THE SUPERIMPRIMITIVE SUBGROUPS OF THE ALTERNATING GROUP OF DEGREE 8

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ABSTRACT

A transitive permutation group \( G \) is called superimprimitive if it is imprimitive with non-trivial block systems of imprimitivity of lengths all the non-trivial divisors of the degree of \( G \); The superimprimitivity concepts was studied first by Omar (2), and later by the authors (3). In the present paper we shall give some results concerning this concept in part 1, and determine in part 2, all superimprimitive subgroups of the alternating group of degree 8.

We proved the following:

Lemma (1): Let \( G \) be a transitive group acting on a set \( X \) and \( m \) is the number of non-trivial divisors of \( |X| \). If \( G \) contains \( m \) intransitive normal proper subgroups each having different orbit lengths then \( G \) is superimprimitive. The orbits of each subgroup form a block system of imprimitivity.

Lemma (2): (a) Let \( G \) be a superimprimitive group. For every non-trivial divisor \( d \) of the degree of \( G \) and for \( x \in X \), there exists a group \( Z \) which lies properly between \( G \) and \( G \) such that the set \( \{x^i\} \) has length \( d \).

(b) If \( G_i \subseteq Z \subseteq G \) holds, where \( Z_i, i=1, \ldots, m \) are proper subgroups of \( G \) and the sets \( \{x^i\} \) have different lengths, then \( G \) is superimprimitive.

Then we show that, among the 48337 subgroups of \( A_8 \), which split into 137 classes there are 4425 superimprimitive subgroups which split into 18 classes, their generators are given.

1. The Superimprimitivity

Let \( G \) be a transitive permutation group acting on a set \( X \), and \( m \) is the number of non-trivial divisors of \( |X| \).

Lemma (1):

If \( G \) contains \( m \) intransitive normal proper subgroups each having different orbit lengths then \( G \) is superimprimitive. The orbits of each subgroup form a block system of imprimitivity.
**Proof:**

Let $N$ be an intransitive proper subgroup of $G$. If $B$ is an orbit of $N$, then $B^g$, $g \in G$ is an orbit of $g^{-1}N g = N$. Thus $G$ can only permute the pairwise disjoint orbits of $N$ among each other. These therefore form blocks of $G$. Because $N \neq \{e\}$ they contain more than one point, because of the intransitivity of $N$ they are proper subsets of $X$, and because of transitivity of $G$ they are conjugate. Since $G$ contains $m$ intransitive normal proper subgroups then there are $m$ different block systems. i.e. There are a block system of imprimitivity for every divisor of the degree of $G$. Thus $G$ is superimprimitive.

**Lemma (2):**

(a) Let $G$ be a superimprimitive group. For every non-trivial divisor $d$ of the degree of $G$ and for $x \in X$, there exists a group $Z$ which lies properly between $G_x$ and $G$ such that the set $\{x^z\}$ has length $d$.

(b) If $G_x \subset Z_i \subset G$ holds, where $Z_i$, $i = 1, \ldots, m$ are proper subgroups of $G$ and the sets $\{x^z\}$ have different lengths, then $G$ is superimprimitive.

**Proof:**

(a) Let $B$ be a non-trivial block of $G$ and $Z$ the set of those $z \in G$ for which $B = B^z$, $Z$ is clearly a proper subgroup of $G$. Since $x = x$ where $g \in G_x$ and $x \in B \rightarrow B^g = B$ then $G_x$ is a subgroup of $Z$. Also $|B| > 1$ and the transitivity of $G$ implies that $G_x$ is a proper subgroup of $Z$, this holds for every non-trivial divisor $d$ of the degree of $G$, since there is a bijection of the subgroups of $G$ containing $G_x$ and the blocks containing $x$.

(b) Let $G_x \subset Z_i \subset G$ and $B_i = x^z$. For $b \in B_i \cap B_i^g$ with $g \in G$, then $b = x^z = x'^g$ (with $z, z' \in Z_i$), therefore $z'^{-1} g \in G_x \subset Z_i$, thus $B_i^g = B_i$ and $B_i$ is a block. Because $G_x \subset Z_i$ does not consist of $x$ alone. Since $B_i = B_i^g$ holds only for $g \in Z_i$, and $Z_i \subset G$ there is a $g \in G$ with $B_i \neq B_i^g$, therefore $B_i \neq X$, hence $B_i$ is a non-trivial block. Since we have $m$ subgroups of $Z_i$ then we have $m$ blocks of orders $|\{x^z\}| = |Z_i : G_x|$. Thus $G$ is superimprimitive.
2. The subgroups of $A_n$

To give examples of the superimprimitive groups, we looked through the subgroups of $A_n$. For $n \leq 7$ there is only one superimprimitive subgroup of $A_4$. For $n=8$, all the subgroups of $A_8$ can be classified into mutually disjoint classes as follows:

<table>
<thead>
<tr>
<th>Subgroup Description</th>
<th>The number of classes</th>
<th>The total number of subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intransitive cyclic</td>
<td>11</td>
<td>6973</td>
</tr>
<tr>
<td>Intransitive abelian</td>
<td>3</td>
<td>1225</td>
</tr>
<tr>
<td>I Transitive abelian</td>
<td>2</td>
<td>630</td>
</tr>
<tr>
<td>Intransitive elementary abelian</td>
<td>11</td>
<td>2590</td>
</tr>
<tr>
<td>II Transitive elementary abelian</td>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>Transitive abelian</td>
<td>2</td>
<td>630</td>
</tr>
<tr>
<td>Intransitive nilpotent</td>
<td>9</td>
<td>5985</td>
</tr>
<tr>
<td>III Transitive nilpotent</td>
<td>13</td>
<td>3990</td>
</tr>
<tr>
<td>Intransitive self-normalizing</td>
<td>13</td>
<td>8030</td>
</tr>
<tr>
<td>Transitive self-normalizing</td>
<td>8</td>
<td>2225</td>
</tr>
<tr>
<td>Intransitive simple</td>
<td>3</td>
<td>252</td>
</tr>
<tr>
<td>Transitive simple</td>
<td>2</td>
<td>121</td>
</tr>
<tr>
<td>Intransitive self-normalizing simple</td>
<td>2</td>
<td>240</td>
</tr>
<tr>
<td>Intransitive self-normalizing simple maximal</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Intransitive self-normalizing maximal</td>
<td>2</td>
<td>84</td>
</tr>
<tr>
<td>Transitive self-normalizing maximal</td>
<td>3</td>
<td>65</td>
</tr>
<tr>
<td>IV Transitive self-normalizing nilpotent</td>
<td>1</td>
<td>315</td>
</tr>
<tr>
<td>Intransitive</td>
<td>38</td>
<td>13189</td>
</tr>
<tr>
<td>Transitive</td>
<td>13</td>
<td>2375</td>
</tr>
</tbody>
</table>

137 48337
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By searching through all the above 137 classes case-by-case check to pick up the superimprimitive subgroups, we have the following table. For the notation of the isomorphism type see (4):

<table>
<thead>
<tr>
<th>Class</th>
<th>The generators</th>
<th>Class length</th>
<th>Subgroup order</th>
<th>Isomorphism type</th>
</tr>
</thead>
<tbody>
<tr>
<td>I 1</td>
<td>&lt;A₁,A₂&gt;</td>
<td>315</td>
<td>8</td>
<td>C₂xC₂</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₃&gt;</td>
<td>315</td>
<td>8</td>
<td>C₂xC₄</td>
</tr>
<tr>
<td>II 1</td>
<td>&lt;A₄,A₅,A₆&gt;</td>
<td>15</td>
<td>8</td>
<td>C₂xC₂xC₂</td>
</tr>
<tr>
<td></td>
<td>&lt;A₄,A₇,A₈&gt;</td>
<td>15</td>
<td>8</td>
<td>C₂xC₂xC₂</td>
</tr>
<tr>
<td>III 1</td>
<td>&lt;A₁,A₆&gt;</td>
<td>360</td>
<td>8</td>
<td>D₄</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₉&gt;</td>
<td>210</td>
<td>8</td>
<td>Q₄</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₁₀&gt;</td>
<td>315</td>
<td>16</td>
<td>Γ₂c₁</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₁₅&gt;</td>
<td>315</td>
<td>16</td>
<td>Γ₂c₁</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₂,A₆&gt;</td>
<td>315</td>
<td>16</td>
<td>C₂xD₄</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₂,A₉&gt;</td>
<td>360</td>
<td>16</td>
<td>Γ₂b</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₃,A₆&gt;</td>
<td>315</td>
<td>16</td>
<td>C₂xD₄</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₄&gt;</td>
<td>315</td>
<td>32</td>
<td>Γ₂c₁</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₁₀,A₁₁&gt;</td>
<td>315</td>
<td>32</td>
<td>Γ₄a₁</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₁₀,A₁₂&gt;</td>
<td>315</td>
<td>32</td>
<td>Γ₄a₁</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₁₀,A₁₃&gt;</td>
<td>105</td>
<td>32</td>
<td>Γ₄a₂</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₁₅,A₁₆&gt;</td>
<td>105</td>
<td>32</td>
<td>Γ₄a₂</td>
</tr>
<tr>
<td></td>
<td>&lt;A₁,A₁₂,A₃,A₅&gt;</td>
<td>105</td>
<td>32</td>
<td>Γ₄a₂</td>
</tr>
<tr>
<td>IV 1</td>
<td>&lt;A₁₇,A₁₈,A₁₉&gt;</td>
<td>315</td>
<td>64</td>
<td>C₂xC₃xC₂/C₂xC₂xC₂</td>
</tr>
<tr>
<td>18</td>
<td></td>
<td>4425</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Where the generators are:

A₁=(1857) (2436), A₂=(1456) (2837), A₃=(1253) (4768),
A₄=(15) (23) (46) (78), A₅=(14) (27) (38) (56), A₆=(13) (25) (48) (67),
A₇=(12) (35) (48) (67), A₈=(14) (28) (37) (56), A₉=(1352) (4768),
A₁₀=(1836) (2457), A₁₁=(1824) (3756), A₁₂=(1856) (2437),
A₁₃=(1824) (3657), A₁₄=(1637) (2854), A₁₅=(1268) (3475),
A₁₆=(1247) (3685), A₁₇=(1826) (3754), A₁₈=(1634) (2857),
A₁₉=(1235) (4768).

So the conclusion is: There are 18 classes of superimprimitive subgroups of A₈.
ACKNOWLEDGEMENT

The authors would like to thank Dr. Volkmart Felsch, for supplying reference (1)

REFERENCES

(1) Felsch, V. 1982. A List of subgroups of A₈, Computer output, RWTH AACHEN University, West Germany, Private Communication.


الزمر الجزئية متعددة غير الأولية من الزمرة A

عبد الرؤوف عمر

يقال لزمرة التبديلات الانتقالية أنها متعددة غير الأولية إذا كانت غير الأولية ولها نظام من البلوكات الفصلية لكل قاسم فعلي من قواسم درجة الزمرة. ولقد قدم هذا البحث نظريتين لشروط مكافئة للتعرف، للتعرف على الزمرة متعددة غير الأولية. ثم وضحنا أنه بين كل الزمرة الجزئية لزمرة التبديلات الزوجية من درجة ثمانية، $A_8$، وعدهم 8737 زمرة جزئية مقسمين إلى 137 فصل تكافؤ، يوجد 4425 زمرة جزئية متعددة غير الأولية مقسمين إلى 18 فصل تكافؤ.