau_n)\right]' F(x,i\tau_n) = -2k\int_x^\infty F^*(t,i\tau_n)\rho(t)F(t,i\tau_n)dt$$

Thus, we have for $x = a_{n-1}$ that

$$F^*(a_{n-1},i\tau_n) - \left[F^*(a_{n-1},i\tau_n)\right]' F(a_{n-1},i\tau_n) = -2i\tau_n\int_{a_{n-1}}^\infty F^*(t,i\tau_n)\rho(t)F(t,i\tau_n)\,dt = -2i\tau_n A_n. \quad (32)$$

Clearly, A_n is a positive definite matrix. Multiplying equation (32) on the left by P_n^* and on the right by H_n and taking into account that $F(a_{n-1},i\tau_n)H_n = 0$ to have

$$P_n^*F^{*'}(a_{n-1},i\tau_n)IF(a_{n-1},i\tau_n)H_n = 2i\tau_n P_n^* A_n H_n. \quad (33)$$

Since the matrix function $F(x,i\tau_n)$ H_n and $\varphi(x,i\tau_n)$ are the solutions of the same equation, thus we have

$$F(x,i\tau_n)H_n = \varphi(x,i\tau_n)F(a_{n-1},i\tau_n)H_n. \quad (34)$$

It follows from (31) and (33) that

$$\begin{aligned} P_n^*F^{*'}(a_{n-1},i\tau_n)\,IF(a_{n-1},i\tau_n)H_n &= \left[E_n - P_n^{*(0)}F^{*'}(a_{n-1},i\tau_n)\right]F(a_{n-1},i\tau_n)H_n \\ &= F(a_{n-1},i\tau_n)H_n - P_n^{*(0)}F^{*'}(a_{n-1},i\tau_n)F(a_{n-1},i\tau_n)H_n \\ &= F(a_{n-1},i\tau_n)H_n. \end{aligned}$$

Therefore, equation (33) takes the form

$$F(a_{n-1},i\tau_n)H_n = 2i\tau_n P_n^* A_n H_n.$$

Now, from (34) we have

$$\begin{aligned} F(x,i\tau_n)H_n &= \varphi(x,i\tau_n)F(a_{n-1},i\tau_n)H_n \\ &= 2i\tau_n\,\varphi(x,i\tau_n)P_n^*A_nH_n \\ &= 2i\tau_n\,\varphi(x,i\tau_n)IP_n^*\left[H_nA_nH_n + E_n - H_n\right] \\ &= 2i\tau_n\,\varphi(x,i\tau_n)IP_n^*D_n, \end{aligned} \quad (35)$$

where

$$D_n = H_nA_nH_n + E_n - H_n.$$

Clearly, $D_nH_n = H_nD_n$ and D_n are positive definite matrices. Thus, there exists a matrix $M_n^2 = H_nD_n^{-1} = H_nD_n^{-1}$ which is nonnegative and its rank is the same rank as H_n i.e. the multiplicity of the zeros of $\det F'(a_{n-1},k)$.

Multiplying both sides of (35) on the left by D_n^{-1} to have $F(x,i\tau_n)M_n^2 = 2i\tau_n\varphi(x,i\tau_n)P_n^*$. We multiply both sides of this formula on the right by $\tilde{F}(t,i\tau_n)$ to give

$$2i\tau_n\,\varphi(x,i\tau_n)I\,\tilde{P}_n\,\tilde{F}(t,i\tau_n) = F(x,i\tau_n)M_n^2F^*(t,i\tau_n).$$

Thus it follows from (29) that

$$\operatorname*{Res}_{k=i\tau_n}\left[2k\int_{a_{n-1}}^\infty R_k(x,t)\rho(t)f(t)dt\right] = F(x,i\tau_n)M_n^2\int_{a_{n-1}}^\infty F^*(t,i\tau_n)\rho(t)f(t)dt. \quad (36)$$

Hence, by (28) and (36) we conclude that

$$f(x) = \frac{1}{2\pi}\int_{a_{n-1}}^\infty dk\int_{a_{n-1}}^\infty u(x,k)u^*(t,-k)\rho(t)f(t)dt + F(x,i\tau_n)M_n^2\int_{a_{n-1}}^\infty F^*(t,i\tau_n)\rho(t)f(t)dt.$$

Then multiplying both sides of the last formula by $f(x)\rho(x)$ and integrating from a_{n-1} to ∞ to obtain Parsevals equation (26) and the theorem is proved.

The collection of quantities $\{S(k), i\tau_n, M_n, n=\overline{1,m}\}$ is called the scattering data of the problem (1)-(2).

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