MORPHISMS OF AFFINE SCHEMES AND EQUIVALENCE OF CATEGORIES

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In this work we construct the relation between the category $R$-filt $(G(R)$-gr) and the category $Q$-tilt $(Q$-Gr).

The quantum (ungraded) version of this result can be obtained by restricting to parts of zero degree. Geometrically, we summarize geometrical properties for the affine scheme Spec$_R G(R)$. Finally, in the language of formal affine schemes we obtain the relation between $R$-filt $(G(R)$-gr) and the category $Q$-Filt $(Q$-Gr).

The results have a special meaning for sheaves of rings of differential operators.

Key words: Micro-Structure Sheaves, Zariski Filtered (rings) Sheaves, Quantum Sheaves, Affine Schemes and Morphisms

ABSTRACT:

In this work we construct the relation between the category $R$-filt $(G(R)$-gr) and the category $Q$-tilt $(Q$-Gr). The quantum (ungraded) version of this result can be obtained by restricting to parts of zero degree. Geometrically, we summarize geometrical properties for the affine scheme Spec$_R G(R)$. Finally, in the language of formal affine schemes we obtain the relation between $R$-filt $(G(R)$-gr) and the category $Q$-Filt $(Q$-Gr). The results have a special meaning for sheaves of rings of differential operators.

INTRODUCTION

For a long time after its introduction by Jerary, sheaf theory was mainly applied successfully to the theory of functions of several complex variables and to algebraic geometry, until it became a basic tool for almost all mathematics, and cohomology a natural language for many people.

In recent years new topological methods, especially the theory of sheaves founded by Jerary, have been applied successfully to algebraic geometry. In the study of micro-structure sheaves $Q$-tilt of filtered rings with Zariskian filtration, the author observed that, in order to find the regular and holonomic solutions of $O_X$ (we come back to this later), it is necessary first to study the application in section 3. Using the filtered micro-structure sheaves $Q$ and its associated graded sheaves $G(Q_X)$, it is possible to introduce the relation (commutative and non-commutative cases are considered) between the category of filtered (graded) affine schemes $(X,O_X)$ with filtered geometric space morphism and the category of Zariski filtered rings with
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filtered ring morphisms. Geometrically, we summarize geometrical properties for \( X = \text{Spec}^\mathbb{G}(R) \) in section 2. Finally, in section 4, in the language of formal affine schemes we obtain the relation between R-filt \((G(R)\text{-gr})\) and the category \( \mathcal{O}_X^\mu \). We can obtain the ungraded and quantum versions of our results, in sections 3 and 4, by restricting to the parts of zero degree. So far, this is the micro, formal and quantum versions of that application introduced in [5] as commutative case or that in [8] as non-commutative case.

I - PRELIMINARIES

We recall some basic notions here but for full details we have to refer to the references. Basic facts concerning schemes, affine schemes, coherent sheaves and formal completion may be found in [5]. Full detail on Zariskian filtration, filtrations and gradations on modules and micro-localizations of filtered modules may be found in [6], [7], [4]. For micro-structure sheaves ~\( Q^\mu \) and quantum sections \( F_0 Q^\mu \) over projective (affine) schemes, coherent sheaves over \( Q^\mu(R) \) and formal schemes and quantum sections over projective (affine) schemes one may use [9], [11], [2]. Finally on Zariskian filtrations on sheaves we have to refer to [3].

In the sequel we assume that \( F \) is a Zariskian filtration such that the associated graded ring

\[
G(R) = \bigoplus_{n \in \mathbb{Z}} (F_n R / F_{n-1} R) \cong R / X R
\]

is a commutative (Noetherian because of the Zariski hypothesis) strongly graded domain, this situation is general in the sense that it allows application of the results to most of the important examples; enveloping Algebras of Lie Algebras, Weyl Algebras, many rings of differential operators as well as the classical commutative Zariski rings.

We consider \( X = \text{Spec}^\mathbb{G}(R) \), the graded prime spectrum of \( G(R) \). Write \( \beta \) for the basis of the Zariski topology on \( X \) consisting of the basic open sets \( X(f) = \{ p \in X, f(p) \} \); \( f \in G(R) \) homogeneous element. We may define structure sheaves on \( X \). First we may associate to \( X(f) \), the graded commutative Noetherian ring \( Q^\mu(f)(G(R)) = G(R)[f^{-1}] \) and we obtain the graded Noetherian structure sheaf \( \mathcal{O}^\mu \) on \( X \) having as the stalk at \( p \in X \) the Noetherian local graded ring \( Q^\mu_{(f)}(G(R)) \).

Secondly we may associate to \( X(f) \) the Noetherian commutative ring \( G(R)[f^{-1}] \) and we obtain the usual structure sheaf \( \mathcal{O}_{G(R)} \) on \( X \) having as the stalk at \( p \in X \) the Noetherian local ring \( Q^\mu_{(f)}(G(R))_0 \). Associating to \( X(f) \) the micro-localizations, \( Q^\mu_{(f)}(R) \), resp. \( Q^\mu_{(f)}(R) \) we obtain sheaves \( \mathcal{O}^\mu_X \), resp. \( \mathcal{O}^\mu_X \), on \( X \), having as the completed stalks at \( p \in X \) the Noetherian rings \( \mathcal{O}^\mu_{(p)}(R) \), resp. \( \mathcal{O}^\mu_{(p)}(R) \). The quantization of the micro-level for \( G(R) \) is obtained by looking at the parts of filtration degree zero \( F_0 \mathcal{O}^\mu_X \). This may also be viewed as parts of degree zero in the graded sense for the corresponding Rees sheaves \( F_0 \mathcal{O}^\mu_X \equiv (\mathcal{O}^\mu_X)_0 \).

1. 1 Theorem: With the above notations, [1], [3]

\( 1^\mu - \mathcal{O}^\mu_X, F_0 \mathcal{O}^\mu_X \) are Zariski coherently filtered sheaves on \( X \).

2. \( \mathcal{O}^\mu_X \) is Noetherian graded sheaf on \( X \).

II - AFFINE GRADED-SHEAVES FOR \( G(R) \)

In this section, we summarize properties for \( X = \text{Spec}^\mathbb{G}(G(R)) \). In general, obviously \( X = \text{Spec}^\mathbb{G}(G(R)) \) is not a scheme. However in case \( G(R) \) is positively graded, then we write \( P(X) = \text{Proj}(G(R)) \) for the Zariski open subset of \( X \) consisting of the graded prime ideals not containing \( G(R)_+ = \bigoplus_{n \geq 0} (G(R))_n \), and in this case the closed set \( \text{V}(G(R)_+) \) in \( X \) is nothing but \( \text{Spec}(G(R)_+) \). We may define the \( P(X) = \text{Proj}(G(R)) \) generally as the subset in \( X \) given by the graded prime ideals \( p \) of \( G(R) \) such that there is an \( n \neq 0 \), \( G(R)_n \subset p \). Hence we have.

2.1 Theorem: \( X = p(X) \Leftrightarrow G(R) \) is quasi-strongly graded.

So, there is no sensitivity in studying many other problems like coherence and theorems of formal completions, ... Only we have to assume the condition that \( G(R) \) is quasi-strongly graded as we will see in section 4.

Let \( R, R' \) be non-commutative Zariski filtered rings and \( \phi : \ R \rightarrow R' \) be as strict epimorphism. Then \( \phi \) induces a continuous map

\[
\alpha G(\phi) : (\text{Spec}^\mathbb{G}(G(R')) \rightarrow (\text{Spec}^\mathbb{G}(G(R))
\]

which is

\[
\alpha G(\phi) : (\text{Spec}^\mathbb{G}(G(R'))) \rightarrow \text{Spec}^\mathbb{G}(G(R))
\]

defined by: \( q \rightarrow G(\phi)^{-1}(q) \), where \( G(\phi) : G(R) \rightarrow G(R') \). Moreover, if \( R'' \) is another non-commutative Zariski filtered ring and \( \phi' : R' \rightarrow R'' \) is another strict epimorphisms, then

\[
\alpha G(\phi') \circ G(\phi) = G(\phi) \circ \alpha G(\phi')
\]

From this it follows that \( \text{Spec}^\mathbb{G}(G) \) is a contravariant graded functor from the category of non-commutative Zariski filtered rings and strict epimorphisms to the category of topological spaces and continuous maps.

Also, we can prove the following properties.

2.2. Theorem: With the same considerations as above,
Spec⁰G(R) is an affine graded-scheme and has a basis for the Zariiski topology of affine Noetherian basic open sets.

2.3 Theorem: X = Spec⁰G(R) is compact and integral such that at each point p ∈ X, the coefficient for G(R) is the union of sections for G(R).

2.4 Theorem: with notations and considerations as above,

a-Since Spec⁰G(R) is path-connected then Spec⁰G(R) is not disconnected.

b-Spec⁰G(R) is irreducible and Noetherian.

2.5 Remark: On X, the graded sheaf G(Q(R)) is nothing but the graded structure sheaf O_X, since, for each X (X) we have

G(Q(R))((X(f)) = G(Q(R))((Y)) = G(Q(R)) = G(R)[f⁻¹] and for each p ∈ X, we have

G(Q(R) p) = G Q(R) p = Q(R).

Hence, as just terminology, we call (Spec⁰G(R), G(Q(R))) as affine graded scheme of G(R). Now a graded-scheme needs not be a scheme, but this is just terminology of course.

III- MORPHISMS OF AFFINE SCHEMES

Let R', R be non-commutative Zariski filtered rings and f : R' → R be strict filtered epimorphism.

Let f ∈ h(G(R)), p ∈ X = Spec⁰G(R) such that a(G)(p) = q ∈ Y = Spec⁰G(R'). By the exactness of the localization functors G(ϕ) induces a graded ring epimorphism G(R') → G(R) with G(ϕ) = G(R'/p).

Thus, G(ϕ) induces a graded ring epimorphism G(ϕ)' : G(R)' → G(R) as required.

Again, by the exactness of the localization functors associated to p, a(G)(p) = q, G(ϕ) induces a graded local ring epimorphism of the stalks G(ϕ) : G(R)' → G(R) which already is compatible with the restriction graded homomorphism, hence we have a graded sheaf epimorphism G(ϕ) : G(R)' → G(R) as required.

Therefore f : R' → R induces a graded morphism

a(G(ϕ)G(ϕ)) : (X, O_X) → (Y, O_Y) of graded affine schemes such that the graded homomorphisms induced on the stalks are graded local homomorphisms.

Conversely, let (ψ, ϕ) : (X, O_X) → (Y, O_Y) be an epimorphism (where X = Spec⁰G(R), Y = Spec⁰G(R') and O_X, O_Y are the graded structure sheaves O_X, O_Y) such that ϕ_q : G(R)'_q → G(R)_p is a graded local epimorphism for each p ∈ X (q = ψ(p)). (ψ, ϕ) determines a graded ring epimorphism G(ϕ) : G(R') → G(R) and then one may define a filtered epimorphism ϕ_i : R' → R such that G(ϕ_i) = G(ϕ).

Since the induced embedding ϕ_q : O_q → O_p of the residue fields is monomorphism, hence

ψ = a(G(ϕ) and ϕ_q : G(R)'_q → G(R)_p.

is graded epimorphism induced by ϕ. Therefore (ψ, ϕ) = (a(G(ϕ), G(ϕ)). We have therefore proved:

3.1 Theorem: with conventions and notations as before, there is a map from the strict non-commutative Zariski filtered ring epimorphism ϕ : R' → R to the graded epimorphisms (ψ, ϕ) : (X, O_X) → (Y, O_Y) such that

ϕ_q : O_q → O_p, p ∈ X, q = ψ(p), is local graded ring epimorphism.

3.2 Theorem: With conventions and notations as before, there is a one-to-one correspondence between the graded epimorphisms G(ϕ) : G(R') → G(R) where R', R are non-commutative Zariski filtered rings and the graded epimorphisms (ψ, ϕ) : (X, O_X) → (Y, O_Y) such that ϕ_q is a local epimorphism for each p ∈ X.

Now let us return to filtered setting. Let I ⊆ G(R) be a graded ideal in G(R), Y(I) be an affine basic Zariski open set in Y = Spec⁰G(R). Since G(ϕ) is a continuous map, hence a(G(ϕ)^{-1}(Y(I)) = X(G(ϕ)(I))) is an affine basic Zariski open set in X = Spec⁰G(R). We write K_1(n) for the kernel functor, induced by K_1, on R'/X^n R and K(n)G(ϕ)(I)) for the kernel functor, induced by K_1(I), on R'/X^n R. By the saturation condition, L(K_1(n))[L(K(n)G(ϕ)(I))] has a filter basis consisting of L/X^n L of R'/X^n R with L ∈ L(K_1) and J/X^n J of R'/X^n R with J ∈ L(K_1(I))).

Write Q_{n+1}(Q_nG(ϕ)(I)) for the localization functor associated to K_1(n)[K(n)G(ϕ)(I)). Since K_1(n)[K(n)G(ϕ)(I)] is perfect, hence ϕ induces the graded homomorphism.

Q_{n+1}(R'/X^n R) → Q_nG(ϕ)(I)(R'/X^n R)

Taking the inverse limit in the graded sense, we have the graded ring homomorphism

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\[ Q^R_k \rightarrow \varphi^R_k \]

Since \( \varphi^R_k(X) \rightarrow Q^R_k(R) \subset (X \rightarrow Q^R_k(R)) \), hence \( \varphi^R_k \) induces a filtered Zariski ring homomorphism

\[ \theta^R_k \]

Therefore, we have the filtered ring epimorphisms

\[ \theta^R_k \]

which are compatible with the restriction homomorphisms.

Now, \( \theta^Y_k \) defines \( \theta^Y_k \) as a filtered sheaf morphism. For every prime ideal \( \mathfrak{p} \in X \), \( q = G(\phi)(\mathfrak{p}) \in Y \) and since the local property exists in case of associated graded functor, hence the stalk sheaf morphism \( \theta^Y_k \) is filtered local homomorphism. All this leads to the following:

3.3 Theorem: There is a map from the strict non-commutative Zariski filtered ring epimorphisms \( \theta^R_k \rightarrow \theta^X_k \), which are compatible with the restriction homomorphisms.

4.1. Note

We can consider the defined sheaves as sheaves over \( Y \), if it is necessary, this always means the extension by zero outside \( Y \). Also, since the open affine Noetherian subsets of \( Y \) form a base of the topology on \( Y \), then the intersection of those with the closed subset \( \hat{Y} \) forms a basis for the topology on \( \hat{Y} \), see 2.2.

4.2. Theorem (Commutative Level)

There is an equivalence of categories between \( G(R)-gr_{\hat{Y}} \) and \( Q^R_{\hat{Y}}-Grcoh_{\hat{Y}} \). The zero-version of this result over \( (Y, Q^Y) \) is in fact corollary 9.9 of (5), Hence it is true.

Hence we consider the following theorem with its proof under the above consideration, but with \( Y = Spec^h(G(R)) \) and \( G(R) \) is I-adically complete, where I defines \( J^Y \).

4.2. Theorem (Commutative Level)

There is an equivalence of categories between \( G(R)-gr_{\hat{Y}} \) and \( Q^R_{\hat{Y}}-Grcoh_{\hat{Y}} \). The zero-version of this result over \( (Y, Q^Y) \) is in fact corollary 9.9 of (5), Hence it is true.
generated graded modules over $G(R)$ and $O^g_Y - Grcoh.$ is the category of coherent graded $O^g_Y$ modules.

Proof:

The functor $F_1: G(M) \rightarrow M^g_{Y}$ from $G(R)-grf$ to $O^g_Y - Grcoh.$ is a graded exact functor. Indeed, let $U \in \mathcal{B}(Y)$; say $U = \text{spec}(B)$ and $G(M) \in G(R)-grf.$ Hence $M^g_{Y}(U) = N \in B- grf$ is finitely generated graded $B$-module. Now $M^g_{Y}(U) = N$. Since we have finitely generated graded modules, hence the completion functor is exact. Therefore $G(M) a M^g_{Y}(U)$ is graded exact functor.

The functor $F_2: [(\hat{\gamma}, Q^g_Y)] \rightarrow (\hat{\gamma}, Q^g_Y)$ from $Q^g_Y - Grcoh.$ into $G(R)-grf$ is exact and $F_1, F_2$ are inverse to each other. It is clear that $\Gamma(\hat{\gamma}, M^g_{\hat{\gamma}}(U)) \equiv G(M).$ Thus, the two compositions of our two functors are the identity. But, to show that the graded functor $\Gamma(\hat{\gamma}, \cdot)$ is exact on the category $O^g_Y - Grcoh.$ let

$$0 \rightarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 0$$

be an exact sequence in $O^g_Y - Grcoh.$ Since $\Gamma(\hat{\gamma}, 1) \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow 0$ is exact. Therefore, $\Gamma(\hat{\gamma}, \cdot)$ is a graded exact functor.

4.3. Remark (Commutative real case)

Hartshorne’s version, [5], of this over $(\hat{\gamma}, Q^g_Y)$ is also true because, in this case, we have real schemes.

Let us consider the behaviour of the result 4.1. on the filtered level. Recall from [2] that for an ideal $I$ or $R$ equipped with the induced (hence good) filtration, we may define the filtered (non-commutative) Noetherian formal scheme over $Y$ as follows. Put $G(I) = I_g$ and $V(I_g) = \hat{\gamma}$. On $\hat{\gamma}$ we now introduce two structure coherently filtered sheaves $O^\mu_Y \equiv \text{Lim}_{\hat{\gamma}} (O^\mu_Y / (I^\mu_Y)^n)$ and $O^\eta_Y \equiv F_0 O^\mu_Y$ (the sheaf of quantum-sections over $\hat{\gamma}$), where we write $\text{Lim}_{\hat{\gamma}}$ to indicate that we have taken into account the filtration defined on $\text{Lim}_{\hat{\gamma}}$ as in [2].

Similarly, for $M \in R$-filtgood; filtered module with good filtration, the coherently filtered formal sheaves of $M^g_{\hat{\gamma}}$ and $M^\mu,g_{\hat{\gamma}}$ may be defined as

$$M^\mu,g_{\hat{\gamma}} = \text{Lim}_{\hat{\gamma}} (O^\mu_Y / (I^\mu_Y)^n)$$

respectively. Of course these define coherently filtered sheaves over $O^\mu_Y$ and $O^\eta_Y$ respectively, i.e. $M^\mu,g_{\hat{\gamma}} \in O^\mu_Y - \text{Filtcoh.}$.

4.4 Note

If $R$ is $I$-adic filtered complete, i.e. $R \equiv \text{Lim}_{\hat{\gamma}} R / I^n$, then

$$O^\mu_Y \equiv \text{Lim}_{\hat{\gamma}} (O^\mu_Y / (I^\mu_Y)^n) \equiv (\text{Lim}_{\hat{\gamma}} R / I^n)^\mu_Y \equiv O^\mu_Y$$

Conversely, if $O^\mu_Y \equiv O^\mu_Y$, we can assume that $\hat{\gamma} = Y$. Then this gives rise to that the ideal $I$ is contained in the prime radial of $R$. Therefore $I^n = 0$ and this implies that $\hat{\gamma} = I$.

Similarly, we can say that $I^\mu_Y$ is formal complete $\Rightarrow$ $I$ is $I$-adic
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filtered complete. It is clear that if $M$ is $I$-adic complete, then $M^\mu_Y$ will be formal complete, i.e. $M^\mu_Y \cong M^\mu_Y$. 

Combined with section 3 in [2] and the conventions assumed in 4.2. this yields the non-commutative filtered level of the result in 4.2.

4.5. Theorem (non-commutative level)

There is an equivalence of categories between $R$-filtgood and $Q^\mu_Y$-Filtcoh., where $R$-filtgood is the category of good filtered $I$-adically complete $R$-modules and $Q^\mu_Y$-Filtcoh. is the category of coherent filtered $Q^\mu_Y$-modules.

Proof:

The functor $F_1 : M \rightarrow M^\mu_\Delta$, from $R$-filtgood into $Q^\mu_Y$-Filtcoh., is exact. Indeed, let $U = Y(f) = \text{spec } G(Q_f(R))$ for suitable $f \in R$ with $\sigma(f) \in G(R)$, for the definition of $\sigma(f)$ see [7]. By the well known exactness property of $Q^\mu_f(-)$ on strict sequences we obtain that $\Gamma(Y(f), -)$ from $R$-filtgood to $Q^\mu_f(R)$-filtgood is exact. Since $\Gamma(Y(f), M^\mu_\Delta) = Q_f^\mu(M)^{\hat{\Lambda}^1}$ and the $Q^\mu_f(I)$-adic completion is exact on good filtration, it follows that $M \rightarrow M^\mu_\Delta$ is filtered exact functor.

The functor $F_2 : Y(\hat{\Sigma}^\mu_Y \Delta^\mu_Y)$ from $Q^\mu_Y$-Filtcoh., to $R$-filtgood, is exact. Indeed, consider

$0 \rightarrow \Sigma^\mu_Y \rightarrow \Sigma^\mu_Y \rightarrow \Sigma^\mu_Y \rightarrow 0$  

(4)

an exact sequence in $Q^\mu_Y$-Filtcoh. Since $\Sigma^\mu_Y \equiv M^\mu_1_Y$ for some $M_1$; $i = 1, 2, 3$ in $R$-filtgood it follows that (4) becomes

$0 \rightarrow M^\mu_1_Y \rightarrow M^\mu_2_Y \rightarrow M^\mu_3_Y \rightarrow 0$  

(5)

Now it is sufficient to compare (5) and (6) to get $M^\mu_4_Y = 0$.

Finally, clearly $F_1F_2 = 1$ and $F_2F_1 = 1$.

From all of this, the theorem follows.

4.6. Final remark

1 - All arguments still hold if $\Delta$ is replaced by $q$ in 4.5. This gives the quantum formal version of the result 4.5.

2 - The fact that the micro-or quantum-version of Serre’s global sections results for coherent sheaves is also true. This follows from theorem 4.5.

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