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KREIN'S METHOD FOR SOLVING THE INTEGRAL EQUATION OF THE FIRST KIND WITH LOGARITHMIC KERNEL

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طريقة كرين لحل معادلة تكاملية من النوع الأول ذات نواه لوغاريتمية

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باعتبار معادلة فريد هولم من النوع الأول ذات النواه المتمثلة في دالة كارلمان قام الباحثان بحل هذه المعادلة باستخدام طريقة كرين واستنتج الباحثان تكافيء هذه النتيجة مع الطرق السابقة (طريقة كوشي) طريقة نظرية الجهد ، طريقة كثيرات الحدود المتعامدة ، طريقة محولات فوربير .

Key Words: Logarithmic Kernel, Krein's method

ABSTRACT

Consider the Fredholm's integral equation of the first kind with logarithmic kernel K(|X|) = (- ln |x|. The aim of this paper is to establish the equivalence of Krein's method of solving the equation with the following methods of solution: method of potential theory, method of singular integral equations, method of orthogonal polynomials and method of Fourier transformation.

INTRODUCTION

In [1] different methods for solving the Fredholm's integral equation of the first kind which may be written in the form.

$$\int_{-a}^{a} \ln \frac{1}{|x-s|} P(s) = f(x), \qquad \text{.................... (1.1)}$$

are considered, where the known function f(x) belongs to C^1 [-a,a] (the class of continuous functions with continuous first derivatives in [-a,a]). Equation (1.1) is solved [2] by using the method of potential theory, and in [7,8] the same equation is considered; using the method of singular integral equations and the theory of boundary value problems for analytic function, the solution is obtained. paPov in his work [6], solved equation (1.1) by using the method of orthogonal Tchebyshev polynomials. The equivalence of these methods is obtained in [1].

In [1], Mkhitarian and Abdou applied M.G. Krein's method for obtaining the basic formulae for the potential functions of (1.1) in the form:

$$P_+(x) = \frac{J(a)}{\ln(2/a)} \frac{1}{\sqrt{a^2-x^2}} - \frac{2}{\pi}\int_x^a \frac{du}{\sqrt{u^2-x^2}} \cdot \frac{d}{du}\left[u \frac{d}{du}\int_0^u \frac{f_+(s)\,ds}{\sqrt{u^2-s^2}}\right]$$

and (1.2)

$$P_-(x) = -\frac{2}{\pi^2}\frac{d}{dx}\int_x^a \frac{u\,du}{\sqrt{u^2-x^2}}\int_0^u \frac{df(s)}{\sqrt{u^2-s^2}} \qquad (1.3)$$

where

$$J(u) = \frac{2}{\pi}\left[\int_0^u \frac{f_+(s)\,ds}{\sqrt{u^2-s^2}} + u \ln\frac{2}{u}\frac{d}{du}\int_0^u \frac{f_+(s)\,ds}{\sqrt{u^2-s^2}}\right] \qquad (1.4)$$

$$f(x) = f_+(x) + f_-(x), \qquad P(x) = P_+(x) + P_-(x)$$

$$f_+(-x) = \pm f_\pm(x), \qquad P_\pm(-x) = \pm P_\pm(x) \quad (-a, a)$$

In this paper, we prove equivalence of the previous methods with Krein's method.

SOLUTIONS OF THE PROBLEM

We start by proving the following lemmas.

Lemma 1: For all positive integers n, the value of the following integral

$$I_n = \int_o^u \frac{T_{2n}(s)\,ds}{\sqrt{u^2-s^2}}, \qquad \text{..... (2.1)}$$

is given in the form

$$I_n(u) = \frac{p}{2} P_n^{(-1,o)}(2u^2-1), \qquad \text{........... (2.2)}$$

where $T_{2n}(s)$ are Tchebyshev Polynomials and $P_n^{(\alpha,\beta)}(x)$ are Jacobe polynomials.

Proof: Using the substitution s = ut, and the two relations [5]

$$\int_{-l}^{l}(l-t^2)^{-1/2}\, T_n(l-t^2 y).\, dt = \frac{p}{2}\left[P_n(l-y) + P_{n-l}(l-y)\right],$$

$$2P_n^{(-l,o)}(x) = P_n(x) - P_{n-1}(x),$$

where P_n (x), n=o, 1, 2, ..., are Legender polynomials, together with $T_{2n}(t) = T_n(2t^2-1)$ the lemma can be proved.

Corollary 1: The first two derivatives of I_n (u) are given by

$$I_n^{(1)}(u) = \frac{dI_n}{du} = n\pi u P_{n-1}^{(o,1)}(2u^2-1), \quad (n=1,2,3,\ldots\ldots\ldots), \qquad (2.3)$$

$$I_n^{(2)}(u) = \frac{d}{du}\left(\frac{dI_n}{du}\right) + \frac{d}{du}\left[u\, I_n^{(1)}(u)\right],$$

which can be written in the form

$$I_n^{(2)}(u) = 2n\,\pi u\, P_{n-1}^{(o,1)}(2u^2-1) + 2n\pi\,(n+1)u^3 P_{n-2}^{(1,2)}(2u^2-1),$$

$$(n=1,2,3,\ldots) \qquad (2.4)$$

Note that $P_n^{(\alpha,\beta)}(x) = 0$, for negative integers.

To find the value of J_n (u) when u = 1, we substitute (2.3) in (1.4), and put u = 1, to obtain

$$J_n(1) = 2\left[\frac{1}{2}P_n^{(-1,o)}(1) + n \ln 2.\, P_{n-1}^{(o,1)}(1)\right].$$

Since it is known [9] that

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)},$$

then we have the following corollary.

Corollary 2: The value of J_n (1) is given by

$$J_n(1) = 2n \ln 2 \; (n=1, 2, \ldots) \qquad (2.5)$$

Lemma 2: The value of the integral.

$$H_n(y) = \int_o^1 (1-t)^{-1/2} P_{n-1}^{(o,1)}[1-(1-y)t]\,dt,$$

can be written in the form

$$H_n(y) = \frac{\sqrt{\pi}(n-1)!}{\Gamma(n+1/2)} P_{n-1}^{(1/2,1/2)}(y) \quad (n=1,2,3,\ldots) \qquad (2.6)$$

Proof: To prove this lemma, we use the relation between the hypergeometric function and the Jacobe polynomial [5].

$$\int_o^1 t^{\lambda-1}(1-t)^{\mu-1} P_n^{(\alpha,\beta)}(1-\gamma t)\,dt =$$

$$= \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\,\Gamma(1+\alpha)\Gamma(\lambda+\mu)}\, {}_3F_2(-n, n+\alpha+\beta+1; \lambda, \alpha+1, \lambda+\mu; \frac{\gamma}{2})$$

$$(\mathrm{Re}\,\lambda > o,\ \mathrm{Re}\,\mu > o), \qquad (2.7)$$

where (x0 is the gamma function and ${}_3F_2(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, Z)$ is the generalized hypergeometric series;

$${}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z) = \sum_{m=o}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_m(\alpha_3)_m}{(\beta_1)_m(\beta_2)_m\, m!} z^m,$$

$$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}.$$

In this case, we can write H_n (y) in the form

$$H_n(y) = 2\, {}_3F_2(-n+1, n+1, 1; 1, \tfrac{3}{2}, \tfrac{1-y}{2})$$

$$= 2F(-n+1, n+1; \tfrac{3}{2}; \tfrac{1-y}{2}), \qquad (2.8)$$

where F(α, β, γ; z) is the hypergeometric Gauss function. It is known [9] that

$$P_n^{(\alpha,\beta)}(y) = \binom{n+\alpha}{n} F(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1-y}{2}), \qquad (2.9)$$

introducing (2.9) into (2.8) the result follows.

Lemma 3: The value of the integral

$$G_n(y) = \int_o^1 (1-t)^{-1/2} P_{n-2}^{(1,2)}[1-(1-y)t]\,dt,$$

is given by

$$G_n(y) = \frac{-2\sqrt{\pi}(n-1)!}{(n-1/2)(n+1)(1-y)} P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(y) + \frac{2}{(n+1)(1-y)}, \qquad (2.10)$$

Proof: Independent of the relation (2.7), we can write G_n (y) in the form

$$G_n(y) = 2(n-1)\, {}_3F_2(-n+2, n+2, 1; 2, \tfrac{3}{2}; \tfrac{1-y}{2})(n=1,2,\ldots) \qquad (2.11)$$

To write the hypergeometric series in the form of Jacobe polynomials, assume that

$$h_n(z) = z\, {}_3F_2(-n+2, n+2, 1; 2, \tfrac{3}{2}; z)(z = \tfrac{1-y}{2}), \qquad (2.12)$$

Differentiating (2.12) with respect to z and using (2.9), we get

$$\frac{dh_n(z)}{dz} = \frac{\sqrt{\pi}\,\Gamma(n-1)}{2\,\Gamma(n-1/2)} P_{n-2}^{(1/2,\,5/2)}(1-2z) \quad (n=2,3,\ldots).$$

Integrating the last equation under the condition $h_n(o) = o$, one easily obtains.

$$h_n(z) = \frac{-\sqrt{\pi}\,\Gamma(n-1)}{2\Gamma(n-1/2).(n+1)} P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(1-2z) + \frac{1}{2(n^2-1)}. \qquad (2.13)$$

Comparing (2.12) with (2.13) lemma 3 follows.

Along the same lines, one may prove the following lemma.

Lemma 4: The value of the integral

$$K_n(y) = \int_o^1 (1-t)^{1/2} P_{n-2}^{(1,2)}[1-(1-y)t]\,dt$$

is given by

$$K_n(y) = \frac{-\sqrt{\pi}(n-1)!}{\Gamma(n+1/2).(n+1)(1-y)} P_{n-1}^{(1/2,1/2)}(y) + \frac{2}{(n+1)(1-y)}. \qquad (2.14)$$

Now, to connect the previous results in one we need the following lemma.

Lemma 5: The value of the integral equation

$$A_n = \int_x^1 \frac{I_n^{(2)}(u)\,du}{\sqrt{u^2-x^2}}$$

is given by

$$A_n(x) = \frac{n\pi(1-T_{2n}(x))}{\sqrt{1-x^2}} \quad (n=1,2,\ldots) \qquad (2.15)$$

where $I_n^{(2)}$ (u) is given in (2.4).

Proof: Substituting for $I_n^{(2)}$ (u) from (2.4) the above integral becomes

$$A_n(x) = 2n\pi \int_x^1 \frac{u\, P_{n-1}^{(o,1)}(2u^2-1).du}{\sqrt{u^2-x^2}} + 2n\pi(n+1)\int_x^1 \frac{u^3 P_{n-2}^{(1,2)}(2u^2-1)du}{\sqrt{u^2-x^2}}$$

Using the parameters $y = 2x^2 -1$ and $v = 2u^2-1$ the last equation becomes

$$A_n(x) = \frac{\pi n}{\sqrt{2}} \int_y^1 \frac{1}{\sqrt{v-y}}\left[P_{n-1}^{(o,1)}(v) + \frac{n+1}{2}\, v\, P_{n-2}^{(1,2)}(v) + \frac{n+1}{2} P_{n-2}^{(1,2)}(v)\right] dv.$$

Also, using the parameter $v = 1-(1-y)\,t$ $(o \le t \ge 1)$ the previous equation takes the form

$$A_n(x) = \frac{n\pi}{\sqrt{2}}\sqrt{1-y}\left[H_n(y) + \frac{n+1}{2}(1+y)\,G_n(y) + \frac{n+1}{2}(1-y)K_n(y)\right]$$

$$(y=2x^2-1,\ n=1,2,\ldots) \quad (2.16)$$

where

$$H_n(y) = \int_0^1 (1-t)^{-1/2} P_{n-1}^{(o,1)}\left[1-(1-y)t\right] dt\ .$$

$$G_n(y) = \int_0^1 (1-t)^{-1/2} P_{n-2}^{(1,2)}\left[1-(1-y)t\right] dt\ .$$

and

$$K_n(y) = \int_0^1 (1-t)^{1/2} P_{n-2}^{(1,2)}\left[1-(1-y)t\right] dt.$$

Using the values of these integrals, obtained in lemmas (2)-(4) we obtain

$$A_n(x) = \frac{\pi^{3/2}\ n!}{\sqrt{2}\,\Gamma(n-1/2)} \cdot \frac{1}{1-y}\left[\frac{1-y}{2n-1} P_{n-1}^{(1/2,1/2)}(y) - (1+y) P_{n-1}^{(-1/2,\frac{3}{2})}(y)\right]$$

$$+ \frac{\sqrt{2}\,n\pi}{\sqrt{1-y}} \quad (y = 2x^2-1\ ;\ n=1,2,\ldots) \quad (2.17)$$

Now it is our aim to find a relation between Jacobe polynomials and Tchebyshev polynomials. For this end we must use these two famous relations ([9], p. 177).

$$P_n^{(\lambda-1/2,-1/2)}(2x^2-1) = \frac{\Gamma(n+1/2)\,\Gamma(\lambda)}{\sqrt{\pi}\,(n+\lambda)}\, C_{2n}^{\lambda}(x),$$

and

$$P_n^{(\lambda-1/2,1/2)}(2x^2-1) = \frac{\Gamma(n+3/2)\,\Gamma(\lambda)}{\sqrt{\pi}\,\Gamma(n+\lambda+1)x}\, C_{2n+1}^{\lambda}(x) \quad (2.18)$$

where $C_n(x)$ is Heigenber polynomials. When $\lambda \to o$ ([5] p. 1044 equation (8) and p. 934 equation (4)),

$$\lim_{\lambda\to o} \Gamma(\lambda)\, C_n^{\lambda}(x) = \frac{2}{n}\, T_n(x) \quad (n=1,2,\ldots) \quad (2.19)$$

The results of equation (2.18) and equation (2.19), can be written in the form

$$P_n^{(-1/2,-1/2)}(2x^2-1) = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\, n!}\, T_{2n}(x)\ ,$$

$$P_n^{(-1/2,1/2)}(2x^2-1) = \frac{2\,\Gamma(n+3/2)}{\sqrt{\pi}\,(2n+1)\, n!\, x}\, T_{2n+1}(x). \quad (2.20)$$

Equation (2.20) gives the relation between the Jacobe polynomials, and Tchebyshev polynomials of the first type, if we want to connect equation (2.20) with (2.17), firstly assume

$$L_n(y) = \frac{1-y}{2n-1} P_{n-1}^{(1/2,1/2)}(y) - (1+y) P_{n-1}^{(-1/2,\,3/2)}(y),$$

which may be written in the form

$$L_n(y) = \frac{2(1-y)}{n(2n-1)} \frac{d}{dy}\left[P_n^{(-1/2,-1/2)}(y)\right] - \frac{2(1+y)}{n} \frac{d}{dy}\left[P_n^{(-3/2,1/2)}(y)\right]$$

$$= \frac{2(1-x^2)}{x} \cdot \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,(2n-1).\,n!}\, U_{2n-1}(x) - \frac{x}{n}\frac{d}{dx}\left[\frac{(n+\frac{3}{2})}{\sqrt{\pi}\,\Gamma(n)x}\right.$$

$$\left. .\left(\lim_{\lambda\to -1} \Gamma(\lambda)\, C_{2n+1}^{\lambda}(x)\right)\right] \quad (n=1,2,\ldots), \quad (2.21)$$

where $U_{n-1}(x)$ is Tchebyshev polynomials of the second kind. According to equation (2.19), and using [9, p. 185 equation (4)], we get

$$\lim_{\lambda\to -1} \Gamma(\lambda)\, C_{2n+1}^{\lambda}(x) = \frac{1}{n}\left[\frac{T_{2n+1}(x)}{2n+1} - \frac{T_{2n-1}(x)}{2n-1}\right] (n=1,2,\ldots). \quad (2.22)$$

Secondary, rewrite (2.21) using (2.20), to get

$$L_n(y) = L_n(2x^2-1) = \frac{-4\,\Gamma(n+1/2)}{\sqrt{\pi}\,(2n-1)\,\Gamma(n)} \cdot T_{2n}(x) \quad (x=1,2,\ldots). \quad (2.23)$$

Introducing (2.23) in (2.17) we obtain (2.15) and the lemma is proved.

Finally to obtain out main result we put $a = 1$ in (1.2) and let $f_+(x) = T_{2n}(x)$ $(n=1,2,\ldots)$; this gives.

$$P_+(x) = \frac{J(1)}{\pi \ln 2} \cdot \frac{1}{\sqrt{1-x^2}} - \frac{2}{\pi^2} \int_x^1 \frac{du}{\sqrt{u^2-x^2}} \frac{d}{du}\left[u \frac{d}{du} \int_0^u \frac{T_{2n}(s)\, ds}{\sqrt{u^2-s^2}}\right]$$

$$(2.24)$$

and hence we have:

Theorem 1: The complete solution of equation (2.24), can be adapted in the form.

$$P_+(s) = \frac{2n\, T_{2n}(x)}{\sqrt{1-x^2}} \quad (o < x < 1,\ n=1,2,\ldots) \quad (2.25)$$

THE RESULTS OF THE PROBLEM

The above results lead to the following two theorems.

Theorem 2: For the Fredholm's integral equation of the first kind when the kernel is in the form of a logarithmic function ($K(|x-y|) = -$ in $(|x-y|)$ which has a singularity at $x=y$, and the known function is even and in the form of Tchebyshev function $T_{2n}(x)$, the special relation has the form

$$\int_{-1}^{1} \ln\frac{1}{|x-s|} \frac{T_{2n}(s)\, ds}{\sqrt{1-s^2}} = \frac{\pi}{2n} T_{2n}(x) \quad (|x|<1;\ n=1,2,\ldots), \quad (3.1)$$

Equation (3.1) is in agreement with (1.14) in [1] when n is replaced by 2n and a = 1.

Also when the known function of Fredholm's integral equation is odd, and has the form $f_-(x) = T_{2n-1}(X)$, we have the spectral relation in the form

$$\int_{-1}^{1} \ln\frac{1}{|x-s|} \cdot \frac{T_{2n-1}(s)\, ds}{\sqrt{1-s^2}} = \frac{\pi}{2n-1} T_{2n-1}(x), \quad (3.2)$$

which is in agreement with (1.14) in [1] when a=1 and n is replaced by 2n-1.

Theorem 3: For solving equation (1.1) when $f(x) \varepsilon C'$ $[-a, a]$, $f''(x)$ satisfies the Dirichler condition in $(-a,a)$, and

$$f(x) = \sum_{n=o}^{\infty} a_n T_n(x,a) \quad (|x| < a),$$

where a_n is a linear parameter and a_n is determined as in [1] then we have the potential function in the form

$$P(x) = \frac{1}{\pi\sqrt{a^2 - x^2}}\left[P + \sum_{n=1}^{\infty} n\, a_n\, T_n(x/a)\right] \quad (|x| < a).$$

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