INDECOMPOSABLE REPRESENTATIONS OF ORDER OF $\tilde{E}_6$

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ABSTRACT

The extended Dynkin diagram

\[ \tilde{E}_6: \]

is a valued graph. We are going to construct a Bachstrom order $A$ associated to $E_6$. We prove, by constructions, that the order $A$ of infinite lattice-type but can be listed (tame-type), i.e., we put all indecomposable $A$ - lattices in finite number of general forms. Finally we give a method to obtain easily and directly the lattices from its associated representations.

1. Bachstrom order of $\tilde{E}_6$

Ringel and Roggenkamp have introduced for each basic Bachstrom order a valued graph (4).

In this section we construct an R-order $A$ for $\tilde{E}_6$, where $R$ is a complete valuation ring. The orientation and the numerical of the vertices of the diagram $\tilde{E}_6$ are given as follows:

1. $\rightarrow .4$
2. $\rightarrow .5$
3. $\rightarrow .6$

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Let its modulation $M$ be given as follows,

$$i S_j = F \text{ and } F_i = F (F = R/\pi \text{ where } \pi \text{ is the maximal ideal of } R), 1 \leq i \leq 3, 4 \leq j \leq 7.$$ 

We construct an $R$-order $\Gamma$, satisfying the conditions:

(i) $M$ is hereditary and (ii) $\Gamma / \text{rad } \Gamma = \Pi (F_j)^n$ as follows

$$\Gamma = \begin{bmatrix}
R & R & R & R & R & R \\
\pi & R & R & R & R & R \\
\pi & \pi & R & R & R & R \\
\pi & \pi & \pi & R & R & R \\
\pi & \pi & \pi & \pi & R & R \\
\pi & \pi & \pi & \pi & \pi & R
\end{bmatrix}$$

Then

$$\text{rad } \Gamma = \begin{bmatrix}
\pi & R & R & R & R & R \\
\pi & \pi & R & R & R & R \\
\pi & \pi & \pi & R & R & R \\
\pi & \pi & \pi & \pi & R & R \\
\pi & \pi & \pi & \pi & \pi & R \\
\pi & \pi & \pi & \pi & \pi & \pi
\end{bmatrix}$$

and $\Gamma / \text{rad } \Gamma = \begin{bmatrix}
F & O & O & O & O & O \\
O & F & O & O & O & O \\
O & O & F & O & O & O \\
O & O & O & F & F & F \\
O & O & O & F & F & F
\end{bmatrix}$

so the simple $\Gamma / \text{rad } \Gamma$-modules are:

$$S_4 = \begin{bmatrix}
F \\
O \\
O \\
O \\
O
\end{bmatrix} \quad S_5 = \begin{bmatrix}
O \\
F \\
O \\
O \\
O
\end{bmatrix} \quad S_6 = \begin{bmatrix}
O \\
O \\
F \\
O \\
O
\end{bmatrix} \quad \text{and } S_7 = \begin{bmatrix}
O \\
O \\
O \\
O \\
F
\end{bmatrix}$$

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Now we construct a Bäckstrom order $\Lambda$ of $\tilde{E}_6$, satisfying the conditions:

(i) $\Lambda \subseteq \Gamma$

(ii) $\Lambda/\text{rad}\, \Lambda = \Pi^3 F_i \quad i = 1$

(iii) $\text{rad}\, \Lambda = \text{rad}\, \Gamma$

(iv) $\mathcal{S}_j = F_i \bigotimes_{\Lambda} S_j = F, \ 1 \leq i \leq 3, \ 4 \leq j \leq 7$,

as follows:

$$\Lambda = \begin{bmatrix}
\alpha & R & R & R & R & R \\
\pi & \beta & R & R & R & R \\
\pi & \pi & \gamma & R & R & R \\
\pi & \pi & \pi & \alpha' & \pi & \pi \\
\pi & \pi & \pi & \pi & \beta' & \pi \\
\pi & \pi & \pi & \pi & \pi & \gamma' \\
\end{bmatrix}$$

where $\alpha = \alpha' \pmod{\pi}$, $\beta = \beta' \pmod{\pi}$, and $\gamma = \gamma' \pmod{\pi}$.

2. The positive roots of $\tilde{E}_6$.

Let $(G,d)$ be an extended Dynkin diagram, and let $c$ be a Coxeter transformation of the vector space $Q^G$ of all vectors $x = (x_i)_i \in G$ over the rational field $Q$. Then all positive roots of negative, positive and zero defect with respect to $c$ are the vectors (see [1]):

(1) $x = c^{-t} P_k, \ O \leq r, \ 1 \leq t \leq n$

(2) $x = c^{-r} Q_t, \ O \leq r, \ 1 \leq t \leq n$ and

(3) $x = x_o + r g \bar{n}, \ O \leq r, \ X_o \leq g \bar{n}, \ \partial_c x_o = 0$,

where $\bar{n}$ is the canonic vector respectively.

In the case of $\tilde{E}_6$ we have

$c = s_1 s_2 \ldots s_7$ the Coxeter transformation,

$C^+ = s^+_1 s^+_2 \ldots s^+_7$ \quad The Coxeter functors,

$C^- = s^-_7 s^-_6 \ldots s^-_1$

and
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$q_t = s_1s_2 \ldots s_{t-1} T\right\}, \ 1 \leq t \leq 7$,

$P_t = s_7s_6 \ldots s_{t-1} T\right\}$

where $T$ is the vector in $Q^o$ defined by:

$T_t = 1$ and $T_i = 0$ for all $i \neq t$.

The defect of $\bar{E}_6$ with the given orientation has the following components:

$$d_c = 3 \begin{array}{c}
 2 \\
 2 \\
 2 \\
 1 \\
 1 \\
 1
\end{array}$$

2.1 The positive roots with negative defect:

These roots are $C^t_q t$, $0 \leq r$, $1 \leq t \leq 7$, we deduce the general forms as follows ($n \geq 0$):

$t = 1$: there are three general forms:

$$\begin{array}{c}
 3n \\
 2n \\
 2n
\end{array} \begin{array}{c}
 2n+1 \\
 n \\
 n
\end{array} \begin{array}{c}
 2n+1 \\
 n+1 \\
 n+1
\end{array} \begin{array}{c}
 2n+1 \\
 n \\
 n
\end{array}$$

$t = 2$: We obtain the roots by interchanging the edges (1-4) and (2-5) in the case ($t = 1$)

$t = 3$: Similarly by interchanging the edges (1-4) and (3-6) in the case ($t = 1$)

$t = 4$: There are six general forms:

$$\begin{array}{c}
 3n+1 \\
 2n+1 \\
 2n+1
\end{array} \begin{array}{c}
 2n+2 \\
 n \\
 n
\end{array} \begin{array}{c}
 2n+2 \\
 n+1 \\
 n+1
\end{array} \begin{array}{c}
 2n+3 \\
 2n+2 \\
 2n+2
\end{array} \begin{array}{c}
 2n+3 \\
 2n+2 \\
 2n+2
\end{array} \begin{array}{c}
 2n+3 \\
 2n+2 \\
 2n+2
\end{array}$$

$t = 5$: We obtain the roots by interchanging the edges (1-4) and (2-5) in the case ($t = 4$)

$t = 6$: Similarly by interchanging the edges (1-4) and (3-6) in the case ($t = 4$)
t = 7: There are two general forms:

\[
\begin{align*}
2n+1 & \longrightarrow n \\
3n+1 & \longrightarrow 2n+1 \longrightarrow n \\
2n+ & \longrightarrow n
\end{align*}
\]

2.2 The positive roots with positive defect:
These roots are \( C_{p_i} \), \( o \leq r \), \( 1 \leq t \leq 7 \).

We deduce the general forms as follows (\( n \leq o \)):

\( t = 1 \): There are three general forms

\[
\begin{align*}
2n+1 & \longrightarrow n+1 \\
3n+1 & \longrightarrow 2n \longrightarrow n \\
2n+ & \longrightarrow n
\end{align*}
\]

(\( 3n+1 \), \( n \) and \( 3n+2 \) interchange)

\[
\begin{align*}
2n+ & \longrightarrow n+1 \\
3n+1 & \longrightarrow 2n+1 \longrightarrow n+1 \\
2n+ & \longrightarrow n+1
\end{align*}
\]

\( t = 2 \): We obtain the roots by interchanging the edges (1-4) and (2-5) in the case (\( t = 1 \)).

\( t = 3 \): Similarly by interchanging the edges (1-4) and (3-6) in the roots of the case (\( t = 1 \)).

\( t = 4 \): There are six general forms:

\[
\begin{align*}
2n & \longrightarrow n+1 \\
3n+1 & \longrightarrow 2n \longrightarrow n \\
2n+ & \longrightarrow n
\end{align*}
\]

\[
\begin{align*}
2n+1 & \longrightarrow n+1 \\
3n+2 & \longrightarrow 2n+1 \longrightarrow n+1 \\
2n+1 & \longrightarrow n+1
\end{align*}
\]

\( t = 5 \): We obtain the roots by interchanging the edges (1-4) and (2-5) in the case (\( t = 4 \)).

\( t = 6 \): Similarly by interchanging the edges (1-4) and (3-6) in the case (\( t = 4 \)).

\( t = 7 \): There are two general forms

\[
\begin{align*}
2n & \longrightarrow n \\
3n+1 & \longrightarrow 2n \longrightarrow n \\
2n+ & \longrightarrow n
\end{align*}
\]

These representations correspond to the roots calculated in the previous sections, we use the following notations:

(i) $\text{FFF} \ldots$ instead of the vector space $F + F + F + \ldots$, for any number of $F$, where $F = \mathbb{R}/\pi$. Also the vector of the representations is denoted by its dimensions, e.g. $\text{FFF} : = 3$.

(ii) The linear mappings of the representations are:

(a) $1: F \rightarrow F$, $11: F F \rightarrow F F \ldots$, ...
   $f \rightarrow f$  $(f_1, f_2) \rightarrow (f_1, f_2)$

(b) $\circ: F \rightarrow \circ$ or $\circ \rightarrow F$, $\circ \circ: FF \rightarrow \circ$ or $\circ \rightarrow FF$, ...

(c) $1 = 1: F \rightarrow FF$, $1 = 1 = 1: F \rightarrow FFF$, ...
   $f \rightarrow (f,f)$  $f \rightarrow (f,f,f)$

(d) $+: F F \rightarrow F$, $++: F F F \rightarrow F F$, ...
   $(f_1,f_2) \rightarrow (f_1+f_2)$  $(f_1,f_2,f_3,f_4) \rightarrow (f_1+f_2, f_3+f_4)$

Moreover, we may also combine the above notations, for example:

10: $F F \rightarrow F$ or $F \rightarrow F F$
   $(f_1, f_2) \rightarrow f_1$  $f_1 \rightarrow (f_1, 0)$

1+: $F F F \rightarrow F F$, 101: $F \rightarrow F F F$, and
   $(f_1,f_2,f_3) \rightarrow (f_1+f_2+f_3)$  $f \rightarrow (f,o,f)$

$(10)^n : 10101010 \ldots$ 10 (10 is repeated n times), similarly

$(+)^n$ and the other $(\ldots)^n$.

Since we have a one-to-one correspondence between all positive roots of non-zero defect and all indecomposable representations of non-zero defect, it is enough to give only the linear mappings

$j^i$, $i = 1, 2, 3, j = 4, 5, 6, 7$ of the general forms.

3.1 The indecomposable representations $C^+Q_t$ of $E_6$.

The general forms of these representations are:

t = 1: There are three general forms:

(a) $4^i 1 = o(10)^n$, $5^i 2 = (01)^n$, $6^i 3 = (+)^n$, ...
\[ 7 \hat{f}_1 = \begin{cases} 0 & \text{for } n = 0 \\ 111 & \text{for } n = 1 \\ 111(011)^{-1} & \text{for } n \geq 2 \end{cases} \]

\[ 7 \hat{f}_2 = \begin{cases} 0 & \text{for } n = 0 \\ (-f_1, f_1 + f_2, f_2) & \text{for } n = 1 \\ (-f_1, f_1 + f_2, f_2, g_1, g_2, \ldots, g_{n-1}) & \text{for } n \geq 2 \end{cases} \]

(note that we have defined the linear mapping with its value of \((f_1, \ldots, f_n)\) where

\[ g_i = f_{2i} - f_{2i+1}, f_{2i+1}, f_{2i+2}, f_{2i+2}, i = 1, 2, \ldots, n-1, \]

and

\[ 7 \hat{f}_3 = \begin{cases} 0 & \text{for } n = 0 \\ 111 = 1 & \text{for } n = 1 \\ f_1, f_2, f_2, g_1, g_2, \ldots, g_{n-1} & \text{for } n \geq 2 \end{cases} \]

where

\[ g_i = f_{2i} + f_{2i+1}, f_{2i+2}, f_{2i+2}, i = 1, 2, \ldots, n-1 \]

(b) \[ 4 \hat{f}_1 = O(10)^n, \quad 5 \hat{f}_2 = O(1)^n, \quad 6 \hat{f}_3 = (+)^nO, \quad 7 \hat{f}_1 = 1(1 = 1)'^n, \quad 7 \hat{f}_2 = 1(110)^n \]

\[ 7 \hat{f}_3 = \begin{cases} 1 & \text{for } n = 0 \\ f_1, g''_1, g''_2, \ldots, g''_{n-1} & \text{for } n \geq 1, \end{cases} \]

where \( g''_i = f_2, -f_{2i-1}, f_{2i-2} + f_{2i} + f_{2i-1}, i = 1, 2, \ldots, n \)

(c) \[ 4 \hat{f}_1 = O(+)^n, \quad 5 \hat{f}_2 = (+)^{n-1}, \quad 6 \hat{f}_3 = (01)^{n+1} \]

\[ 7 \hat{f}_1 = \begin{cases} 01 & \text{for } n = 0 \\ 0, f_1, g'''_1, g'''_2, \ldots, g'''_{n-1} & \text{for } n \geq 1 \end{cases} \]

where \( g'''_i = 0, f_2, f_2, f_{2i-2} + f_{2i}, f_{2i+2}, i = 1, 2, \ldots, n \)

\[ 7 \hat{f}_2 = 11(1 = 11)^n \]

and

\[ 7 \hat{f}_3 = \begin{cases} 11 & \text{for } n = 0 \\ f_1, f_2, g_1, g_2, \ldots, g_{n-1} & \text{for } n \geq 1 \end{cases} \]

where \( g_i = f_{2i+1}, f_{2i+1}, f_{2i+2}, f_{2i+2}, i = 1, 2, \ldots, n \).

\( t = 2, t = 3 \): by the same interchanging as in the roots.
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$t = 4$: There are six general forms:

(a) $4^1 = (+)^n$, $5^2 = O(+)$

$6^3 = \begin{cases} 
O & \text{for } n = 0 \\
 f_1 + f_3 & \text{for } n = 1 \\
f_1 + f_3, h_2, h_3, \ldots, h_i, \ldots, h_n & \text{for } n \geq 2
\end{cases}$

where $h_i = f_{2i-2} + f_{2i+1}$, $i = 2, 3, \ldots, n$,

$7^1 = 0(101)^n$, $7^2 = 1(110)^n$ and $7^3 = 1(011)^n$

(b) $4^1 = (+)^{n+1}$, $5^2 = 6^3$ in case (a), $6^3 = O(+)$, $7^1 = 11(101)^n$, $7^2 = 10(110)^n$ and $7^3 = 10(011)^n$

(c) $4^1 = O(+)^0$, $5^2 = (+)^{n+1}$, $6^3 = (+)^{n+1}$, $7^1 = (101)^{n+1}$, $7^2 = (110)^{n+1}$ and $7^3 = (011)^{n+1}$

(d) $4^1 = 0(10)^n$, $5^2 = 1(10)^n$, $6^3 = 1(10)^n$, $7^1 = 11 = 111 = \ldots = 111$ (111 repeated $n$ times), $7^2 = 1(1 = 11)^n$ and $7^3 = 1(11 = 1)^n$

(e) $4^1 = 0(10)^n$, $5^2 = 1(10)^n$, $6^3 = 1(10)^n$, $7^1 = 1(111)^n$, $7^2 = 1(1 = 11)^n$, and $7^3 = 1(111)(111)\ldots(111)(111)$

(f) $4^1 = 1(10)^n$, $5^2 = (10)^{n+1}$, $6^3 = (01)^{n+1}$, $7^1 = 1(111)^n$, $7^2 = 11(111)(111)\ldots(111)(111)$, and $7^3 = 11(11 = 1)^n$

$t = 5$, $t = 6$: by the same interchanging as in the roots.

$t = 7$: We have two general forms:

1. $4^1 = (+)^n$

$5^2 = \begin{cases} 
o & \text{for } n = 0 \\
f_1 + f_3 & \text{for } n = 1 \\
f_1 + f_3, l_2, \ldots, l_i, \ldots, l_n & \text{for } n \geq 2
\end{cases}$
where \( l_i = f_{2i-2} + f_{2i-1} \), \( i = 2, 3, \ldots, n \);

\( 6 \hat{f} 3 = 5 2, 7 \hat{f} 1 = 1(01)^n, 7 \hat{f} 2 = 1(110)^n \), and \( 7 \hat{f} 3 = 1(011)^n \)

(b) \( 4 \hat{f} 1 = (10)^{n+1}, 5 \hat{f} 2 = 6 \hat{f} 3 = (01)^{n+1}, 7 \hat{f} 1 = \frac{1111111 \ldots 111}{n} \)

\( 7 \hat{f} 2 = \frac{1111111 \ldots 1111111}{n} \), \( 7 \hat{f} 3 = 11111111111 \ldots 111 \).

3.2 The indecomposable representations \( \tilde{C}Q_t \) of \( \tilde{E}_6 \).

The general forms of these representations are:

\( t = 1 \) We have the following three general forms:

(a) \( 4 \hat{f} 1 = 1(01)^n, 5 \hat{f} 2 = (10)(01)^{n-1}, 6 \hat{f} 3 = (10)^n \),

\( 7 \hat{f} 1 = 1101 (1 1 = 1)^{n-1}, (n \neq 0) \)

\[
7 \hat{f} 2 = \begin{cases} 
O & \text{for } n = 0 \\
 f_1, o, f_2, f_1 & \text{for } n = 1 \\
 f_1 + f_4, f_4, f_2, f_4, o, f_3, f_4 & \text{for } n = 2 \\
 f_1, f_4, f_4, f_2, f_1, m_1, \ldots, m_{n-2}, o, f_2n-1, f_{2n} & \text{for } n \geq 3 
\end{cases}
\]

where \( m_i = -f_{2i+4}, f_{2i+1}, f_{2i+2}, i = 1, 2, \ldots, n-2, \)

\( 7 \hat{f} 3 = \begin{cases} 
1 = 1 1 = 1 & \text{for } n = 0 \\
 f_1, f_1, f_2+f_3, f_2, f_3, o, f_4 & \text{for } n = 1 \\
 f_1, f_1, f_2+f_3+f_6, f_2, f_3, f_5, f_4, f_5, o, f_6 & \text{for } n = 2 \\
 f_1, f_1, f_2+f_3+f_6, f_2, m_1, \ldots, m_{n-2}, f_{2n-3}, f_{2n-1}, f_{2n-2}, f_{2n-1}, o, f_{2n} & \text{for } n \geq 3 
\end{cases}
\]

where \( m_i = f_{2i-1}, f_{2i+1}+f_{2i+4}, f_{2i}, i = 2, \ldots, n-2 \)

(b) \( 4 \hat{f} 1 = o(+)n, \)

\[
5 \hat{f} 2 = \begin{cases} 
1 & \text{for } n = 0 \\
 f_1 + f_3, f_2 & \text{for } n = 1 \\
 f_1 + f_3 - f_5, m_i, \ldots, m_{i-1}, f_{2n} & \text{for } n \geq 2 
\end{cases}
\]

where \( m_i = -f_{2i} + f_{2i+3}, 1, 2, \ldots, n-1 \)
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$6 \Phi 3 = \begin{cases} 
1 & \text{for } n = 0, 7 \Phi 1 = 01(011)^n, \\
+1 (01)^{n-1} & \text{for } n \geq 1 
\end{cases}$

$7 \Phi 2 = \begin{cases} 
10 & \text{for } n = 0 \\
10101 & \text{for } n = 1 \\
f_1, o, f_2, f_4, f_3, f_4, o, f_5 & \text{for } n = 2 \\
f_1, o, f_2, f_4, f_5, m_i^0, ..., m_i^n, f_2n, o, f_{2n+1} & \text{for } n \geq 3 
\end{cases}$

where $m_i^0 = f_{2i} + f_{2i+2i+3}, f_{2i-2}, f_{2i-1}, i = 3, ..., n-1$
and $7 \Phi 3 = (1 = 1 (110)^n)$

(c)

$4 \Phi 1 = 1(10)^n, 5 \Phi 2 = \begin{cases} 
01 & \text{for } n = 0 \\
010 & \text{for } n = 1 \\
010(+)^{n-1} & \text{for } n \geq 2 
\end{cases}$

$6 \Phi 3 = (+)^{n-1}, 7 \Phi 1 = (1 = 1 (101)^n)$

$7 \Phi 2 = \begin{cases} 
011 & \text{for } n = 0 \\
o, f_1, f_2 + f_3, m_1, ..., m_i, ..., m_n & \text{for } n \geq 1 
\end{cases}$

where $m_i = f_{2i-1}, f_{2i-2}, o, i = 1, 2, ..., n$

and

$7 \Phi 3 = \begin{cases} 
101 & \text{for } n = 0 \\
f_1, f_4, f_2, o, f_3, f_4 & \text{for } n = 1 \\
f_1, f_4, f_2, f_5 + f_6, f_3, f_4, o, f_5, f_6 & \text{for } n = 2 \\
f_1, f_4, f_2, f_5 + f_6, f_3, f_4, n_3, ..., n_i, ..., n_n, b, f_{2n+1}, f_{2n+2} & \text{for } n \geq 3 
\end{cases}$

where $n_i = -(f_{2i+1} + f_{2i+2}), f_{2i-1}, f_{2i}, i = 3, ..., n.$

$t = 2, t = 3$: by the same interchanging as in the roots.

$t = 4$: we have the following six general forms:

(a)

$4 \Phi 1 = \begin{cases} 
o & \text{for } n = 0 \\
11 & \text{for } n = 1 \text{, } 5 \Phi 2 = (+)^n \\
11(+)^{n-1} & \text{for } n \geq 2 
\end{cases}$

$6 \Phi 3 = \begin{cases} 
o & \text{for } n = 0 \\
+ & \text{for } n = 1 \\
o_1, ..., o_1, ..., o_{n-1}, f_1, f_{2n} & \text{for } n \geq 2 
\end{cases}$
where $o_i = f_{2i} + f_{2i+1}$, $i = 1, 2, \ldots, n-1$,

$7 \Phi 1 = (110)^n$, $7 \Phi 2 = (011)^n$ and $7 \Phi 3 = (101)^n$

(b) $4 \Phi 1 = 0(01)^n$, $5 \Phi 2 = (10)^n$, $6 \Phi 3 = (10)^n$, $7 \Phi 1 = 1(11 = 1)^n$, for $n = 0$

$$7 \Phi 2 = \begin{cases} 0 \\ \frac{1011}{1011 = 111 = 111 = \ldots = 111 = 111} \\ \text{for n} \geq 2 \end{cases}$$

and $7 \Phi 3 = (1 = 11)^n$.

(c)

$$4 \Phi 1 = (+)^n, \quad 5 \Phi 2 = \begin{cases} 1 \\ f_2, o_1', \ldots, o_i', \ldots, o_{n-1}', f_1 + f_{2n-1} \\ \text{for n} \geq 1 \end{cases}$$

where $o_i = f_{2i-1} + f_{2i-2}$, $i = 1, 2, \ldots, n-1$,

$6 \Phi 3 = 1(+)^n$, $7 \Phi 1 = 0(011)^n$, $7 \Phi 2 = 1(101)^n$ and $7 \Phi 3 = 1(110)^n$

(d) $4 \Phi 1 = 1(10)^n$, $5 \Phi 2 = 0(10)^n$, $6 \Phi 3 = 0(10)^n$,

$$7 \Phi 1 = \begin{cases} 1 = 1 \\ 11 = 110 \\ 11 = 110(110)^{n-1} \\ \text{for n} = 0, 1, 2 \ldots \end{cases}$$

$$7 \Phi 3 = \begin{cases} 10 \\ \frac{10\ 11 = 1+1}{10\ 11 = 1-1 = 11 = 1+1} \\ \frac{10\ 11 = 1-1 = 11 = 1-1 = \ldots = 11 = 1-1 = 11 = 1-1}{10\ 11 = 1-1 = 11 = 1-1 = \ldots = 11 = 1-1 = 11 = 1-1} \end{cases}$$

for $n = 0$,

for $n = 1, 2 \ldots$.

(e)

$$4 \Phi 1 = \begin{cases} + \quad \text{for n} = 0 \\ 0(+)^n1 \quad \text{for n} \geq 1 \end{cases}, \quad 5 \Phi 2 = 1(+)^n, \quad 6 \Phi 3 = 1(+)^n$$

$$7 \Phi 1 = \begin{cases} 11 \\ 11(101)^{n-1}101+1 \\ \text{for n} = 0, \quad 7 \Phi 2 = 10(110)^n, \quad \text{for n} \geq 1 \end{cases}$$

and $7 \Phi 3 = 01(110)^n$.

(f) $4 \Phi 1 = O(10)^n$, $5 \Phi 2 = (01)^{n+1}$, $6 \Phi 3 = (10)^{n+1}$, $7 \Phi 1 = 001(1 = 11)^n$, $7 \Phi 2 = (11 = 1)^{n+1}$

and
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$$7 \xi_3 = \begin{cases} 1111 & \text{for } n = 0 \\ \frac{111 = 111 = 111 = \ldots = 111 = 111}{n+1} & \text{for } n \geq 1 \end{cases}$$

$t = 5, t = 6$: By the same interchanging as in the roots.

$t = 7$ We have following two general forms:

(a) $4 \xi 1 = (+)^
u$, $5 \xi 2 = (01)^
u$, $6 \xi 3 = (10)^
u$,

$$7 \xi 1 = \begin{cases} 0 & \text{for } n = 0 \\ 1 = 1 = 1+1- (1 = 1+1 - 1)^
u & \text{for } n \geq 1 \end{cases}$$

$$7 \xi 2 = \begin{cases} 0 & \text{for } n = 0 \\ 1 = 110 (110)^
u & \text{for } n \geq 1 \end{cases}$$

and

$$7 \xi 3 = \begin{cases} 0 & \text{for } n = 0 \\ \frac{011 (01 +1) (011+1) \ldots (01 + 1)}{n-1} & \text{for } n \geq 1 \end{cases}$$

(b) $4 \xi 1 = 1(+)^
u$, $5 \xi 2 = \begin{cases} 1 & \text{for } n = 0 \\ 1+ & \text{for } n = 1, \\ 1(01)^
u & \text{for } n \geq 2 \end{cases}$

$$7 \xi 1 = 1 = 1(101+1)^
u , 7 \xi 2 = \begin{cases} 10 & \text{for } n = 0 \\ 10-1 -1 1 & \text{for } n = 1 \\ 10(1 = 1 = 1+1)^
u & \text{for } n \geq 2 \end{cases}$$

and

$$7 \xi 3 = \begin{cases} 01 (011+1)(011+1) \ldots (011+1) \\ n \end{cases}$$

4. The regular representations of $\widetilde{E}_6$:

The regular representations of $\widetilde{E}_6$ include the homogeneous and the nonhomogeneous regular representations. Therefore we give first the simple regular representations and then the indecomposable regular representations.
4.1: The simple regular representations of $\tilde{E}_6$:

For $E_6$ we have the following eight simple regular representations:

$$
\begin{align*}
E_0 &= 1 \\
E_1 &= 2 \\
E'_0 &= 1 \\
E'_1 &= 1 \\
E''_0 &= 1 \\
E''_1 &= 1
\end{align*}
$$

4.2: The indecomposable regular homogeneous representations of $\tilde{E}_6$.

We construct these representations for $n \geq 2$, they can be summarized in the following two cases:

Case 1: $n$ is odd

$$
\xi_1 = (F \oplus F \oplus \ldots \oplus F) \otimes F \to F \oplus F \oplus F \oplus \ldots \oplus F \oplus F
$$

$$(f_1, f_2, \ldots, f_n) \times m_1 \to (f_1 + f_2), f_3, f_4, \ldots, f_n, f_1) m_1$$

$$(C_1)_n = \{(f_1, f_2, f_3, f_4, \ldots, f_n, f_1), (f_1', f_2', f_3', f_4', \ldots, f_n', f_1'), (f_1'' f_2'', f_3'', f_4', \ldots, f_n'', f_1''), \ldots, (\bar{0}, \bar{0}, \ldots, \bar{0}) | (f_1', f_2', \ldots, f_n') \in F^n \} + (C_1)_n$$

$$(C_i)_n = C_i (i = 1, 2) \text{ in the case } \dim U = \dim V = n$$

Case 2: $n$ is even.

$$
\xi_2 = (F \oplus F \oplus \ldots \oplus F) \otimes F \to F \oplus F \oplus F \oplus \ldots \oplus F \oplus F
$$

$$(f_1, f_2, \ldots, f_n) \times m_2 \to (f_1 + f_2, 0, f_3, f_4, 0, f_5 + f_6, \ldots, f_{n-1}, f_n) m_2$$

$$(C_1)_n = \{(f_1, f_2, \ldots, f_n), (f_1, f_2, \ldots, f_n), (f_1, f_2, \ldots, f_n), \ldots, (\bar{0}, \bar{0}, \ldots, \bar{0}) | (f_1, f_2, \ldots, f_n) \in F^n \} + (C_1)_n$$

$$(C_i)_n = C_i (i = 1, 2) \text{ in the case } \dim U = \dim V = n$$

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$\mathfrak{f}_2 : (F \bigoplus F \bigoplus \ldots \bigoplus F) \otimes F \to F \bigoplus F \bigoplus F \bigoplus F \bigoplus F \bigoplus \ldots \bigoplus F \bigoplus F \bigoplus F$

$$(f_1, f_2, \ldots, f_n) \times n \to (o, f_1 + f_2, o, f_3 + f_4, o, \ldots, f_{n-3} + f_{n-2}, o, f_1 + f_2 + \ldots + f_n)_{m_2}$$

$C_1 = \{(f'_1 + f'_2, o, f'_3 + f'_4, o, \ldots, f'_{n-1} + f'_n, o), (f'_1, f'_2, \ldots, f'_n)\}$

$$(0, \ldots, 0) \mid (f'_1, \ldots, f'_n) \in F^n$$

$C_2 = \{(0, f_1 + f_2, o, f_3 + f_4, o, \ldots, f_{n-3} + f_{n-4}, o, f_1 + f_2 + \ldots + f_n), (f_1, f_2, \ldots, f_n) | (f_1, \ldots, f_n) \in F^n\} + C_1$

5. A method of constructing the $\Lambda$-lattices:

One can construct at once the $\Lambda$-lattices, where $\Lambda$ is the Baechstrom order of $\tilde{E}_6$.

Using the following method:

Let $x = (x_1, x_2, \ldots, x_7, j \mathfrak{m}_i, i = 1, 2, 3, j = 4, 5, 6, 7)$ be a representation of $\tilde{E}_6$, and let

$\dim x = (\dim x_i) = (n_i), i = 1, 2, \ldots, 7.$

Then the $\Lambda$-lattice $M$, which corresponds to $x$ has the following form:

$$
M = \begin{bmatrix}
R^{n_4} & R^{n_5} & R^{n_6} & R & \ldots & R \\
\pi^{n_4} & R^{n_5} & R^{n_6} & R & \ldots & R \\
\pi^{n_4} & \pi^{n_5} & R^{n_6} & R & \ldots & R \\
\pi^{n_4} & \pi^{n_5} & \pi^{n_6} & R & \ldots & R \\
\pi^{n_4} & \pi^{n_5} & \pi^{n_6} & \pi^{n_6} & \ldots & N \\
\pi^{n_4} & \pi^{n_5} & \pi^{n_6} & \pi^{n_6} & \ldots & N
\end{bmatrix}
$$

where $N$ is the $3 \times n_7$-matrix $(\text{Im } 7 \mathfrak{m}_1, \text{Im } 7 \mathfrak{m}_2, \text{Im } 7 \mathfrak{m}_3)^T$. It is clear that $R^{n_4}$ and $\text{Im } 7 \mathfrak{m}_1$ are related by $4 \mathfrak{m}_1$, $R^{n_5}$ and $\text{Im } 7 \mathfrak{m}_2$ are related by $5 \mathfrak{m}_2$, and $R^{n_6}$ and $\text{Im } 7 \mathfrak{m}_3$ are related by $6 \mathfrak{m}_3$.

Some examples of $\Lambda$-lattices: It is enough to give for each $\Lambda$-lattice the block $N$ and the relations indicated above.

(1) The following $\Lambda$-lattices are the lattices, which are correspond to the representations included in the general form (a) in 3.1 (t = 1), i.e. the representations $\mathcal{C}^+\mathcal{Q}_1$, $\mathcal{C}^+\mathcal{Q}_1$, \ldots. See also the general form (a) of roots in 2.1 (t = 1).
Note that we have used the following notations:

(i) $R \xrightarrow{R_i} R$ means $r \times r_i \ (\pi)$ for all $r \in R$, $r_i \in R_i$

(ii) $R \xrightarrow{R_j} R_{r_i}$ means $r = (r_i r_1 \ldots + r_t) \ (\pi)$ for all $r \in R$ and $r_s \in R_s$, $s = i, \ldots, t$

(iii) $R_i = R$ for all $r = 1, 2, \ldots$ and the $R''$ with the same index means there exists the relation $= (\pi)$ between the elements in $R''$.

(2) The following $\Lambda$-lattices are the lattices, which correspond to the representations included in the general form (b) in 3.1 ($t = 1$), i.e. the representations $C^+Q_1$, $C'^+Q_1$, $\ldots$. See also the general form (b) of roots in 2.1 ($t = 1$)
(3) $C^{+3}Q_1, C^{+5}Q_1, \ldots$ are:

\[
\begin{align*}
N &= \begin{bmatrix}
\pi & R & R & \pi & R & R & \pi & R & R \\
R & R & R & R & R & R & R & R & R
\end{bmatrix} \\
&\quad \text{at } n = 0 \\
&\quad \text{at } n = 1 \\
&\quad \text{at } n = 2
\end{align*}
\]

The following $\Lambda$ - lattices are the lattices, which correspond to the representations included in the general forms (a), (b) and (c) in 3.1 ($t = 4$) see also (a), (b), (c) in 2.1 ($t = 4$), i.e. the representations $C^{+4}Q_4, C^{+7}Q_4, \ldots, C^{+9}Q_4, C^{+11}Q_4, \ldots$ and $C^{+6}Q_4, C^{+8}Q_4, \ldots$

(4)

\[
\begin{align*}
N &= \begin{bmatrix}
\pi & R & R & \pi & R & R & \pi & R & R \\
R & R & R & \pi & R & R & \pi & R & R \\
R & R & R & R & R & R & R & R & R
\end{bmatrix} \\
&\quad \text{at } n = 0 \\
&\quad \text{at } n = 1 \\
&\quad \text{at } n = 2
\end{align*}
\]
REFERENCES


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