A FAMILY OF NONRAMIFIED FUNCTIONS, DEFINED IN Z\bar{Z}<1

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ABSTRACT
In this article a family of nonramified functions, defined in z\bar{z}<1, is obtained. It is characterized by an achieved distortion theorem; its Bloch constant and the limit of convexity are determined. Peschl's principle is mainly used throughout this work.

INTRODUCTION
One of the interesting questions in complex analysis is the study of analytic functions based on the nature of their produced mappings. H. A. Schwartz and C. Carthéodory [6] gave a powerful geometrical development in this line. E. Study [13] considered, firstly, convex domains; that is the tangent at their boundary points of each domain always turns in the same direction as we move around the boundary, and obtained the lemma: if f(z) maps the units disk |z|<1, in a convex domain, then f(z) maps every disk |z|<r<1 onto a domain of this type. Then he generalized the behaviour of the boundary curvature in theory of conformal mapping.

E. Peschl [7] investigated the problem of conformal mapping onto regions with boundary curvature which is bounded from below. He continued and succeeded in concluding and publishing in 1955 his principle [8]. Then, one of the invaluable tools in different branches of mathematics (e.g. differential geometrical theory of function of a complex variable [9] has been introduced. It is worth mentioning that applying that principle has led to some interesting results (see e.g. [1,2,4,5,9,10,12,14,15]).

In this article, the first section is devoted to review briefly, Peschl's principle and its role in complex analysis. As a result of applying it, a new family of nonramified functions**) is obtained in the second section. It is characterized, mainly, by the value of its Bloch constant, the limit of convexity and the nature of the image of the unit disk under each function of the family.

(1) Peschl's principle and its Role

Of interest to us here are functions w(z) which are nonramified in the unit disk of the z-plane. D={ z: |z|< 1}.

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** i.e. holomorphic functions with nonvanishing derivative
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Suppose the hyperbolic metric $ds_z = |dz|/(1-|z|^2)$ and the euclidean metric $ds_w = |dw|$ are considered in $D$ and $w$-plane respectively. Recall in addition that:

$$\delta_w = (1-|z|^2)v_z^2$$ and $$\delta_z v = (1-|z|^2)v_z^2$$

are the Beltrami differential operators, while the associated mixed Beltrami differential operator is defined as:

$$\delta_1(u,v) = (1-|z|^2)u_z v_z^2$$

Using the differential invariants,

$$\alpha = \log |w'| (1-|z|^2)$$

and

$$\beta = (1-|z|^2) \alpha$$

$$= \frac{w''}{2w'} (1-|z|^2) - z,$$

we form the function $v = \beta \tilde{\beta} - f(\alpha)$, where $f(\alpha)$ is initially unknown function and differentiable twice at least. Calculation of $\delta_2 v$ yields:

$$\delta_2 v = \frac{1}{A} \delta_1 v + B \{ \delta_1 (v, \alpha) + \delta_2 (\alpha, v) \} + D_1 v + D_0$$

(1)

where

$A = \beta \tilde{\beta}$ and $B = A(1 + f')$;

$D_1 = -f'' - 2$ and $D_0 = -ff'' + f'2 + 3f' - 2f + 2$.

Now, let $f(\alpha)$ is suggested to the following conditions:

$$D_0 = 0, D_1 \neq 0 \text{ and } v < 0 \text{ in a neighbourhood } U(\partial D), \text{ where } \partial D \text{ is the boundary of } D$$

(2)

Hence it follows that $v \leq 0$ at every point $z \in \{ DU \partial D \}$.

This principle has been obtained by E. Peschl and its proof is sketched as follows (see e.g. [7], [10]):

Suppose that the function $v$ takes its maximum at a point $z^* \in D$, where $v(z^*) = \max_D. (v) > 0$ and $\delta_2 v \leq 0$. Moreover, since at $z^*$, $\delta_1 v = 0$ and $\delta_1 (\alpha, v) = \delta_1 (v, \alpha) = 0$, it follows that equation (1) reduces to $\delta_2 v = D_1 v$.

But from the preceding, $D_1$ is either $>0$ or $<0$. On one hand if $D_1 >0$, this leads, immediately, to a contradiction ($\delta_2 v$ is not allowed to be positive at a maximum situation). On the other hand, if $D_1 <0$, using a modified maximum principle[10], a similar contradiction follows.
Now, it is of importance to define $f(\alpha)$, such that $D_0 = 0$, $D_1 \neq 0$ and $v < 0$ in a neighbourhood $U(\partial D)$. These functions are obtainable in Ref. [11].

Considering $v = \beta \bar{\beta} - f(\alpha) \leq 0$, $\beta = (1 - |z|^2)\alpha$, it follows:

$$J \frac{d\alpha}{\sqrt{f(\alpha)}} \leq J \frac{1}{\sqrt{f(\alpha)}} \left| \frac{d\alpha}{dr} \right| dr \leq 2 J \frac{dr}{1 - r^2},$$

where $r = |z|$. Upon integration, one has estimations for $\alpha$, integrating once more one has:

$$H_1(r) \leq |w(z) - w(0)| \leq H_2(r)$$

where $H_1(r)$ and $H_2(r)$ depend on $f(\alpha)$. Without loss of generality, $w(z)$ is normalized such that $w(0) = 0$. Thus, it is clear that: corresponding to any function, $f(\alpha)$, satisfying the above conditions, (2), and by applying this approach, one obtains estimations for the differential invariant $\alpha$ and $|w(z)|$. So, Peschl's principle and the main method of its application is represented. Its role has been introduced in different works, and results are obtained. For example:

(i) Since the above differential invariant $\alpha$, satisfies the equation $\mathcal{S}_2 \alpha = -1$, E. Raupach [12] gave different estimations for $\alpha$.

(ii) Considering the hyperbolic and elliptic metric in the $w$-plane, K.-W. Bauer [1] proved the principle and gave estimations for the solutions of the equation, $\mathcal{S}_2 \alpha = -1 \pm e^{i\alpha}$ where $\alpha$ is the relevant differential invariant. Also distortion theorems are obtained.

(iii) K. J. Wirths [14] considered the principle as a maximum principle and generalized it. Furthermore, using the original principle, he solved the following problem [15] which was stated in 1965 (see [16 problem N]):

For all $z \in D$, which value of $\beta \in \mathbb{R}$ under which $|f'(z)| < (1 - |z|^2)^{-1}$ holds?, where $f(z)$ is holomorphic and $|f(z)| < (1 - |z|^2)^{-1}$; also what is the coefficients ranges for the class of these functions?

(iv) In [2] some families of analytic functions are defined and a set of distortion theorems are obtained considering hyperbolic and elliptic metric in addition to the euclidean metric.

(v) In [3] families of ramified and non-ramified functions and a relation governing them are obtained. Moreover, in [4] considering hyperbolic metric, and using Julia theorem a covering theorem is deduced.

(vi) Lastly, in [5], a relation governing solutions of $\mathcal{S}_2 w = -1$ and Joukowski's profile is induced.

(2) A family of non-ramified functions

Let $\mathcal{F}$ denotes the family of all non-ramified functions, $w(z)$, which are defined in the unit disk $D$ and are considered in the definition of the above differential invariant $\alpha$. In correspondance
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with a selected solution of the nonlinear differential equation \( D_0 = 0 \). The family \( \mathcal{F} \) is characterized by conditions hold for each \( w(z) \in \mathcal{F} \).

Let us select the following solution of \( D_0 = 0 \):

\[ \alpha = \log(t+b) - t, \quad f = \{(t+b) - 1\}^2, \quad (b \text{ arbitrary and } t > -b) \]

which satisfies the conditions (2).

To make use of Peschl's principle, inequality (3) is applicable for \( t \in [1-b, \infty] \). Since

\[ \alpha(0) \int \frac{d\alpha}{-\text{vf}(\alpha)} = -t \int_{t+b}^{1} \frac{dt}{t+b} = \log(t_1+b)/(t+b), \quad (4) \]

relevant calculations show that \( t_1 \) is determined through \( w'(0) = (t_1+b)e^{x_1} \):

\[ QR - b \leq t \leq Q/R - b, \]

where \( Q = t_1 + b \) and \( R = \frac{1-r}{1+r} \).

The above two intervals of \( t \) are consistent iff:

1. \( 1 - b \leq QR - b \Rightarrow r \leq \frac{Q-1}{Q+1} = r^* \)
2. \( \frac{O}{R} - b \leq \infty \) (trivial).

Since \( e^\alpha \) is a monotonic decreasing function of \( t \) then, by applying (4), one has:

\[ \exp \left\{ b - \frac{Q}{R} \right\} (QR) \leq e^{\alpha(0)} \leq \exp \left\{ b - Qr \right\} (Q/R), \]

in this and in the next inequalities the R.H.S. holds only under the restriction \( r \leq r^* \).

Since \( e^\alpha = |w'| (1-|z|^2) \), the above inequality implies:

\[ ((1-r)^{-1} \exp \{\alpha(t_1) - \frac{2Or}{1-r} \} \leq |w'| \leq (1+r)^{-1}\exp\{\alpha(t_1) + \frac{2Or}{1+r} \} \]

which yields by integration:

\[ \frac{1}{2} \exp(-t_1)(1-\exp\{-\frac{2Or}{1-r}\}) \leq |w| \leq \frac{1}{2} \exp(-t_1) \{\exp\{\frac{2Or}{1+r}\} - 1\} \]

*) Bloch constant of \( \mathcal{F} \) is defined \([6]\) as the infimum of the radii of the largest one-sheeted disks lying on the Riemann surface, onto which \( w(z) \in \mathcal{F} \) map \( D \) dijectively.
The obtained inequalities hold for each \( w(z) \in \mathcal{J} \), corresponding to the selected solution of \( D_0 = 0 \).

Hence Bloch constant \( B_\mathcal{J} \) of the family is calculated from the last inequality by taking the limit of the L.H.S. as \( r \to 1 \) which gives \( B_\mathcal{J} = \frac{1}{2} \exp(-t_1) \).

Furthermore, as a conclusion from Peschl's principle \([10]\), the family \( \mathcal{J} \) is characterized by the fact that:

the boundary curvature of the image of the unit disk \( D \) under \( w \in \mathcal{J} \),

namely \( \frac{1}{|w'|} \text{Re}\{1 + \frac{W''}{W'}\} \), is bounded from below by \( k \) (say),

where \( k \) is given by \( \lim_{r \to 1} \left(1 - \beta \mathcal{B}\right) e^{\infty} = 2e^{-\gamma} (>0) \). This means

\[
\frac{1}{r^2} e^{\infty} \leq \frac{1}{r^*} \leq \frac{1}{1+r^2} \exp\left(\frac{2Qr}{1-r}\right)
\]

that the image of the unit disk \( D \) under each \( w(z) \in \mathcal{J} \) is always convex. Furthermore, \( w(z) \) maps every disk \( |z| < r < 1 \) onto a convex domain \([13]\). The suprimum of the radii \( r \) of these disks yields the limit of convexity for the family \([6]\); it can be shown that it is equal to unity.

Hence, the family \( \mathcal{J} \), which is defined in correspondence with the above selected solution of the differential equation \( D_0 = 0 \), is characterized by the following distortion theorem.

**Theorem**

For each \( w(z) \in \mathcal{J} \) the following conditions are satisfied:

1. \( w(z) \) is nonramified in the unit disk \( D \) and \( w(0) = 0 \).

2. \[
\frac{e^{\infty}(t_1)}{1-r^2} \exp\left(\frac{2Qr}{1-r}\right) \leq |w'| \leq \frac{e^{\infty}(t_1)}{1+r^2} \exp\left(\frac{2Qr}{1+r^2}\right)
\]

3. \[
\frac{1}{2} \exp\left(-t_1\right) \left(1 - \exp\left(-\frac{2Qr}{1-r}\right)\right) \leq |w| \leq \frac{1}{2} \exp\left(-t_1\right) \left(\exp\left(\frac{2Qr}{1+r}\right) - 1\right)
\]

where \( t_1 \), \( Q \) and \( r^* \) are defined as above.

4. Bloch constant \( B_\mathcal{J} \) is equal to \( \frac{1}{2} \exp(-t_1) \) and the limit of convexity is unity.

5. The boundary curvature of the image of the unit disk \( D \) at least equal to \( \exp(-2b) \).

**COROLLARY**

It can be proved that the function \( w(z) = \frac{1}{2} \left(\exp\left(\frac{2Qz}{1+z}\right) - 1\right) \) belongs to the family \( \mathcal{J} \). This function is the majorant function of the inequality 3 in the theorem.

\[\dagger\] The branch \(+\sqrt{f}\) and the parameter \( t \) lies on \([-b,1-b]\).
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REFERENCES


عائلة من دوال متفرقة ومعرفة في قرص الوحدة

محمد كامل جبر

في هذا البحث حصلنا على عائلة من دوال غير متفرقة ومعرفة في قرص الوحدة. ميزت هذه العائلة بنظرية تشبه مع تبديد ثابت بلوخ ونهاية التحذب لها. تم استخدام مبدأ بشف بصفة أساسية في هذا البحث.