ELLiptic Surfaces over a genus 1 curve with exactly the pair (I, IV) of singular fibers

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ABSTRACT
In this paper we classify all minimal elliptic surfaces \( \pi : E \to C \) over a genus 1 curve \( C \), with a section and exactly the pair (I, IV) of singular fibers.
Elliptic surfaces over a genus 1 curve with exactly the pair (I\textsubscript{n}, IV) of singular fibers

1. INTRODUCTION

The serious study of elliptic surfaces was started by Kodaira (see [6]). He listed all possible types of singular fibers, gave their invariants and analyzed an important invariant called the J-map. Beauville (see [2]) has studied elliptic surfaces over \( P^1 \), in fact he classified the semi-stable cases (i.e., the cases in which all singular fibers are of type I\textsubscript{n}). He proved that there are 6 semi-stable cases with the minimal number (=4) of singular fibers. In 1985 Schmüdgen-Hirzebruch wrote Weierstrass equations for all elliptic fibrations with at most three singular fibers (see [5]). In 1986, R. Miranda and U. Persson have listed all extremal rational elliptic surfaces (see [8]). In 1988, Stiller (see [12]) has classified all elliptic surfaces over a genus 1 curve with exactly one singular fiber necessarily of type \( I_n \). In 1989, R. Miranda and U. Persson has classified all possible configurations of singular fibers on elliptic \( K3 \) surfaces (see [9]). In 1990, U. Persson has classified all possible configurations of singular fibers on rational elliptic surfaces (see [10]). Also, R. Miranda has analyzed the same problem by giving a more combinatorial and less geometric analysis (see [7]).

In this paper we let \( C \) denote a genus one curve and study minimal elliptic surfaces \( \pi : E \to C \) with a section and exactly the pair \((I_n, IV)\) of singular fibers. In this paper the notation \([J'(x)] = (n_1, \ldots, n_t)\) will be used to indicate that \( J(x) \) consists of \( t \) points say \( \{x_1, \ldots, x_t\} \) such that the multiplicity of \( J \) at \( x \) (\( m_x(J) \)) is \( n \) for all \( i \in \{1, \ldots, t\} \).

The plan of the paper goes as follows: First, we review some important ideas which will be used in the text of this paper, then, we give the possible ramification of the J-map and prove its existence, and then we construct the J-map and the required surfaces.

Remark 1.1: To build a minimal elliptic surface with a section and a given number of singular fibers, it is enough to build the J-map associated to this surface (for more details see [7] Section 3).

The following theorem which we call the monodromy theorem is just a restatement of Corollary 3.5 of [7].

**Theorem 1.2:** Let \( C \) be a curve and let \( \pi \) be a finite subset of \( P^1 \) say \(|\pi| = n \), then there is a one-to-one correspondence between

\[
\begin{align*}
\{ J : C \to P^1 \text{ such that } deg(J) = d \text{ and } J \text{ is branched at most over } \pi \} & \quad \text{and} & \{ \text{permutations } \sigma_1, \ldots, \sigma_t \in S_t \} \\
\{ \text{such that } \sigma_1 \ldots \sigma_t = id, \text{ and the } \sigma_i \text{'s generate a transitive subgroup of } S_t \} & \quad \text{such that } \sigma_1 \ldots \sigma_t = id, \text{ and the } \sigma_i \text{'s generate a transitive subgroup of } S_t
\end{align*}
\]

Where the first set is taken up to isomorphism (fixing \( P^1 \) and the second set is taken up to conjugation).

2. MAIN RESULTS

There are the following types of singular fibers (see [6]):

\( I_6, I_4, I_3, II, III, IV, IV\ast, III\ast, II\ast(\geq 1) \). If \( e(F) \) denotes the Euler number of the fiber \( F \), then the Euler numbers of the above list are: \( 6, n, n + 6, 2, 3, 4, 8, 9, 10 \) respectively.

**Lemma 2.1:** Let \( C \) be a genus 1 curve, suppose \( \pi : E \to C \) is a minimal elliptic surface with a section and exactly two singular fibers. If the degree of the line bundle \( L \) is 1 (i.e., \( L \) is the conormal bundle to the section), then there are exactly five possible pairs \((F_1, F_2)\) of singular fiber types such that the sum of the Euler numbers is 12.

**Proof.** Immediate from the fact that if \((F_1, F_2)\) is a possible pair of singular fibers, then \( e(F_1) + e(F_2) = 12 \).

Notice that the pair \((I_2, IV)\) is one of these possible pairs, and this case cannot occur if the genus of the base curve is 0 (i.e., \( C \equiv P^1 \)).

**Lemma 2.2:** Let \( C \) be a genus 1 curve, suppose \( \pi : E \to C \) is a minimal elliptic surface with a section and exactly the pair \((I_n, IV)\) of singular fibers. If \( J : C \to P^1 \) is the J-map associated to this fibrations, then degree \((J) = 12 \) and \( J \) is ramified as follows: \([J'(0)] = (3, 3, 2), [J'(1)] = (2, 2, 2, 2) \) and \([J'((\infty))] = (8)\).

**Proof.** \( \deg(J) = \sum_{i=1}^{n} m_i(J) - \#(\text{fibers} + \#(\text{fibers}) = 8 \) (see [7], p. 194).

Let \( R = \{ \text{ramification points of } J \} \) and \( m_i(J) \) denote the multiplicity of \( J \) at the point \( x \). By Hurwitz's formula for the genus of a curve we have: \( 16 = \sum_{i=1}^{\#J \cap P^1} (m_i(J) - 1) \). Now over \( 0 \) we have the \( IV \) fiber, hence the minimum ramification of \( J \) over 0 is obtained if \([J'(0)] = (3, 3, 2) \) (see [7]), over 1 we have smooth fibers, hence the minimum ramification of \( J \) over 1 is obtained if \([J'(1)] = (2, 2, 2) \) and over \( \infty \) we have \( I_2 \)-fiber, hence \([J'((\infty))] = (8)\).

Thus

\[
\sum_{x \in J \cap (0,1,\infty)} (m_i(J) - 1) = 5 + 4 + 7 = 16,
\]

hence \( R = \{0, 1, \infty\} \) and there is not other ramification of \( J \).

**Theorem 2.3:** Under the hypothesis of Lemma 2.2 above the degree 8 map \( J : C \to P^1 \) (ramified as in Lemma 2.2), exists and is unique. Moreover, the genus 1 curve \( C \) is unique.

**Proof.** To prove this theorem is enough to find three permutations \( \sigma_0, \sigma_1, \) and \( \sigma_2 \) in \( S_t \) representing the monodromy of
J around 0, 1 and \( \infty \) respectively, such that: \( \sigma \sigma_1 = \sigma_1 \sigma, \sigma_2, \) and \( \sigma_3 \) are unique up to conjugation, the triple \((\sigma_0, \sigma_1, \sigma_2)\) generates a transitive subgroup of \( S_3 \) and such that the cycle structure of \( \sigma_0 \) is \((3^2, 2)\), that of \( \sigma_1 \) is \((2^2)\) and that of \( \sigma_2 \) is \((8)\).

To this end let \( \sigma_0 = (1 2) (3 4 5) (6 7 8) \) and \( \sigma_1 = (a b) (c d) (e f) (g h) \). Since 1 has to appear in one of the 2-cycles of s1 we may assume \( a = 1 \), hence \( b \neq 2 \) (otherwise we would have a fixed element in the product \( \sigma_0 \sigma \) which is not allowed); therefore we assume that \( b = 3 \). Now 2 has to appear in \( \sigma_1 \), so assume \( c = 2 \), and hence \( d \neq 1, 2, 3 \), hence \( d = 4 \) or 5 or we may assume \( d = 6 \), but clearly if \( d = 4 \) or 5, then this forces \((7 8)\) to be in \( \sigma_1 \) which is not valid, hence \( d \neq 1, 2, 3, 4, 5 \), so assume \( d = 6 \). Now it is easy to check that \((e f) = (4 7) \) or \((4 8)\) and if \((e f) = (4 8)\), then we get the cycle \((5, 8)\) in \( \sigma_0 \sigma_1 \) which is not allowed, hence \((e f) \neq (4 8)\); therefore, we get:

\[
\sigma_0 = (1 3) (2 6) (4 7) (5 8) \\
\sigma_0 \sigma_1^{-1} = (1 4 8 3 2 7 5 6)
\]

and clearly the permutations \( \sigma_0, \sigma_1, \) and \( \sigma_2 \) satisfy all the conditions stated in the beginning of the proof; thus \( J : C \rightarrow P \) (ramified as in Lemma 2.2) exists and is unique and the curve \( C \) is unique.

**Corollary 2.4:** If \( C \) is the unique genus 1 curve of Theorem 2.3 above, then there is a unique (up to analytic isomorphism) minimal elliptic surface \( \pi : E \rightarrow C \) with a section and exactly the pair \((I_4, IV)\) of singular fibers.

**Proof.** This is clear since the J-map exists and is unique, and this guarantees the existence and uniqueness of the desired surface.

3. THE J-MAP AND THE SURFACE

Next we construct the J-map \( J : C \rightarrow P \) associated to a minimal elliptic surface \( \pi : E \rightarrow C \) with a section and exactly the pair \((I_4, IV)\) of singular fibers, where \( J \) and \( C \) are the unique J-map and the unique curve of Theorem 2.3 above.

The plan here is to realize this J-map as a composition of two maps: a degree 4 map \( f : C \rightarrow P \), and a degree 4 map \( J_1 : P \rightarrow P \) (i.e., \( J = J_1 \circ f \)), now we proceed with this construction.

**Remark 3.1:** If \( C \) is a genus 1 curve, then clearly a degree 2 map \( f : C \rightarrow P \) exists, and by Hurwitz’s formula for the genus of a curve \( f \) must be branched over exactly 4 points of \( P \), in fact by a suitable change of coordinates in \( P \) we may assume that these four brach points to be any four points of \( P \).

Moreover, we may assume that the curve \( C \) is given by \( y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \), and the map \( f : C \rightarrow P \) is given by \( f \) (\( y, x \) = \( x \)), hence \( a_1, a_2, a_3 \) and \( \infty \) are the four ramification points of \( f \).

In the next remark we give the degree 4 map \( J_1 : P \rightarrow P \), in fact we have

**Remark 3.2:** Let \( P : S \rightarrow P \) be the rational elliptic surface which has the following singular fibers: the pair \((III, III)\) over 1, a fiber of type \( II \) over 0, and a fiber of type \( IV \) over \( \infty \). This surface has a geometric realization \( M(1, 1, 1, 0) \) (see [10], page 10), and this surface is constructed on page 36 of [10]. If \( J_1 : P \rightarrow P \) is the J-map associated to this surface, then deg \( J_1 = 4 \) and \( J : P \rightarrow P \) is ramified as follows (see [7], page 207):

\[
\|J_1\|_{x=\infty} = (4), \|J_1\|_{x=1} = (1, 1, 2) \text{and} \|J_1\|_{x=0} = (1, 3).
\]

Moreover we may assume that \( J_1 : P \rightarrow P \) is given by \( J_1(x) = 4x^3 - x^4 \), and to clarify this moreover statement, let \( x = 0 \) be the point multiplicity 3 over 0, and \( x = \infty \) be the point of multiplicity 4 over \( \infty \), hence \( J_1 : P \rightarrow P \) must be of the form \( J_1(x) = c_4x^4 - x^5 \), where \( a_4^3 \) and \( c_4 \) are constants. Now \( J_1(x) \) must have a critical point over 1 (since \( [J_1(1)] = 1 \)), and clearly \( J_1(x) = 0 \) if and only if \( x = 3a_4 \), thus \( J_1(3a_4) = 0 \), and hence \( 27a_4c_4 + 256 = 0 \). Clearly \( a_4 = \frac{3}{4} \) and \( c_4 = -3 \) is a solution of this equation; therefore, \( J_1(x) = 4x^3 - x^4 \).

**Lemma 3.3:** The degree 4 map \( J_1 : P \rightarrow P \) ramified as in (3.2.1) above is unique.

**Proof:** To prove this, it is enough to find a set of three permutations \( \sigma_0, \sigma_1, \) and \( \sigma_2 \) representing the monodromy of \( J \) around 0, 1, and \( \infty \) respectively such that \( \sigma_0 \sigma_1 = \sigma_1 \sigma_2 \), the triple \((\sigma_0, \sigma_1, \sigma_2)\) generates a transitive subgroup of \( S_4 \), \( \sigma_1, \sigma_2 \), and \( \sigma_3 \) are unique up to conjugation, and such that the cycle structure of \( \sigma_0 \) is \((3)\), that of \( \sigma_1 \) is \((2)\) and that of \( \sigma_2 \) is \((4)\).

To this end assume that \( \sigma_0 = (2 3 4) \) and \( \sigma_1 = (a b) \). Notice that if \( (a b) \) consists of two elements of \( \sigma_0 \), then the product \( \sigma_0 \sigma_1 \) must have a fixed element which is not allowed; hence we may assume \( a = 1 \) and \( b = 2 \); therefore, we get \( \sigma_0 \sigma_1 = \sigma_2 = (1 3 4 2) \). Moreover, it is clear that these permutations satisfy all the conditions stated in the beginning of the proof, hence \( J_1 : P \rightarrow P \) is unique.

**Lemma 3.4:** Let \( C \) be a genus 1 curve, let \( f : C \rightarrow P \) be a degree 2 map, let \( J_1 : P \rightarrow P \) be the degree 4 map defined in Remark 3.2 above. If \( J : C \rightarrow P \) is defined by \( J = J_1 \circ f \), then
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deg(J) = 8, and by a suitable change of coordinates in $P^1$ (= range of $f$) $J$ is ramified as in Lemma 2.2, and hence is the unique J-map of Theorem 2.3.

**Proof:** $\text{Deg}(J) = \text{deg}(J)$, $\text{deg}(f) = 9$. Let $S_0$ be the point whose $J_1$-value is $\infty$, let $S_1$ and $t_1$ be the points whose $J_1$-value is 1, and let $S_0$ be the point whose $J_1$-value is 0, where the second subscript is used to indicate the multiplicity of $J_1$ and these points. Now change coordinates in $P^1$ so that $S_0, S_1, t_1$, and $S_2$ are the four branch points of $f$, hence it is easy to see that $J = J_1 \circ f$ is ramified as required, and hence it is the unique J-map of Theorem 2.3.

**Theorem 3.5:** Given the unique genus 1 curve $C$ of Theorem 2.3, and if $J: C \to P^1$ is the unique J-map defined in Lemma 3.4 above, then this data can be used to build the unique minimal elliptic surface (see Corollary 2.4). $\pi: E \to C$ with a section and exactly the pair $(I, IV)$ of singular fibers.

**Proof:** Let $J = J_1 \circ f$ (see Lemma 3.4), then it is clear that $f$ is just a base change of order 2, let $J: C \to P^1$ be the pull-back of the surface $P: S \to P^1$ (see Remark 3.2.) via $f$, then $\pi: E \to C$ is a minimal elliptic surface with a section and the following singular fibers: two $I_0$ fibers over 1, a fiber of type $IV$ over 0, and a fiber of type $I_0$ over $\infty$ (see [8], Table 7.1.). Now using the process of deflating two *'s, we deflate the two *'s from the two $I_0$-fibers (see [7], Section 3), so that they becomes smooth fibers, and notice that the rest of the fibers remains unchanged, and hence the resulting surface $\pi: E \to C$ is minimal elliptic surface with a section and exactly the pair $(I_0, IV)$ of singular fibers; thus $\pi: E \to C$ must be the required (up to analytic isomorphism) surface.

We end this paper with the following remark:

**Remark 3.6:** another way to get our surface is to consider the rational elliptic surface $\alpha: S \to P^1$, whose Weierstrass equation is given by: $y^2 = x^3 + 3t(t-1)x^2 + 2t(t-1)^3$. This surface has $J$-map given by $J(t) = t$, and it has exactly three singular fibers (see [7], page 203): a fiber of type $II$ over $t$ = 0, a fiber of type $III_0$ over $t$ = 1, and a fiber of type $I_0$ over $t$ = $\infty$.

Let $\pi: E \to C$ be the pull-back of the rational elliptic surface $a: S \to P^1$ via $J = J_1 \circ f$, then use Table 7.1 of [8] to get the required surface.

**REFERENCES**


