A PROBLEM IN THERMOELASTICITY WITH TWO RELAXATION TIMES FOR AN INFINITE THERMOELASTIC LAYER

By

Hany H. Sherief* and Heba A. Saleh**

* Faculty of Science, University of Qatar, Doha , Qatar
** Faculty of Science, University of Alexandria, Alexandria, Egypt

مسألة في المرونة الحرارية ذات الزمنين الإسترخائيين لشريحة سميكة لا نهائية

هاني حسن شريف و هبه علي صالح

تم في هذا البحث حل مسألة لشريحه سميكه محدود وممتدة إلى ما لانهاية تخضع لمعادلات نظرية المرونة الحرارية المعممة ذات الزمنين الإسترخائيين . أخذ السطح العلوي للشريحه خاليا من الإجهاد ومعرضاً لصدمه حرارية . السطح السفلي للشريحه مستقر على قاعده صلبه معزولة حراريا . أستخدم في الحل طريقة تحويل لابلاس . حلت المسألة بطريقة مباشرة في مجال تحويل لابلاس . تم إيجاد تحويلات لابلاس العكسية بطريقة تحليلية وذلك باستخدام مفكوكات تقاربية صالحه لقيم الزمن الصغيره . يحتوي البحث أيضاً على حسابات عددية لقيم دوال الحرارة والإزاحة والإجهاد كما تم تمثيل هذه الدول بيانياً .

Key Words : Thermoelasticity - Generalized Thermoelasticity - Thermal Stresses.

ABSTRACT

The problem of a thermoelastic layer of finite thickness and infinite extent is considered within the context of the theory of thermoelasticity with two relaxation times. The upper surface of the layer is taken as stress free and is suddenly subjected to a thermal shock. The lower surface of the layer rests on a rigid base that is thermally insulating. Laplace transform techniques are used. The problem is solved by using a direct approach. The inverse Laplace transforms are obtained analytically by using asymptotic expansions valid for small values of time. Numerical computations for the temperature, the displacement and stress distributions are carried out and represented graphically.

NOMENCLATURE

λ , μ Lamé's constants

α_t coefficient of linear thermal expansion

$\beta = [(\lambda + 2\mu) / \mu]^{1/2}$

$\gamma = (3 \lambda + 2\mu) \alpha_t$

ρ = density

$V_1 =$ speed of propagation of isothermal longitudinal waves = $[(\lambda + 2 \mu)/\rho]^{1/2}$

$\sigma = \sigma_{xx}$ component of the stress tensor in x-direction

u   component of displacement in x-direction

c_E specific heat at constant strain

k   thermal conductivity

$\eta = \rho c_E / k$

t   time

T   absolute temperature

T_0 reference temperature chosen so that $[(T - T_0)/T_0] \ll 1$

$b = \gamma T_o / \mu$

$g = \gamma / \rho c_E$

τ, ν relaxation times

q the heat flux normal to the surface of the layer

INTRODUCTION

Biot [1] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equations for both theories are of the diffusion type predicting infinite speeds of propagation for heat waves contrary to physical observations.

Lord and Shulman [2] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. This theory was extended [3] by Dhaliwal and Sherief to include the anisotropic case. In this theory a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. Uniqueness of solution of this theory was proved under different conditions by J. Ignaczak in [4], [5], by Sherief and Dhaliwal in [6], [3] and by Sherief in [7]. The state space approach to this theory was developed by Anwar and Sherief in [8] and by Sherief in [9]. The boundary integral equation formulation was conducted by Anwar and Sherief in [10]. The fundamental solution for this theory was obtained by Sherief in [11].

Green and Lindsay [12] developed the theory of generalized thermoelasticity with two relaxation times which is based on a generalized inequality of thermodynamics. This theory does not violate Fourier's law of heat conduction when the body under consideration has a centre of symmetry. In this theory both the equations of motion and of heat conduction are hyperbolic but the equation of motion is modified and differs from that of the coupled thermoelasticity theory. This theory was initiated by Müller [13]. It was further extended by Green and Laws [14]. The final form used in the present work is that of Green and Lindsay [12]. This theory was also obtained independently by Şuhubi [15]. Longitudinal wave propagation for this theory was studied by Erbay And Şuhubi in [16]. Ignaczak investigated a strong dis-

continuity wave and proved a decomposition theorem for this theory in [17] and [18], respectively. Sherief has obtained the fundamental solution for this theory in [19] Sherief has also formulated the state space approach in [20] and solved a thermo-mechanical shock problem in [21]. The boundary integral equation formulation was done by Anwar and Sherief in [22].

FORMULATION OF THE PROBLEM

In this work we shall consider a homogeneous, isotropic, thermoelastic solid occupying the region of a layer $0 \leq x \leq h$. The lower surface of the layer (x=h) is taken to be thermally insulated and has a rigid foundation. The upper surface of the layer (x=0) is suddenly heated and kept at a constant temperature and is stress free.

We assume that there are no external forces or heat sources acting inside the region. Since the layer is extending to infinity in both y and z directions, the problem is essentially one-dimensional. The displacement components thus have the form

$$u_x = u(x,t) \quad , \quad u_y = u_z = 0.$$

The strain tensor components are given by

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{xy} = e_{yz} = e_{zx} = e_{yy} = e_{zz} = 0.$$

The cubical dilatation e is equal to

$$e = \frac{\partial u}{\partial x}. \tag{1}$$

The solid is assumed to obey the equations of thermoelasticity with two relaxation times. These equations are [12]

(1) The equation of motion

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - \gamma \left[\frac{\partial T}{\partial x} + \nu \frac{\partial^2 T}{\partial x \, \partial t} \right]. \tag{2}$$

(2) The energy equation

$$k \frac{\partial^2 T}{\partial x^2} = \rho \, c_E \left[\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} \right] + \gamma \, T_o \frac{\partial^2 u}{\partial x \partial t}. \tag{3}$$

(3) The constitutive equations

$$\sigma = \sigma_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} - \gamma \left[T - T_o + \nu \frac{\partial T}{\partial t} \right], \tag{4}$$

292

$$\sigma_{yy} = \sigma_{zz} = \lambda \frac{\partial u}{\partial x} - \gamma \left[T - T_o + v \frac{\partial T}{\partial t} \right], \quad (5)$$

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0.$$

These equations can be put into a more convenient form by using the following non-dimensional variables :

$$x^* = v_1 \eta x , \quad t^* = v_1^2 \eta t , \theta = (T - T_o) / T_o,$$

$$u^* = v_1 \eta u , \tau^* = v_1^2 \eta \tau, v^* = v_1^2 \eta v , \sigma_{ij}^* = \sigma_{ij} / \mu .$$

Using the above variables equations (2) - (5), take the following form where we have dropped the asterisks for convenience .

$$\frac{\partial^2 u}{\partial x^2} - \frac{b}{\beta^2} \frac{\partial}{\partial x} \left[\theta + v \frac{\partial \theta}{\partial t} \right] = \frac{\partial^2 u}{\partial t^2} , \quad (6)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} + \tau \frac{\partial^2 \theta}{\partial t^2} + g \frac{\partial^2 u}{\partial x \partial t} , \quad (7)$$

$$\sigma = \beta^2 \frac{\partial u}{\partial x} - b \left[\theta + v \frac{\partial \theta}{\partial t} \right] , \quad (8)$$

$$\sigma_{yy} = \sigma_{zz} = (\beta^2 - 2) \frac{\partial u}{\partial x} - b \left[\theta + v \frac{\partial \theta}{\partial t} \right]. \quad (9)$$

Since the upper surface is stress free and is kept at a constant temperature θ_0, the boundary conditions there take the form

$$\sigma (x, t) \big|_{x=0} = 0, \quad (10)$$

$$\theta (x, t) \big|_{x=0} = \theta_0 H(t), \quad (11)$$

where H(t) is the Heavyside unit step function.
The lower surface has a rigid base and is thermally insulated so the boundary conditions there take the form

$$u (x, t) \big|_{x=h} = 0, \quad (12)$$

$$q \big|_{x=h} = 0.$$

Using Fourier's law of heat conduction, namely $q = -k \, \partial\theta/\partial x$, the last condition reduces to

$$\frac{\partial \theta}{\partial x} \bigg|_{x=h} = 0. \quad (13)$$

The initial conditions are taken to be homogeneous, i.e.

$$u (x, t) \big|_{t=0} = \frac{\partial u (x, t)}{\partial t} \bigg|_{t=0} = 0, \quad (14)$$

$$\sigma (x, t) \big|_{t=0} = \frac{\partial \sigma (x, t)}{\partial t} \bigg|_{t=0} = 0, \quad (15)$$

$$\theta (x, t) \big|_{t=0} = \frac{\partial \theta (x, t)}{\partial t} \bigg|_{t=0} = 0, \quad (16)$$

SOLUTION IN THE LAPLACE TRANSFORM DOMAIN

Differentiating equation (2) with respect to x and using equation (1), equations (2)-(3) can be written as

$$\frac{\partial^2 c}{\partial t^2} = \frac{\partial^2 c}{\partial x^2} - \frac{b}{\beta^2} \frac{\partial^2}{\partial x^2} \left[\theta + v \frac{\partial \theta}{\partial t} \right] , \quad (17)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} + \tau \frac{\partial^2 \theta}{\partial t^2} + g \frac{\partial c}{\partial t} . \quad (18)$$

Introducing the Laplace transform defined by the formula

$$\bar{f} (p) = \int_0^\infty e^{-pt} f(t) \, dt ,$$

into equations (17), (18), (8) and (9), we get upon using the initial conditions (14)-(16)

$$\beta^2 (D^2 - p^2) \bar{c} = b (1 + vp) D^2 \bar{\theta}, \quad (19)$$

$$(D^2 - p - \tau p^2) \bar{\theta} = g p \bar{c}, \quad (20)$$

$$\bar{\sigma} = \beta^2 \bar{c} - b (1 + vp) \bar{\theta}, \quad (21)$$

$$\bar{\sigma}_{yy} = \bar{\sigma}_{zz} = (\beta^2 - 2) \bar{c} - b (1 + vp) \bar{\theta}, \quad (22)$$

where D stands for $\partial / \partial x$.

Eliminating $\bar{\theta}$ between equations (19) and (20), we

get the following fourth-order differential equation for the function $\bar{e}$

$$\{[D^4 - D^2[\,(1+\varepsilon)\,p + (1+\tau+\varepsilon v)\,p^2] + p^3\,(1+\tau p)\}\bar{e} = 0. \qquad (23)$$

The general solution of equation (23) can be written as

$$\bar{e} = \sum_{i=1}^{2} A_i \cosh k_i\,(h-x) + B_i \sinh k_i(h-x) \qquad (24)$$

where A_1 , A_2 , B_1 and B_2 are parameters depending on p only to be determined from the boundary conditions and k_1, k_2 are the roots with positive real parts of the characteristic equation

$$k^4 - k^2\,[(1+\varepsilon)\,p + (1+\tau+\varepsilon v\,)\,p^2] + p^3\,(1+\tau p) = 0 \qquad (25)$$

Similarly, eliminating $\bar{e}$ between equations (19) and (20), we see that the function $\bar{\theta}$ satisfies the differential equation (23), we thus have .

$$\bar{\theta} = \sum_{i=1}^{2} A_i' \cosh k_i\,(h-x) + B_i' \sinh k_i\,(h-x) \qquad (26)$$

where A_1', A_2', B_1' and B_2' are parameters depending on p only.

From equation (19), it follows that the parameters must satisfy the compatibility conditions

$$\beta^2\,(k_i^2 - p^2)\,A_i = b\,(1+v\,p)\,k_i^2\,A_i',$$

$$\beta^2\,(k_i^2 - p^2)\,B_i = b\,(1+v\,p)\,k_i^2\,B_i', \quad i = 1,2 \qquad (27)$$

Substituting from equations (27) into equation (26), we obtain

$$\bar{\theta} = \frac{\beta^2}{b\,(1+v\,p)} \sum_{i=1}^{2} \left[\frac{A_i\,(k_i^2 - p^2)}{k_i^2} \cosh k_i\,(h-x) \right.$$

$$\left. + \frac{B_i\,(k_i^2 - p^2)}{k_i^2} \sinh k_i\,(h-x) \right]. \qquad (28)$$

Integrating both sides of equation (1) with respect to x and using equation (24), we obtain

$$\bar{u} = \sum_{i=1}^{2} \frac{A_i}{k_i} \sinh k_i\,(h-x) + \frac{B_i}{k_i} \cosh k_i\,(h-x). \qquad (29)$$

Using the Laplace transform of the boundary conditions (10)-(13) together with equations (21), (24) (28) and (29), we arrive at the following set of linear equations

$$\sum_{i=1}^{2} \left[\frac{A_i \cosh k_i\,h + B_i \sinh k_i\,h}{k_i^2} \right] = 0,$$

$$\sum_{i=1}^{2} \left[\frac{k_i^2 - p^2}{k_i^2} \right] \{A_i \cosh k_i\,h + B_i \sinh k_i\,h\}$$

$$= \frac{\theta_0\,b\,(1+vp)}{\beta^2\,p} ,$$

$$\sum_{i=1}^{2} \frac{B_i}{k_i} = 0,$$

$$\sum_{i=1}^{2} \frac{B_i\,(k_i^2 - p^2)}{k_i^2} = 0,$$

Solving the above system of linear equations, we obtain

$$B_1 = B_2 = 0$$

$$A_1 = \frac{\theta_0\,b\,(1+v\,p)\,k_1^2}{\beta^2\,p\,(k_1^2 - k_2^2)\cosh k_1 h} , \qquad (30)$$

$$A_2 = \frac{-\theta_0\,b\,(1+v\,p)\,k_2^2}{\beta^2\,p\,(k_1^2 - k_2^2)\cosh k_2 h} .$$

Substituting from equations (30) into equations (24), (28) and (29), we get

$$\bar{e} = \frac{\theta_0\,b\,(1+vp)}{\beta^2\,p\,(k_1^2 - k_2^2)} \left[\frac{k_1^2 \cosh k_1\,(h-x)}{\cosh k_1 h} - \frac{k_2^2 \cosh k_2\,(h-x)}{\cosh k_2 h} \right], \quad (31)$$

$$\bar{\theta} = \frac{\theta_0}{p\,(k_1^2 - k_2^2)} \left[\frac{(k_1^2 - p^2)\cosh k_1\,(h-x)}{\cosh k_1 h} - \frac{(k_2^2 - p^2)\cosh k_2\,(h-x)}{\cosh k_2 h} \right],$$

$$\qquad (32)$$

$$\bar{u} = \frac{-\theta_0\,b\,(1+vp)}{\beta^2\,p\,(k_1^2 - k_2^2)} \left[\frac{k_1 \sinh k_1\,(h-x)}{\cosh k_1 h} - \frac{k_2 \sinh k_2\,(h-x)}{\cosh k_2 h} \right]. \quad (33)$$

From equations (31), (32) and (21), we obtain

$$\bar{\sigma} = \frac{\theta_0\,bp\,(1+vp)}{\beta^2\,p\,(k_1^2 - k_2^2)} \left[\frac{\cosh k_1\,(h-x)}{\cosh k_1 h} - \frac{\cosh k_2\,(h-x)}{\cosh k_2 h} \right]. \quad (34)$$

294

Equations (31) - (33) complete the solution of the problem in the Laplace transform domain. To obtain the solution in the physical domain, we shall obtain the inverse Laplace transforms using asymptotic expansions valid for small values of time. This method was used by Sherief in [19] and [21].

INVERSION OF THE LAPLACE TRANSFORMS

Let us now determine the inverse transforms for the case of small values of time. By the initial value theorem of the Laplace transforms [23] this corresponds to large values of p. We note first that the roots k_1^2 and k_2^2 of equation (25) have the form.

$$k^2_{1,2} = \frac{p}{2} \left[1 + \varepsilon + p(1 + \tau + \varepsilon v) \right.$$
$$\left. \pm \sqrt{\{1 + \varepsilon + p(1 + \tau + \varepsilon v)\}^2 - 4p(1 + \tau p)} \right]. \quad (35)$$

Taking $y = p^{-1}$ (y is small) then equations (35) can be written as

$$k_i^2 = p^2 f_i(y) , \quad i = 1,2. \quad (36)$$

where

$$f_1(y) = \tfrac{1}{2} \left[(1+\varepsilon) y + 1 + \tau + \varepsilon v + f(y) \right], \quad (37a)$$

$$f_2(y) = \tfrac{1}{2} \left[(1+\varepsilon) y + 1 + \tau + \varepsilon v - f(y) \right]. \quad (37b)$$

In the above equations, the function $f(y)$ is given by

$$f(y) = \left[(1 + \varepsilon)^2 y^2 + 2 \left[(1 + \varepsilon)(\varepsilon v + \tau) + \varepsilon - 1 \right] y \right.$$
$$\left. + 1 + (\varepsilon v + \tau)^2 + 2(\varepsilon v - \tau) \right]^{1/2}. \quad (38)$$

Expanding the function $f(y)$ into a Maclaurin series of which the first five terms are retained, we obtain after some manipulations.

$$f_i(y) = a_{i0} + a_{i1} y + a_{i2} y^2 + a_{i3} y^3 + a_{i4} y^4 , i = 1,2 \quad (39)$$

where

$$a_{10} = \frac{1 + \tau + \varepsilon v + A}{2} , \quad a_{20} = \frac{1 + \tau + \varepsilon v - A}{2} ,$$

$$a_{11} = \frac{(1 + \varepsilon) A + B}{2 A} , \quad a_{21} = \frac{(1 + \varepsilon) A - B}{2 A} ,$$

$$a_{12} = \frac{\varepsilon C}{A^3} , \quad a_{22} = -\frac{\varepsilon C}{A^3} ,$$

$$a_{13} = \frac{-\varepsilon B C}{A^5} , \quad a_{23} = \frac{\varepsilon B C}{A^5} ,$$

$$a_{14} = \frac{-\varepsilon C D}{4 A^7} , \quad a_{24} = \frac{\varepsilon C D}{4 A^7} ,$$

and

$$A = \sqrt{1 + (\varepsilon v + \tau)^2 + 2(\varepsilon v - \tau)} ,$$

$$B = (\varepsilon + 1)(\varepsilon v + \tau) + \varepsilon - 1 ,$$

$$C = 1 + (\varepsilon + 1)(v - \tau) ,$$

$$D = (\varepsilon + 1)^2 A^2 - 5 B^2 .$$

From equations (36) and (37a,b), we get

$$\frac{1}{k_1^2 - k_2^2} = \frac{1}{p^2 f(y)} . \quad (40)$$

This can be written as

$$\frac{1}{k_1^2 - k_2^2} = \frac{1}{p^2 A} \left[1 + \frac{2By + (1 + \varepsilon)^2 y^2}{A^2} \right]^{-1/2} . \quad (41)$$

We shall use the binomial expansion

$$(1 + z)^{-1/2} = 1 - \frac{z}{2} + \frac{3z^2}{8} - \frac{5z^3}{16} + \frac{35z^4}{128} , \, |z| < 1, \quad (42)$$

with $z = \dfrac{2By + (1 + \varepsilon)^2 y^2}{A^2}$.

We note that since y and ε are very small and A,B are close to unity then it follows that $|z| < 1$ and the above binomial expansion is valid. Thus, equations (41) and (42) yield after some algebraic manipulations (neglecting terms of higher order than y^4)

$$\frac{1}{k_1^2 - k_2^2} = \frac{1}{p^2} \sum_{j=0}^{4} b_j y^j, \quad (43)$$

where

$$b_0 = \frac{1}{A} \ ,$$

$$b_1 = -\frac{B}{A^3} \ ,$$

$$b_2 = \frac{3B^2 - (1 + \varepsilon)^2 A^2}{2 A^5} \ ,$$

$$b_3 = \frac{B [3(1 + \varepsilon)^2 A^2 - 5B^2]}{2 A^7} \ ,$$

$$b_4 = \frac{3(1 + \varepsilon)^4 A^4 - 30A^2 B^2 (1 + \varepsilon)^2 + 35 B^4}{8 A^9} \ ,$$

From equations (36) and (39), we obtain

$$k_i = p \sqrt{a_{i0}} \left[1 + \frac{a_{i1} y + a_{i2} y^2 + a_{i3} y^3 + a_{i4} y^4}{a_{i0}} \right] \ ,$$
$$i = 1,2 \ . \qquad (44)$$

We shall use the binomial expansion

$$(1 + z)^{1/2} = 1 + \frac{z}{2} - \frac{z^2}{8} + \frac{z^3}{16} - \frac{5 z^4}{128} \ , \ |z| < 1 \ , \quad (45)$$

with $z = \dfrac{a_{i1} y + a_{i2} y^2 + a_{i3} y^3 + a_{i4} y^4}{a_{i0}}$.

As before, from the values of the parameters a_{ij} , i = 1,2 , j = 1,2,3,4, we conclude that $|z| < 1$ and the expansion (45) is valid for this choice of z .

performing the necessary calculations and neglecting terms of order higher than y^4, we obtain.

$$k_i = p \sum_{j=0}^{4} b_{ij} y^j \ , \ i = 1,2 \ , \qquad (46)$$

where

$$b_{i0} = \sqrt{a_{i0}} \ ,$$

$$b_{i1} = \frac{a_{i1}}{2 b_{i0}} \ ,$$

$$b_{i2} = \frac{4 a_{i0} a_{i2} - a_{i1}^2}{8 a_{i0} b_{i0}} \ ,$$

$$b_{i3} = \frac{8 a_{i0}^2 a_{i3} - 4 a_{i0} a_{i1} a_{i2} + a_{i1}^3}{16 a_{i0}^2 b_{i0}} \ ,$$

$$b_{i4} = \frac{64 a_{i0}^3 a_{i4} - 32 a_{i0}^2 a_{i1} a_{i3} - 16 a_{i0}^2 a_{i2}^2 + 24 a_{i0} a_{i1}^2 a_{i2} - 5 a_{i1}^4}{16 a_{i0}^3 b_{i0}} \ ,$$

Since the parameter p is large, it follows from equation (46) that the roots k_i , i = 1,2 also take large values. Thus, the expression.

$$\frac{\cosh k_i (h - x)}{\cosh k_i h} \qquad i = 1, 2$$

can be approximated as follows

$$\frac{\cosh k_i (h - x)}{\cosh k_i h} = \cosh k_i x - \tanh k_i h \sinh k_i x$$
$$\approx \cosh k_i x - \sinh k_i x = e^{-k_i x} \ , i = 1, 2 \qquad (47)$$

since for large k_i, $\tanh k_i x \approx 1$. Substituting for k_1 and k_2 from equaions (46) and retaining only the first three terms, we obtain

$$\frac{\cosh k_i (h - x)}{\cosh k_i h} = e^{-x(b_{i0} p + b_{i1} + b_{i2}/p)} \ , i = 1,2 \qquad (48a)$$

In a similar manner, it can be shown that for large p, we have

$$\frac{\sinh k_i (h - x)}{\cosh k_i h} = e^{-x(b_{i0} p + b_{i1} + b_{i2}/p)} \ , i = 1,2 \qquad (48b)$$

STRESS DISTRIBUTION

Substituting from equations (43) and (48a) into equation (34), we obtain

$$\overline{\sigma} = \theta_0 b \sum_{j=0}^{4} \frac{c_j}{p^j} [e^{-x(b_{10} p + b_{11} + b_{12}/p)} \qquad (49)$$
$$- e^{-x(b_{20} p + b_{21} + b_{22}/p)}]$$

where
$$c_0 = v \, b_0 \ , \ c_j = v \, b_j + b_{j-1} \ , j = 1,2,3,4.$$

Taking the inverse Laplace transform of both sides of equation (49), we arrive at

$$\sigma = \theta_0 b \left[e^{-b_{11} x} \left(c_0 \mathcal{L}^{-1} [e^{-b_{10} x p} e^{-b_{12} x/p}] \right. \right.$$
$$\left. \left. + \sum_{j=0}^{3} c_{j+1} \mathcal{L}^{-1} \left\{ \frac{e^{-b_{10} x p} e^{-b_{12} x/p}}{p^{j+1}} \right\} \right) \right.$$

296

$$- e^{-b_{21}x} \left(c_0 \mathcal{L}^{-1} \left[e^{-b_{20}xp} e^{-b_{22}x/p} \right] \right.$$

$$\left. - \sum_{j=0}^{3} c_{j+1} \mathcal{L}^{-1} \left\{ \frac{e^{-b_{20}xp} e^{-b_{22}x/p}}{p^{j+1}} \right\} \right) \Bigg]$$

$$(50)$$

We shall make use of the convolution theorem of the Laplace transform [23], namely

$$\mathcal{L}^{-1} \left[\bar{g}_1(p) \, \bar{g}_2(p) \right] = \int_0^t g_1(z) \, g_2(t-z) \, dz \ ,$$

and the following relations from [24]

$$\mathcal{L}^{-1} \left[e^{-\alpha p} \right] = \delta(t - \alpha) \ ,$$

$$\mathcal{L}^{-1} \left[e^{-\alpha/p} \right] = \delta(t) - \sqrt{\alpha/t} \ J_1(2\sqrt{\alpha t}) \ , \ \alpha > 0 \ ,$$

$$\mathcal{L}^{-1} \left[e^{\alpha/p} \right] = \delta(t) + \sqrt{\alpha/t} \ I_1(2\sqrt{\alpha t}) \ , \ \alpha > 0 \ ,$$

$$\mathcal{L}^{-1} \left[\frac{e^{-\alpha/p}}{p^{j+1}} \right] = \left(\frac{t}{\alpha} \right)^{j/2} J_j(2\sqrt{\alpha t}) , \ \mathrm{Re}\,(j) > -1 \, , \alpha > 0 \ ,$$

$$\mathcal{L}^{-1} \left[\frac{e^{\alpha/p}}{p^{j+1}} \right] = \left(\frac{t}{\alpha} \right)^{j/2} I_j(2\sqrt{\alpha t}) , \ \mathrm{Re}\,(j) > -1 \, , \alpha > 0 \ ,$$

where J_j and I_j are the Bessel and the modified Bessel functions of the first kind of order j, respectively.

It can be easily shown that the Dirac delta function satisfies the following relation.

$$\int_0^t \delta(z - \alpha) \, f(z) \, dz = f(\alpha) \ H(t - \alpha) \ H(\alpha) \ .$$

Using the above formulae and inverting each term in equation (50) separately, we obtain .

$$\mathcal{L}^{-1} \left[\frac{e^{-b_{10}xp} e^{-b_{12}x/p}}{p^{j+1}} \right] = \int_0^t \left\{ \mathcal{L}^{-1} \left[e^{-b_{10}xp} \right] \right\}_{t=t-z} \mathcal{L}^{-1} \left[\frac{e^{-b_{12}z/p}}{p^{j+1}} \right]_{t=z} dz$$

$$= \int_0^t \delta(t - z - b_{10}x) \left[\frac{z}{b_{12}x} \right]^{j/2} J_j(2\sqrt{b_{12}x \, z}) \, dz$$

$$= \left[\frac{t - b_{10}x}{b_{12}x} \right]^{j/2} J_j \left\{ 2\sqrt{b_{12}x(t - b_{10}x)} \right\} H(b_{10}x) \, H(t - b_{10}x) \ .$$

In the above expression, we have used the easily checked fact that $b_{12} > 0$.

Using the abbreviations

$$x_1 = \frac{t - b_{10}x}{b_{12}x} \ , \ z_1 = 2\sqrt{b_{12}x(t - b_{10}x)} \ ,$$

and the fact that $b_{10} > 0$, the last result reduces to

$$\mathcal{L}^{-1} \left[\frac{e^{-b_{10}xp} e^{-b_{12}x/p}}{p^{j+1}} \right] = x_1^{j/2} J_j(z_1) H(t - b_{10}x) . \quad (51)$$

Similary, noting that $b_{22} < 0$ and $b_{20} > 0$, we obtain

$$\mathcal{L}^{-1} \left[\frac{e^{-b_{20}xp} e^{-b_{22}x/p}}{p^{j+1}} \right] = x_2^{j/2} I_j(z_2) H(t - b_{20}x) . \quad (52)$$

where $x_2 = \dfrac{t - b_{20}x}{-b_{22}x} \, , z_2 = 2\sqrt{-b_{22}x(t - b_{20}x)} \ .$

Using similar techniques, we obtain

$$\mathcal{L}^{-1}\left[e^{-b_{10}xp} e^{-b_{12}x/p} \right] = \left[\delta(t - b_{10}x) - \frac{1}{\sqrt{x_1}} J_1(z_1) \right] H(t - b_{10}x) . \quad (53)$$

$$\mathcal{L}^{-1}\left[e^{-b_{20}xp} e^{-b_{22}x/p} \right] = \left[\delta(t - b_{20}x) + \frac{1}{\sqrt{x_2}} I_1(z_2) \right] H(t - b_{20}x) . \quad (54)$$

Substituting from equations (51) - (54) into equation (50), we obtain the final form of the stress distribution σ valid for short times in the form.

$$\sigma = \theta_0 b \left\{ e^{-b_{11}x} H(t - b_{10}x) \left[c_0 \left(\delta(t - b_{10}x) - \frac{1}{\sqrt{x_1}} J_1(z_1) \right) + \sum_{j=0}^{3} c_{j+1} x_1^{j/2} J_j(z_1) \right] \right.$$

$$\left. - e^{-b_{21}x} H(t - b_{20}x) \left[c_0 \left(\delta(t - b_{20}x) + \frac{1}{\sqrt{x_2}} I_1(z_2) \right) + \sum_{j=0}^{3} c_{j+1} x_2^{j/2} I_j(z_2) \right] \right\}$$

$$(55)$$

TEMPERATURE DISTRIBUTION

Substituting from equations (36), (39), (43) and (48a) into equation (32), we obtain.

$$\bar{\theta} = \theta_0 \sum_{i=1}^{2} (-1)^{i+1} e^{-b_{i1}x} \sum_{j=0}^{3} \frac{c_{ij}}{p^{j+1}} e^{-b_{i0}xp} e^{-b_{i2}x/p} ,$$

$$(56)$$

where

$$c_{i0} = b_0(a_{i0} - 1) \ , c_{ij} = b_j(a_{i0} - 1) + \sum_{k=0}^{j-1} b_k \, a_{i(j-k)}$$

$$, i = 1, 2 \, , \ j = 1, 2, 3 \ .$$

Taking the inverse Laplace transform of both sides of equation (56), and using equations (51) and (52), we obtain the temperature distribution in the form .

$$\theta = \theta_0 \left\{ e^{-b_{11}x} H(t-b_{10}x) \sum_{j=0}^{3} c_{1(j+1)} x_1^{j/2} J_j(z_1) \right.$$

$$\left. - e^{-b_{21}x} H(t-b_{20}x) \sum_{j=0}^{3} c_{j+1} x_2^{j/2} I_j(z_2) \right\}$$

(57)

DISPLACEMENT DISTRIBUTION

Substituting from equations (43), (46) and (48b) into equation (33), we obtain .

$$\bar{u} = \frac{-\theta_0 b}{\beta^2} \sum_{i=1}^{2} (-1)^{i+1} e^{-b_{i1}x} \sum_{j=0}^{3} \frac{d_{ij}}{p^{j+1}} e^{-b_{i0}xp} e^{-b_{i2}x/p},$$

(58)

where

$$d_{i0} = v\, b_0 b_{i0} \ , \quad d_{ij} = v \sum_{k=0}^{j} b_k\, b_{i\,(j-k)} + \sum_{k=0}^{j-1} b_k\, b_{i\,(j-k-1)} \ ,$$

$$i = 1, 2, j = 1,2,3 .$$

Taking the inverse Laplace transform of both sides of equation (58), and using equations (51) and (52), we obtain the displacement distribution in the form

$$u = \frac{-\theta_0 b}{\beta^2} \left\{ e^{-b_{11}x} H(t-b_{10}x) \sum_{j=0}^{3} d_{1j} x_1^{j/2} J_j(z_1) \right.$$

$$\left. - e^{-b_{21}x} H(t-b_{20}x) \sum_{j=0}^{3} d_{2j} x_2^{j/2} I_j(z_2) \right\} .$$

(59)

NUMERICAL RESULTS

The copper material was chosen for purposes of numerical evaluations. The constants of the problem were taken as.

$$\varepsilon = 0.0168 \ , \quad \beta^2 = 3.5 \ \text{and} \ \tau = v = 0.02.$$

The computations were carried out for three values of time, namely for $t = 0.05$, 0.1 and 0.15. The results are illustrated graphically in figures (1), (2) and (3) for the temperature, stress and displacement distribution, respectively. We should note here that as $x \to 0$, $x_1 \to \infty$. Numerical evaluations of the functions at $x = 0$ were done using the relation

$$\lim_{x \to 0} x_1^{j/2} J_j(z_1) = \lim_{x \to 0} x_1^{j/2} I_j(z_1) = \frac{t^j}{j!} \ ,$$

and a similar one for x_2 . These relations follow easily from the fact that .

$$J_j(x), I_j(x) = \frac{x^j}{2^j j!} [1 + O(x^2)]$$

All the functions considered have two discontinuities at $x=t/b_{10}$ and $x=t/b_{20}$ and vanish identically for $x > t/b_{20}$. The stress has infinite discontinuities at these points. The temperature has jumps equal to

$$- \theta_0 c_{10} e^{-b_{11} t/b_{10}}, \ \theta_0 c_{20} e^{-b_{21} t/b_{20}}$$

at the two points of discontinuity. The corresponding values for the displacement are

$$\frac{\theta_0 b\, d_{10}}{\beta^2} e^{-b_{11} t/b_{10}}, \ \frac{\theta_0 b\, d_{20}}{\beta^2} e^{-b_{21} t/b_{20}}.$$

The numerical values of these of these jumps and their locations are shown in table 1 .

	jump 1 t = 0.05	jump 2 t = 0.05	jump 1 t = 0.1	jump 2 t = 0.1	jump 1 t = 0.15	jump 2 t = 0.15
x	0.049991	0.353614	0.099983	0.707228	0.149974	1.060842
θ	-0.000349	-0.286530	-0.000349	-0.082128	-0.000349	-0.023540
u	-0.005609	0.000227	-0.005606	0.000065	-0.005604	0.000019
σ	- ∞	∞	- ∞	∞	- ∞	- ∞

Table 1

REFERENCES

[1] Biot, M., 1956. Thermoelasticity And Irreversible Thermo-Dynamics, J. Appl. Phys., 27, 240 -253.

[2] Lord, H., and Y. Shulman, 1967. A Generalized Dynamical Theory Of Thermoelasticity, J Mech. Phys. Solid, 15, 299-309.

[3] Dhaliwal, R. S., and H. Sherief, 1980. Generalized Thermoelasticity for Anisotropio Media, Quart. Math., 33, 1-8 .

[4] Ignaczak, J., 1979. Uniqueness in Generalized Thermoelasticity, J. Thermal Stresses, 2, 171-175.

[5]   Ignaczak, J., 1982. A Note On Uniqueness In Thermoelasticity With One Relaxation Time, J. Thermal Stresses, 5, 257-263.

[6]   **Sherief, H., and R. Dhaliwal, 1980. A Uniqueness Theorem And A Variational Principle for Generalized Thermoelasticity, J. Thermal Stresses, 3, 223-230.**

[7]   Sherief, H., 1987. On Uniqueness And Stability In Generalized Thermoelasticity, Quart. Appl. Math., 45, 773-778.

[8]   **Anwar, M., and H. Sherief, 1988. State Space Approach To Generalized Thermoelasticity, J. Thermal Stresses, 11, 353-365, 1988 .**

[9]   **Sherief, H., 1993. State Space Formulation for Generalized Thermoelasticity With One Relaxation Time Including Heat Sources, J. Thermal Stresses, 16, 163-180.**

[10]   **Anwar, M., and H. Sherief, 1988. Boundary Integral Equation Formulation of Generalized Thermoelasticity In A Laplace Transform Domain, Appl. Math Modelling, 12, 161 - 166.**

[11]   **Sherief, H., 1986. Fundamental Solution Of The Generalized Thermoelastic Problem for Short Times, J. Thermal Stresses, 9, 151-164.**

[12]   **Green A. E., and K. A. Lindsay, 1972. Thermoelasticity, J. Elast., 2, 1-7.**

[13]   **Müller, I., 1971. The Coldness, A Universal Function In Thermo-elastic Solids, Arch. Rat. Mech. Anal., 41, 319-332.**

[14]   **Green A. E., And N. Laws, 1972. On The Entropy Production Inequality, Arch. Rat. Mech. Anal., 45, 47-53.**

[15]   **Şuhubi, E S., 1975. Thermoelastic Solids, In A. C. Eringen (ed), Continuum Physics, II, chap. 2, Academic, New York.**

[16]   **Erbay, S., and E. S. Şuhubi, 1986. Longitudinal Wave Propagation In A Generalized Thermoelastic Cylinder, J. Thermal Stresses, 9. 279-295.**

[17]   Ignaczak, J., 1985. A Strong Discontinuity Wave In Thermoelasticity With Relaxation Time, J. Thermal Stresses, 8, 25-40.

[18]   Ignaczak, J., 1978. Decomposition Theorem For Thermoelasticity With Finite Wave Speeds, J. Thermal Stresses, 1, 41-52.

[19]   **Sherief, H., 1992. Fundamental Solution for Thermoelasticity With Two Relaxation Times, Int. J. Engng. Sci., 30, 861-870 .**

[20]   **Sherief, H., 1993. State Space Approach To Thermoelasticity With Two Relaxation Time, Int. J. Engng. Sci., 31, 1177 - 1189.**

[21]   **Sherief, H., 1994. A Thermo-Mechanical Shock Problem for Thermoelasticity with Two Relaxation Time, Int. J. Engng. Sci., 32, 313-325.**

[22]   **Anwar, M., and H. Sherief, 1994. Boundary Integral Equation Formulation for Thermoelasticity With Two Relaxation Time, J. Thermal Stresses, 17, 257-270.**

[23]   **Churchill, R. V., 1972. Operational Mathematics, Third edition, McGraw-Hill Book Company, New York.**

[24]   **Oberhettinger, F., and L. Badii, 1973. Tables of Laplace Transforms, Springer Verlag, New York.**