Qatar Univ. Sci. J. (2004), 24 : 37 - 43

On The Classical Multicolor Ramsey Number R(3,3,3)

Jihad Mohamad Jaam and Ahmad Mojahed Hasnah
Department of Computer Science and Engineering
Qatar University
B.O. Box 2713, Doha, Qatar

حول عدد رمسي الكلاسيكي متعدد الألوان R(3,3,3)

جهاد محمد الجعم و أحمد مجاهد حسنة
كلية الهندسة، قسم هندسة وعلوم الحاسب
جامعة قطر، ص . ب : ٢٧١٣ الدوحة - قطر

في هذا البحث، نحن مهتمون بمشكلة تقييم عدد رمسي الكلاسيكي متعدد الألوان R(3,3,3) في المرحلة الأولى، نبين كيف يمكن تحويل هذه المشكلة إلى نظام ساتي من ٣ متغييرات، بعد ذلك، نشرح طريقة Greenwood و Gleason الجبرية، التي تعتمد على المحدود، F_{16} من أجل إنشاء شبكة كاملة من ١٦ نقطة خالية من المثلثات آحادية اللون، بعد تلوينها بثلاثة ألوان مختلفة، حيث أن مثل هذه الشبكة الملونة تحدد قيمة عدد رمسي الكلاسيكي السابق الذكر، ثم نقترح طريقة جديدة وبسيطة، مختلفة تماماً عن طريقة Greenwood و Gleason لإنشاء شبكة كاملة من ١٦ نقطة ملونة بثلاثة ألوان، وخالية من المثلثات ذات اللون الواحد.

Keywords: *Ramsey Graph, Coloring, Satisfiability.*

ABSTRACT

In this paper, we are interested in the problem of evaluation of the classical multicolor Ramsey number $R(3,3,3)$. We first convert it successfully into a system of clauses of 3-literals each, i.e., a 3-SAT instance. We then describe the algebraic method of Greenwood and Gleason [2], which is based on the finite field F_{16}, that constructs a monochromatic triangle-free edge-coloring with three colors of the Ramsey graph K_{16} associated with the number $R(3,3,3)$. We propose a simple and new coloring method, completely different from that of Greenwood and Gleason, which colors the edges of the Ramsey graph K_{16} with three colors without any monochromatic triangle.

1. Introduction

The classical multicolor Ramsey number $R(3,3,3)$ is the smallest n such that in each 3-coloring of the edges of the complete graph K_n with three colors, there is a monochromatic K_3 (i.e., a monochromatic triangle joining three vertices). In 1955, *Greenwood* and *Gleason* [2] has proven that this number $R(3,3,3)$ is equal to 17, and this is the only non-trivial multicolor known Ramsey number [12]. In [6, 7] we have proposed a stochastic optimization algorithm that constructs a good 3-coloring of the edges of K_{16} with three colors without any monochromatic K_3. Other construction techniques have also been proposed in [2, 3, 8, 9].

In this paper, we show how the problem of evaluation of the classical multicolor Ramsey number $R(3,3,3)$ can be converted into a satisfiability system having 2160 clauses of 3-literals each and 360 variables (i.e., a 3-**SAT** instance). We describe the algebraic method of *Greenwood* and *Gleason* [2], which is based on the finite field F16, that constructs a monochromatic K_3-free edge-coloring with three colors of the Ramsey graph K_{16} associated with the number $R(3,3,3)$. We then propose a simple and new coloring method, completely different from that of *Greenwood* and *Gleason*, which colors the edges of the Ramsey graph K_{16} with three colors without producing any monochromatic triangle.

2. Ramsey Number R(3,3,3) Upper Bound

In this section, we prove that the classical multicolor Ramsey number $R(3,3,3)$ is less than or equal to 17. This turns out to prove that any coloring of the edges of the complete graph of 17 vertices, K_{17}, with three colors, must contain at least one monochromatic K_3 (i.e., one monochromatic triangle).

Theorem 1: *The classical multicolor Ramsey number* $R(3,3,3)$ *is less than or equal to 17. In other words, no good 3-colorings exist for the edges of* K_{17} *without any monochromatic* K_3.

Proof: Let v_0 be a vertex of K_{17}, and let the 16 edges incident with it be colored with three different colors α, β, and γ. So there are at least $\lceil \frac{16}{3} \rceil = 6$ monochromatic edges of color α incident with v_0 and join it with the vertices $v_1, ..., v_6$. The edges between $v_1, ..., v_6$ should be colored with β and γ only, in order to avoid the appearance of a monochromatic K_3 of color α with the vertex v_0. Following the same previous reasoning, there are at least $\lceil \frac{5}{3} \rceil = 3$ monochromatic edges of color β incident with the vertex v_1. Let these edges be $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$. In order to avoid a monochromatic K_3 of color β between the vertices v_1, v_2 and v_3, the edge $\{v_2, v_3\}$ should be colored with γ. Also, to avoid a monochromatic K_3 of color β, with the vertices v_1, v_3 and v_4, the edge $\{v_3, v_4\}$ should be colored with γ. Now, if the edge $\{v_2, v_4\}$ is colored with β, we get a monochromatic K_3 of color γ between the vertices v_1, v_2 and v_4; and if it is colored with γ, we get a monochromatic K_3 of color γ between the vertices v_2, v_3 and v_4. Thus, no good 3-colorings exist for K_{17} without any monochromatic triangle.

3. Converting Ramsey Number to 3-SAT

In this section, we show how the problem of evaluation of the multicolor Ramsey number $R(3,3,3)$ can be converted into a satisfiability system of 2160 clauses of 3-literals each and 360 variables (i.e., a 3-**SAT** instance).

The satisfiability problem, **SAT**, is a fundamental problem in mathematical logic, inference, automated reasoning and computing theory. This problem is known to be NP-complete problem [1, 11]. It is also considered as a typical case of a large family of computationally intractable problems. The problem can be presented as follows:

Let f be a *boolean function*, $f = \{0, 1\}^n \rightarrow \{0,1\}$. So, f can be expressed in a **C**onjunctive **N**ormal **F**orm (CNF) as follows: $f(x) = C_1(x) \wedge C_2(x) \wedge \ldots \wedge \; C_i(x)$ with $C_i(x) = x_{i,1} \vee x_{i,2} \vee \ldots x_{i,j}$, and $j, i \in IN$. Each C_i is called a *clause* and each $x_{i,j}$ is called a *literal* that can be set to *True* or *False*. The negation of $x_{i,j}$ is denoted by $\neg x_{i,j}$. A set of clauses is *satisfiable* if there exists a solution that satisfies all clauses. A clause is *satisfied* if at least one of its literals is set to *True*. Thus, a *solution* is an assignment of truth values to the literals that satisfies all clauses [1, 11]. We denote by *k*-**SAT** the set of clauses in CNF form having exactly k literals in each clause.

Constructing a good *3*-coloring of the edges of K_{16} without any monochromatic K_3, turns out to satisfy a 3-*SAT* instance F having 360 variables ($\overset{c}{x}_{1,2}, \overset{c}{x}_{1,3}, \ldots, \overset{c}{x}_{15,16}$, with color $c = \alpha, \beta, \gamma$) and a system of 2160 clauses in CNF where the variable $\overset{c}{x}_{i,j}$ represents an edge of K_{16} of color c joining two different vertices v_i and v_j. These clauses are as follows:

1. 120 clauses to say that each edge has at least one color, either α or β or γ. These clauses are of the following form: $\overset{\alpha}{x}_{i,j} \vee \overset{\beta}{x}_{i,j} \vee \overset{\gamma}{x}_{i,j}$), $\forall_{i,} j = 1, \ldots, 16$ with $i < j$.

2. 120 x 3 = 360 clauses to say that an edge cannot have two different colors i.e., each edge has only one color. These clauses are of the following form: $(\neg \overset{\alpha}{x}_{i,j} \vee \neg \overset{\beta}{x}_{i,j})$, $(\neg \overset{\alpha}{x}_{i,j} \vee \neg \overset{\gamma}{x}_{i,j})$, $(\neg \overset{\beta}{x}_{i,j} \vee \neg \overset{\gamma}{x}_{i,j})$, $\forall_{i,} j = 1, \ldots,$ with $i < j$.

3. 560 x 3 = 1680 clauses to say that no three edges joining three different vertices are of the same color (i.e., no monochromatic K_3). Note that, the number of K_3 in K_{16} is equal to $\binom{16}{3} = 560$. These clauses are of the following form:

$(\neg \overset{\alpha}{x}_{i,j} \vee \neg \overset{\alpha}{x}_{j,k} \vee \neg \overset{\alpha}{x}_{i,k})$, $(\neg \overset{\beta}{x}_{i,j} \vee \neg \overset{\beta}{x}_{i,k} \vee \neg \overset{\beta}{x}_{i,k})$, $(\neg \overset{\gamma}{x}_{i,j} \vee \neg \overset{\gamma}{x}_{j,k} \vee \neg \overset{\gamma}{x}_{i,k})$, $\forall\, i,j,k = 1, \ldots 16$ with $i < k$).

Thus, the system of clauses F is the following ($i,j,k = 1, \ldots 16$ with $i < j < k$):

$$F = \begin{cases} \overset{\alpha}{x}_{i,j} \vee \overset{\beta}{x}_{i,j} \vee \overset{\gamma}{x}_{i,j} \\ \neg \overset{\alpha}{x}_{i,j} \neg \vee \overset{\beta}{x}_{i,j} \\ \neg \overset{\alpha}{x}_{i,j} \vee \neg \overset{\gamma}{x}_{i,j} \\ \neg \overset{\beta}{x}_{i,j} \vee \neg \overset{\gamma}{x}_{i,j} \\ \neg \overset{\alpha}{x}_{i,j} \vee \neg \overset{\alpha}{x}_{j,k} \vee \neg \overset{\alpha}{x}_{i,k} \\ \neg \overset{\beta}{x}_{i,j} \vee \neg \overset{\beta}{x}_{j,k} \vee \neg \overset{\beta}{x}_{i,k} \\ \neg \overset{\gamma}{x}_{i,j} \vee \neg \overset{\gamma}{x}_{j,k} \vee \neg \overset{\gamma}{x}_{i,k} \end{cases}$$

Constructing a good *3*-coloring of the edges of K_{16} consists of satisfying the above system of clauses F. We mention that F is really a hard *3*-**SAT** instance and cannot be solved with the complete resolution methods of *Davis-Putnam* [4] and *Duboi et al.* [5]. However, the incomplete algorithms of *Selman et al.* [13] and *Minton et al.* [10] are currently the most appropriate methods to solve F.

4. Greenwood and Gleason Coloring Method

In this section we describe the algebraic method of *Greenwood* and *Gleason* [2] based on the finite field F_{16} that constructs a good *3*-coloring of the edges K_{16} without any monochromatic K_3.

Let F_{16} be a finite field of 16 elements that can be generated from the polynomial $P(x) = x^4 + x + 1$ which is based on a field of only two elements 0 and 1 (with 1 +1 = 0). These 16 elements of the polynomial $P(x)$ are the following: 0, 1, x, x^2, x^3, x^4, ..., x^{14} and can be determined as shown in the following table by doing some algebraic calculations:

0	$x^7 = x^3 + x + 1$
1	$x^8 = x^2 + 1$
x	$x^9 = x^3 + x$
x^2	$x^{10} = x^2 + x + 1$
x^3	$x^{11} = x^3 + x^2 + x + 1$
$x^4 = x + 1$	$x^{12} = x^3 + x^2 + x + 1$
$x^5 = x^2 + x$	$x^{13} = x^3 + x^2 + 1$
$x^6 = x^3 + x^2$	$x^{14} = x^3 + 1$

These 16 elements are used as labels of the 16 vertices of the complete graph K_{16}. Thus, each edge of K_{16} is colored according to the sum of its two vertices, which belongs to only one of the following classes: $\alpha = x^{3k}$, $\beta = x^{3k+1}$, $\gamma = x^{3k+2}$, (with α, β and γ classes representing the three different colors). For example the edge joining the two vertices x^6 and x^9 is of color γ because the sum of x^6 and x^9 is $x^3 + x^2 + x^3 + x = 2x^3 + x^2 + x = x^5$ (with $2x^3 = 0$, because our field has only two elements 0 and 1) which is of the form x^{3k+2}. Also, the edge joining the vertices x^5 and x^8 is of color β because their sum is equal to x^4 which is of the form x^{3k+1}, and so on. We notice that the classes are as follows: $\alpha = \{1, x^3, x^6, x^9, x^{12}\}$, $\beta = \{x, x^4, x^7, x^{10}, x^{13}\}$, $\gamma = \{x^2, x^5, x^8, x^{11}, x^{14}\}$, and the sum of two elements of a given class is not in the class as shown in the following tables:

Elements of the class α	sum in different classes β or γ
$1 + x^3$	$= x^{14}$
$1 + x^6$	$= x^{13}$
$1 + x^9$	$= x^7$
$1 + x^{12}$	$= x^{11}$
$x^3 + x^6$	$= x^2$
$x^3 + x^9$	$= x$
$x^3 + x^{12}$	$= x^{10}$
$x^6 + x^9$	$= x^5$
$x^6 + x^{12}$	$= x^4$
$x^9 + x^{12}$	$= x^8$

Elements of the class β	sum in different classes α or γ
$x + x^4$	$= 1$
$x + x^7$	$= x^{14}$
$x + x^{10}$	$= x^8$
$x + x^{13}$	$= x^{12}$
$x^4 + x^7$	$= x^3$
$x^4 + x^{10}$	$= x^2$
$x^4 + x^{13}$	$= x^{11}$
$x^7 + x^{10}$	$= x^6$
$x^7 + x^{13}$	$= x^5$
$x^{10} + x^{13}$	$= x^9$

Elements of the class γ	sum in different classes α or β
$x^2 + x^5$	$= x$
$x^2 + x^8$	$= 1$
$x^2 + x^{11}$	$= x^9$
$x^2 + x^{14}$	$= x^{13}$
$x^5 + x^8$	$= x^4$
$x^5 + x^{11}$	$= x^3$
$x^5 + x^{12}$	$= x^{12}$
$x^8 + x^{11}$	$= x^7$
$x^8 + x^{14}$	$= x^6$
$x^{11} + x^{14}$	$= x^{10}$

Thus no monochromatic K_3 exists in such coloring (i.e, α, β, and γ partition).

5. Our Coloring Method

In this section, we present our new method of constructing a good *3*-coloring of the edges of K_{16} without any monochromatic K_3. The vertices of K_{16}, are denoted by $v_0, \ldots, v_{15}$ and the *3* colors are denoted by α, β and γ. We write $v_i \rightarrow v_k$ to indicate that the edge connecting v_i and v_k in K_{16} is of color α.

Definition 1 Let **S** be a set of vertices of K_{16}, $S = \{v_1, v_2, \ldots, v_m\}$. *The distance* **d** *between two different vertices* v_i *and* v_j of **S** is $\mathbf{d} = |v_i - v_j|$. *We denote by* $\mathbf{D}^{\alpha}\{_{d1}, _{d2}, \ldots, _{dl}\}$ *the set of all possible distances of the vertices of* **S** *of color* α.

Our coloring technique consists of splitting K_{16} into three different K_5 (plus the first vertex v_0) and to color each one separately and correctly as follows:

(1) Coloring the first K_5 of vertices $v_1, \ldots, v_5$.
This coloring is denoted by $\mathbf{C}_{\beta,\alpha}(v_1, \ldots, v_5)$ and defined by $\mathbf{D}^{\beta}_{\{1,4\}}$ and $\mathbf{D}^{\alpha}_{\{2,3\}}$.
(2) Coloring the second K_5 of vertices $v_6, \ldots, v_{10}$.
This coloring is denoted by $\mathbf{C}_{\alpha,\gamma}(v_6, \ldots, v_{10})$ and defined by $\mathbf{D}^{\alpha}_{\{1,4\}}$ and $\mathbf{D}^{\gamma}_{\{2,3\}}$.
(3) Coloring the third K_5 of vertices $v_{11}, \ldots, v_{15}$.
This coloring is denoted by $\mathbf{C}_{\gamma,\beta}(v_{11}, \ldots, v_{15})$ and defined by $\mathbf{D}^{\gamma}_{\{1,4\}}$ and $\mathbf{D}^{\beta}_{\{2,3\}}$.

We then color the edges connecting (*1*) and (*3*); (*1*) and (*2*); and (*2*) and (*3*) in the following manner:

1. Coloring (*1*) and (*3*). This coloring is denoted by $\mathbf{X}_{\alpha,\beta,\gamma}(v_1, \ldots, v_5, v_{11}, \ldots, v_{15})$ and defined by $\mathbf{C}_{\beta,\alpha}(v_1, \ldots, v_5)$ and $\mathbf{C}_{\gamma,\beta}(v_{11}, \ldots, v_{15})$ and $\mathrm{D}^{\beta}_{\{10\}}$, $\mathrm{D}^{\alpha}_{\{7,8,12,13\}}$, $\mathrm{D}^{\gamma}_{\{6,9,11,14\}}$.
2. Coloring (*1*) and (*2*). This coloring is denoted by $\mathbf{X}_{\gamma,\alpha,\beta}(v_1, \ldots, v_5, v_6, \ldots, v_{10})$ and defined by $\mathbf{C}_{\beta,\alpha}(v_1, \ldots, v_5)$ and $\mathbf{C}_{\alpha,\gamma}(v_6, \ldots, v_{10})$ and $\mathrm{D}^{\gamma}_{\{2,3,7,8\}}$, $\mathrm{D}^{\beta}_{\{1,4,6,9\}}$, and $\mathrm{D}^{\gamma}_{\{5\}}$. Note that, the distances are calculated only between the vertices $(v_1, \ldots, v_5)$ and $(v_6, \ldots, v_{10})$.
3. Coloring (*2*) and (*3*). This coloring is denoted by $\mathbf{X}_{\beta,\gamma,\alpha}(v_6, \ldots, v_{15})$ and defined by $\mathbf{C}_{\alpha,\gamma}(v_6, \ldots, v_{10})$ and $\mathbf{C}_{\gamma,\beta}(v_{11}, \ldots, v_{15})$ and $\mathrm{D}^{\alpha}_{\{1,4,6,9\}}$, and $\mathrm{D}^{\beta}_{\{2,3,7,8\}}$, and $\mathrm{D}^{\gamma}_{\{5\}}$. Also, the distances are calculated only-between the vertices $(v_6, \ldots, v_{10})$ and $(v_{11}, \ldots, v_{15})$.

Finally, we color the edges connecting the first vertex v_0 with all other vertices $v_1, \ldots, v_{15}$) as follows:

1. $\mathbf{Y}_{\alpha,\beta,\gamma}(v_1, \ldots, v_5)$:
   $\mathbf{X}_{\alpha,\beta,\gamma}(v_1, \ldots, v_5, v_{11}, \ldots, v_{15})$ and
   $\mathbf{X}_{\gamma,\alpha,\beta}(v_1, \ldots, v_5, v_6, \ldots, v_{10})$ and
   $\mathbf{X}_{\beta,\gamma,\alpha}(v_6, \ldots, v_{15})$ with $v_6 \xrightarrow{\gamma} (v_1, \ldots, v_5)$ and $v_0 \xrightarrow{\beta} (v_6, \ldots, v_{10})$ and $v_0 \xrightarrow{\alpha} (v_{11}, \ldots, v_{15})$.

Figure 1 represents a good edge-coloring monochromatic K_3-free of the Ramsey graph K_{16} with three colors constructed by our method.

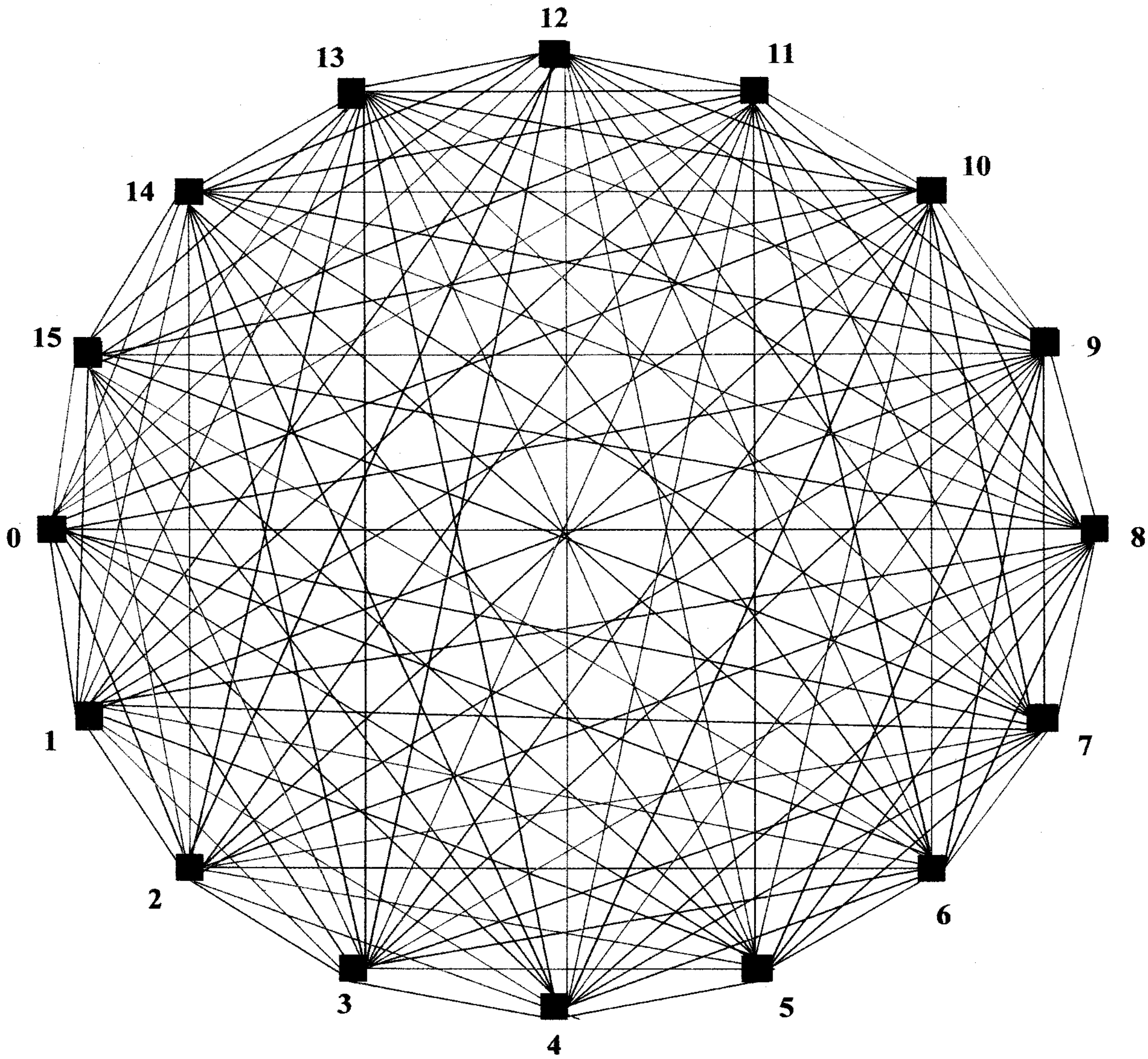

Figure 1: A good *3*-coloring triangle-free of the edges of K_{16} associated with the Ramsey number *R(3,3,3)* with three colors constructed by our method.

6. Conclusion

We have shown how the Ramsey graph coloring problem associated with the multicolor Ramsey number $R(3,3,3)$ can be converted easily into a *3*-**SAT** instance of 2160 clauses and 360 variables. Thus, it can be solved using the incomplete stochastic algorithms proposed in [6, 7, 10, 13]. This conversion technique may be generalized and applied on all Ramsey graphs associated with the classical Ramsey numbers $R\ (k_1, k_2, ..., k_n)$. We have described the algebraic method of *Greenwood* and *Gleason*, which is based on the finite field F_{16}, that constructs a good *3*-coloring of the edges of k_{16} without any monochromatic k_3. We have also proposed a simple and new coloring method of the edges of the Ramsey graph k_{16} associated with the number R (3,3,3) without any monochromatic triangle.

REFERENCES

[1] M.R. Garey and D.S. Johnson: *Computers and Intractability: A Guide to The Theory of NP-Completeness*, W.H. Freeman and Company, New York, (1979).

[2] R.E. Greenwood and A.M. Gleason: *Combinatorial Relations and Chromatic* Graphs, *Canadian Journal of Mathematics*, 7, (1955), pp. 1-7.

[3] B. Benhamou: *Etude des Symétries et de la Cardinalité en Calcul Propoaitionel: Application aux Algorithmes Syntaxiques*. Ph.D. Thesis, University of Aix-Marseilles I, France, (1993).

[4] M. Davis and H. Putnam: *A Computing Procedure for Quantification* Theory, *ACM Journal*, 7, (1960), pp. 201-215.

[5] O. Dubois, P. Andr , Y. Boufkhad, and J. Carlier: SAT vs UNSAT. In *DIMACS Challenge on Satisfiability Testing*, (1994).

[6] J. Jaam: Méthode d'Optimisation Stochastique Pour Résoudre des Problèmes Combinatoires, "Application Sur Les Nombres de Ramsey", In *Proceedings of Secondes Rencontres National des Jeunes Chercheurs en Intelligence Artificielle*, France, (1994), pp. 109-110.

[7] J. Jaam, T. Fliti and D. Husain: New Bounds of Ramsey Numbers Via a Top-Down Algorithm. In *Proceedings of SSRHP at First International Conference on Constraints Programming*, France, (1995), pp. 110-118.

[8] J.G. Kalbfleish and R.G. Stanton: On The Maximal Triangle-Free Edge-Chromatic Graphs in Three Colors, *Journal of Combinatorial Theory*, 5, (1968), pp. 9-20.

[9] C. Laywine, and L.P. Mayberry: A Simple Construction Giving The Two Non-Isomorphic Triangle Free 3-Colored K16's, *Journal of Combinatorial Theory*, Series B, (1988), 120-124.

[10] S. Minton, M.D. Johnston, A.B. Philips, and P. Laird: Minimizing Conflicts: A Heuristic Repair Method for Constraint Satisfaction and Scheduling Problems. *Artificial Intelligence*, 58, (1992), 161-205.

[11] C.H. Papadimitriou: *Computational Complexity*. Addison Wesley, (1994).

[12] S.P. Radziszowski: Small Ramsey Numbers, *Electronic Journal of Combinatorics*, (1999), pp. 1-35.

[13] B. Selman, H.J. Levesque, and D.G. Mitchell: A New Method for Solving Hard Satisfiability Problems. *In Proceedings of AAAI-92*, San Jose, CA, (1992), pp. 440-446.