CONDITIONALLY EXPONENTIAL CONVEX FUNCTIONS ON LOCALLY COMPACT GROUPS

By

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الدوال الأساسية المشروطة على الزمرات محلة التضاغط

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The main objectives of this study are:

1) The construction of a compact base for the convex cone of all conditionally exponential convex functions.

2) The determination of the extreme parts of this cone.

INTRODUCTION

Conditionally exponential convex functions have been introduced and studied in references [2,8]. Berg [2] named it "negative-definite". The set of these functions, denoted \( E_0 (G) \), is a convex cone, hence is amenable to an analysis by Choquet theory. For the real line, such a study was done by Johansen[7]. We follow an idea of Johansen to construct a compact base for \( E_0 (G) \). This leaves us with the task of finding the extreme points of the base. To do so, we first study a weight, called the Levy weight, in its abstract form.

Now, let \( C^* (G) \) be the enveloping C*-algebra of \( L^1 (G) \) equipped with # involution defined by: \( f# (x) = \Delta (x^{-1}) f^* (x) \), where \( \Delta \) is a modular function and \( f^* (x) = f (x^{-1}) \). The dual Banach space of \( C^* (G) \) is \( B (G) \) [4]. The set of positive linear functionals in \( B (G) \) is \( P (G) \) and it is identified with the set of exponentially convex functions on \( G \); i.e., the set of functions satisfying

\[
\sum_{i,j=1}^{n} \psi (g_i, g_j) c_i c_j \geq 0,
\]

where \( g_1 , ..., g_n \in G \) and \( c_1 , ..., c_n \in \mathbb{R} \). The set of elements form \( P_1 (G) \) with norm equals 1 is \( P_1 (G) \); this is a convex set whose set of extreme points is denoted by \( \text{ext} P_1 (G) \)[7]. Also, we write \( E_0 (G) \) for the set of all conditionally exponential convex functions defined on \( G \) and vanishing at the group identity; i.e., functions satisfying

\[
\sum_{i,j=1}^{n} \psi (g_i, g_j) + \psi (g_j, g_i) c_i c_j \geq 0,
\]

where \( g_1 , ..., g_n \in G \) and \( c_1 , ..., c_n \in \mathbb{R} \).

Now, if \( a \in C^* (G) \) and \( p \in P (G) \), define the translation of \( a \) by \( p \) to be the unique element \( T_p a \) of \( C^* (G) \) such that for all \( b \in B (G) \), \( (b, T_p a) = (b, T_q a) \). The translation operator \( T_p \) is a completely positive linear map on \( C^* (G) \) and is norm decreasing if \( p \in P_1 (G) \). If \( p, q \in P (G) \) and \( \lambda \geq 0 \), then \( T_{pq} = T_p T_q \) and \( T_p + \lambda \eta q \) \( = T_p + \lambda T_q \), so that \( T(.) \) is a homomorphism from \( P (G) \) into the set of completely positive linear maps on \( C^* (G) \)[3].

Given the notion of translation, we can define differentiation as a limit of difference quotients. In this way, we obtain a tangent space at each point of \( P_1 (G) \).

A semitangent vector to \( P_1 (G) \) at the identity is any continuous real valued function \( \psi \) on \( G \) satisfying

\[
\psi = \lim_{k \to \infty} \lambda_k \psi (1-p_k),
\]

where \( \{\lambda_k\}_{k>1} \) is a diverging increasing sequence of
nonnegative numbers and \( \{p_k\}_{k>1} \) is a uniformly convergent sequence belongs to \( P_1(G) \). It can be seen that the collection of such semitangent vectors is identified with the set \( \mathcal{E}_0(G) \), then \( \psi \) is called a tangent vector to \( P_1(G) \) at the identity \( \{1\} \).

**LEVY WEIGHTS**

Let \( \psi \in \mathcal{E}_0(G) \) and let \( \delta : C^*(G) \rightarrow C^*(G) \) be a linear functional such that \( \delta(a \alpha^#) \geq \delta(a \alpha^#) a + a \alpha^#(\delta a) \) if \( a \alpha^# \delta a \) belong to \( D(\delta) \), the domain of \( \delta \). Define a linear functional on \( C^*(G) \), denoted by \( \psi \), by

\[
\psi(a) = (1, \delta \psi a), a \in D(\delta).
\]

Clearly, \( \psi \) is densely defined and \( \psi \mid (\ker 1)^+ \geq \alpha; i.e., \( \psi \) is a weight for the elements of \( \mathcal{E}_0(G) \) and it is called a Levy weight.

Let \( W^*(G) \) be the double dual of \( C^*(G) \). We may consider \( G \) and \( C^*(G) \) to be contained in \( W^*(G) \); when it is necessary to emphasize that a measure \( \mu \), say, belongs to \( W^*(G) \), we will write \( \omega(\mu) \), where \( \omega \) is the universal representation \([6]\).

**Lemma 2.1**

Let \( \mathcal{M}_c \) be the set of compactly supported Borel measures on \( G \) of total mass zero. For \( \mu \in \mathcal{M}_c(G) \) and \( a \in C^*(G) \) we have

\[
\psi(\omega(\mu^#) \omega(\mu)) = (-\psi \mu, a), \text{ where } \psi \mu = \mu \ast \psi \ast \mu^+.
\]

**Proof:**

First we suppose that \( a \) takes the form \( \omega(\mu) \), \( f \) belongs to the set \( C_c(G) \) of compactly supported continuous functions on \( G \), then

\[
\psi(\omega(\mu^#) \ast f \ast \mu) = (1, \delta \psi \omega(\mu^#) \ast f \ast \mu)) = -\int \psi(x) \mu^# \ast f \ast \mu) dx = -\int \psi(\mu) f(x) dx.
\]

Now, we prove the lemma in the general case. Let \( a \in C^*(G) \) be the strong limit of the sequence \( a_n = \omega(f_n), f_n \in C_c(G) \). Then

\[
\psi(\omega(\mu^#) \omega(\mu)) = \psi(\omega(\mu^#) a_n \omega(\mu)) + \psi(\omega(\mu^#)(a - a_n) \omega(\mu)).
\]

Applying Cauchy-Schwarz inequality to the second term of the right side and then taking the limit we get

\[
\psi(\omega(\mu^#) \omega(\mu)) = \lim_{n} \psi(\omega(\mu^#) a_n \omega(\mu))\]

\[
= \lim_{n} (\psi \mu, a_n) = (\psi \mu, a).
\]

Now let \( Z_1 \) be the central support of the weak closure of \( \ker 1 \) in \( W^*(G) \) and let \( (\ker 1)^+ \) be the unit ball of \( \ker 1 \). The proof of the following theorem, which is similar to that of proposition 1.11 of \([3]\) is omitted.

**Theorem 2.2**

The necessary and sufficient condition for the function to \( \psi \in \mathcal{E}_0(G) \) to be lower semi-continuous is that there exist positive linear functional \( \{p_k\}_{k>1} \) on \( \ker 1 \) such that \( \psi(a) = \infty \)

\[
\sum_{k=1}^{\infty} (f_k, a) \text{ for } a \in (\ker 1)^+.
\]

Now we reformulate this theorem in a more concrete form. Choose \( a \in (\ker 1)^+ \) to be of the form \( \mu^# \ast h^# \ast h \ast \mu \) where \( h \in C_c(G) \), and let \( p_k \in P(G) \) be the extension of \( f_k \) with the same norm. Finally, put \( \mu = \delta_c - \delta_y \) where \( \delta \), denotes the point mass at \( y \). Then

\[
\psi(a) = (-\psi \mu, h^# \ast h) = \infty \sum_{k=1}^{\infty} (p_k, a) \ast \mu^# \ast h^# \ast h),
\]

and we get that \( -\psi \mu \) is the monotonic limit of the exponentially convex functions

\[
\sum_{k=1}^{\infty} \mu \ast p_k \ast \mu^#.
\]

**A COMPACT BASE FOR \( \mathcal{E}_0(G) \)**

In this section we construct a base for \( \mathcal{E}_0(G) \). First we begin with the following definition.

**Definition 3.1**

If \( \psi, \varphi \in \mathcal{E}_0(G) \), we say that \( \psi \) dominates \( \varphi \) if \( \psi \varphi \in \mathcal{E}_0(G) \). If \( \psi \) and \( \varphi \)dominate each other, they are said to be equivalent. They are weakly equivalent if one is equivalent to a positive multiplier of the other.

Now we construct a base for \( \mathcal{E}_0(G) \) which is compact in some suitable topology. This can be done by the selection of an element on each equivalence class in \( \mathcal{E}_0(G) \). Let \( G \) be compactly generated, \( N \) be a compact symmetric neighborhood of the identity in \( G \) which generates the group and \( K = \{ \psi \in \mathcal{E}_0(G) \mid \int N^3 \psi(x) dx = 1 \} \). If \( \psi \in \mathcal{E}_0(G) \) satisfies \( \int_{N^3} \psi(x) dx = 0 \), then \( \psi \mid N^3 = 0 \). Since \( N \) is a generating set, we have \( \psi = 0 \) everywhere. So, for \( \psi \in \mathcal{E}_0(G) \) we can devide it by \( \int N^3 \psi(x) dx \) to obtain an element of \( K \), which implies that \( K \) is a base for \( \mathcal{E}_0(G) \).

**Lemma 3.2**

Suppose that \( X_{A_n} \) is the indicator function of the set \( A_n \), where \( A_1 = N, A_n = N \setminus \{1 + \Delta(y) \mid y \in N \} \) for \( n \geq 2 \) and \( c = \alpha^{-1} \sup \{1 + \Delta(y) \mid y \in N \} \) where \( \alpha \) is the left Haar measure of \( N \). Then for \( \psi \in K \) we have \( \psi(x) \leq f(x) \) where

\[
f = c \sum_{n=1}^{\infty} n^2 \cdot X_{A_n}.
\]
Proof:

Let \( y \in G \) and \( \mu = \delta_c + \delta_y \), then for each \( K \) we have

\[
\psi^{(k)}(y) - \psi^{(k)}(e) = \psi(x) + \psi(xy) + \psi(yx) + \psi(xy) - 2\psi(y) \in E_0(G) \text{ and by integration we get}
\]

\[
2\psi(y) \leq \alpha^{-1} \int \psi(x) + \psi(xy) + \psi(yx) + \psi(xy) \, dx.
\]

If \( y \in N \), then

\[
2\psi(y) \leq \alpha^{-1} \{ \int_N \psi(x) \, dx + \int_{\mathbb{N}} \psi(x) \, dx + \int_{\mathbb{K}} \psi(x) \, dx \}
\]

\[
\leq 2\alpha^{-1} \{ 1 + \Delta(y) \} \int_{\mathbb{N}} \psi(x) \, dx = 2\alpha^{-1} \{ 1 + \Delta(y) \},
\]

by symmetry of \( N \) and \( \sup \{ \psi(y) \mid y \in N, \psi \in K \} \leq C \).

Now, if \( x \in G \), then \( x \in A_n \) for fixed \( n \) and hence it belongs to \( N \). So, there exist \( Y_1, ..., Y_n \in N \) such that \( x = Y_1 Y_2 ... Y_n \). Then

\[
\psi^{1/2}(x) = \psi^{1/2}(Y_1, ..., Y_n) \leq \sum_{k=1}^{n} \psi^{1/2}(y_k) \leq nc,
\]

and we get

\[
\psi(x) = \int_{x}^{\infty} n^2 \chi_{A_n}(x).
\]

Let \( S \) be a separable compact convex set. A subset \( F \) of \( S \) is called a face if each line segment in \( S \) whose interior intersects \( F \) is contained in \( F \). The complementary set \( F' \) of \( F \) is the union of all faces of \( S \) disjoint from \( F \). If \( F \) is a face and \( F' \) is a closed face, then \( F' \) is called a closed split face. In the latter case, \( S \) is the direct convex sum of \( F \) and \( F' \) which means that every element \( x \in S \) can be written uniquely in the form

\[
x = \lambda y + (1 - \lambda)z, \, 0 \leq \lambda \leq 1, \, y \in F, \, z \in F'.
\]

Now we notice that \( K \) is compact only if \( G \) is discrete, but we can compactify \( K \) by adding a point at \( \infty \). Let \( \psi^{\infty}(x) = |N|^{-1} \). It is clear that \( \int_{N} \psi^{\infty}(x) \, dx = 1 \), and \( \psi^{\infty} \) belongs to \( E_0(G) \). Let \( \tilde{K} \) be the convex hull of \( K \) and \( \psi^{\infty} \).

**Theorem 3.3**

Suppose that \( L^\infty(G)_1 \) is the unit ball of \( L^\infty(G) \) equipped with the \( (L^\infty, L^1) \) topology and suppose also that \( p: \tilde{K} \to L^\infty(G)_1 \) is given by \( p(\psi) = \psi/f \), where \( f \) is defined as in Lemma 3.2 and \( K \) is compact in the topology induced by \( p \). Then the set \( \{ \psi^{\infty} \} \) is a closed split face of \( \tilde{K} \).

**Proof**

It is clear that \( L^\infty(G)_1 \) is compact in the \( L^\infty(G) \) topology, and we only need to prove that \( p(\tilde{K}) \) is closed. Suppose \( \{ \psi_n \}^{\infty}_{n=1} \in \tilde{K} \) and such that \( p(\psi_n) \to \psi \in L^\infty (G)_1 \) and suppose also that \( \psi = \phi f \). For \( g \in L^1(G) \) and \( h \in C_c^0(G) \), where \( g = (h\# + h) f \), we have

\[
\int \psi(x) h\# + h \, dx = \int \phi(x) g(x) \, dx = \lim_{n} \int p(\phi_n) \, g(x) \, dx
\]

and hence \( \phi(x) \geq 0 \) a.e. and the same is true for \( \psi \). Hence \( \psi \) is a.e. equals to a unique element of \( E(G) \) and it remains to prove only that for \( \psi \in E_0(G) \) such that \( \int_{N} \psi(x) \, dx = 1 \), \( \psi \in \tilde{K} \). In fact, since \( \psi(e) \leq |N|^{-1} = \psi^{\infty} \) we put \( \lambda = \psi(e)/\psi^{\infty} \). If \( \lambda = 0 \) then \( \psi \in E_0(G) \) and hence \( \psi \in \tilde{K} \). If \( \lambda = 1 \), then \( \int_{N} \psi(x) \, dx = 1 \), from which we get \( \psi(x) = \psi(e) = \psi^{\infty} \) and \( \psi \in \tilde{K} \). Finally, we suppose that \( 0 < \lambda < 1 \) and we put \( \psi(x) = \lambda \psi^{\infty} + (1 - \lambda) \phi(x) \), where \( \phi(x) = (\phi(x) - \psi(e))/(1 - \lambda) \) \( \in E_0(G) \). Then

\[
\int_{N} \psi(x) \, dx = (1 - \lambda)\int_{N} \psi^{\infty} \phi(x) \, dx = 1,
\]

so that \( \psi \in \tilde{K} \). The decomposition of \( \psi \) into a convex sum of \( \psi^{\infty} \) and an element of \( K \) is easily seen to be unique.

**EXTREME RAYS OF \( E_0(G) \)**

Since \( E_0(G) \) is a convex cone, one way to understand its structure is to characterize its extreme rays.

**Definition 4.1**

We say that \( \psi \in E_0(G) \) generates an extreme ray in \( E_0(G) \) if each of its dominated elements is either a tangent vector or weakly equivalent to \( \psi \).

**Theorem 4.2**

Suppose that \( \psi \in E_0(G) \) has a lower semi-continuous Levy Weight. The necessary and sufficient condition for \( \psi \) to generate an extreme ray is that it takes the form

\[
\lambda (1 - p) + h \text{ with } \lambda > 0, \, p \in \text{ext } P_1(G) - \{1\} \text{ and } h \in \text{hom } (G, R).
\]

**Proof**

First we suppose that \( \psi \) is weakly equivalent to \( 1 - p \) for \( p \in P_1(G) - \{1\} \). If \( \phi \in E_0(G) \) is dominated by \( (1 - p)_1 \) then
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\( \varphi \) is bounded. Hence, there exists \( P_1 \in \{1\}' \) \( \lambda \geq 0 \) and \( h \in \text{Hom}(G, \mathbb{R}) \) such that \( \varphi = \lambda \cdot (1 - P_1) + h \). So \( 1 - P \) dominates \( \lambda \cdot (1 - P_1) \) in \( E_0(G) \). Now supposing that \( \lambda \geq 1 \) and since \( (1 - p) \cdot \lambda \cdot (1 - p_1) \in E_0(G) \) then we can find a constant \( K \geq 0 \) such that

\[
K + \lambda \cdot (1 - p_1) - (1 - p) = K + \lambda - 1 - \lambda p_1 + p \in p_1(G);
1 - \lambda \geq 0.
\]

This implies that \( K + p - \lambda p_1 \in p_1(G) \). Since \( p \in \text{ext} p_1(G) \), \( p_1 \) is a convex combination of the orthogonal exponentially convex functions \( 1, p \). But \( p_1 \in \{1\}' \) then \( p_1 = p \) and we have \( \psi \) is extreme.

Conversely, let \( \psi \) be lower semi-continuous. Using Theorem 2.2 in its reformulated form, we can see that \( \psi \) is bounded if it generates an extreme ray in \( E_0(G) \). Hence, \( \psi \) is weakly equivalent to an element of \( E_0(G) \) in the form \( 1 - p, p \in p_1(G) \). Noting that \( \{1\} \) is a closed split face in \( p_1(G) \), we have to prove that \( p \in \{1\}' \). In fact, if \( p = \lambda + (1 - \lambda) p_1 \) is the representation of \( p \) in \( \{1\} \) and \( \{1\}' \) then \( 1 - p = (1 - \lambda) (1 - p_1) \) and we get \( 1 - p_1 \) is weakly equivalent to \( \psi \).

Now suppose that \( p \in \text{ext} p_1(G) - \{1\} \). Then \( p \) can be written in the form \( p = (p_1 + p_2)/2 \) with \( p_1 \neq p_2 \) from \( p_1(G) \). Hence \( 1 - p = (1 - p_1)/2 + (1 - p_2)/2 \) contradicting the extremity of \( \psi \) and we have \( p \in \text{ext} p_1(G) - \{1\} \).

REFERENCES