Translation-invariant Positive-definite
Generalized Kernels of Infinite Number of Variables

by

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النواة اللامتغيرة بالإزاحة والمعمّة
ذو متغيرات لا نهائية العدد

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وفي هذه الورقة تدرس نواة ذات متغيرات لا نهائية بعد في فراغات نووية وتحقق الخواص اللاتغيرة بالنسبة لللازاحة وفي الوقت نفسه معمّمة وموجبة ويوجد الشرط الكافي للحصول على تثيل تكامل لها.
Introduction

In the theory of spectral analysis of positive-definite kernels, there exist well-developed methods based on the ideas of Krein connected with the construction of a Hilbert space by means of a kernel \([1,11]\). In the main, our formulation deals with translation-invariant positive-definite generalized kernels.

Consider the rigged Hilbert spaces \(H_\text{r} \supset H_0 \supset H_+ [1]\) with the involution \(\omega \to \overline{\omega}\) defined in \(H_\text{r}\) and also in \(H_0\) and \(H_+\). Let \(K \in H_\text{r} \otimes H_\text{r}\) be a generalized kernel. If 

\[(K, u \otimes u)_{H_0} (\otimes)_{H_0} \geq 0,
\]

then \(K\) is said to be positive-definite (p.d.).

Now, let \(\phi\) be a topological space of functions on \(X\), \(\phi'\) be its adjoint, \(K \otimes \phi'\) a generalized kernel and \(G\) a commutative group in \(X\).

The kernel \(K \in \phi' \otimes \phi'\) is said to be \(G\) quasi-invariant if there exists a function \(\rho(x, a), x \in X, a \in G\) which for each \(a \in G\) is a multiplier in \(\phi\) such that

\[ (K, u(a+) \rho(.,a) \otimes u(a+) \rho(.,a)) = (K, u \otimes \overline{u}), u, v \in \phi; a \in G \]  \hspace{1cm} (0.1)

Now, let \(H_k\) be the Hilbert space constructed from the quasi-scalar product \(\langle u, v \rangle_k = (K, v \otimes u)\) by means of completion and factorization. Let \(B\) be the continuous operator in \(\phi\) which commutes with involution, and let \(B^+\) be its adjoint in \(\phi'\). We say that \(B\) is \(K\) symmetric if

\[ (B^+ \otimes I)K = (I \otimes B^+)K \] \hspace{1cm} (0.2)

which is equivalent to the symmetry of \(B\) in \(H_k\) : \(\langle Bu, v \rangle_k = \langle u, Bv \rangle_k\).

The formula \((T_{a}u)(x) = u(a + x)\rho(x, a)\) makes sense for the representation of \(G\) in \(\phi\), for which the generalized kernel \(K\) is translation-invariant, i.e. \((T_{a} \otimes T_{a})K = K\), and thus \(T_{a}\) is unitary in \(H_k\).

In what follows, we apply the preceding theory to obtain an integral representation of p.d. translation-invariant kernels on spaces of \(R_\infty\) and of the type \(S_\infty \otimes S_\infty\) and \(\sigma_\infty \otimes \sigma_\infty\) (\(R_\infty \subset R_\infty\) is the space of finite sequences), where

\[ S_{\infty}(R_\infty) = \bigcap_{(m_i)_{i=1}^\infty} S_{\infty}^m (R_\infty), \text{ and} \]

\[ S_{\infty}^m (R_\infty) = \left\{ u(t) = \sum_{k=0}^{\infty} u_k e^{ikt} \mid \|u\|_m^2 = \sum_{k=0}^{\infty} |u_k|^2 \left(1 + |k|^2\right)^m < \infty \right\} \]

Also,

\[ \sigma_{\infty}^m (R') = \left\{ u(t) = \sum_{k=0}^{\infty} u_k e^{ikt} \mid \|u\|_m^2 = \sum_{k=0}^{\infty} |u_k|^2 \left| |k| < \infty \right\}, \]

and

\[ \sigma_{\infty} (R') = \bigotimes_{i \in [1, \infty)} \sigma_{\infty} (R') \text{, where } \sigma_{\infty} (R') = \bigcap_{1}^{\infty} \sigma_{\infty}^m (R'), \]

(see [7,12]).
1. The case of $K \in S'_g \otimes S'_g$

Consider the kernel $K \in S'_g \otimes S'_g$ which satisfies the following conditions:

a) p.d., i.e. $(K, u \otimes u) \geq 0$, $u \in S_g (R^n)$.

b) $R_\infty$ quasi-invariant with density

$$
\rho (x, a) = \exp \left\{ - \sum_{i=1}^{n(a)} (2a_i x_i + a_i^2) \right\}, \quad x \in R^n, \ a(a_1, \ldots, a_n) \in R_\infty^n
$$

(1.1)

As in [8], we can show that the density $\rho(x, a)$ which takes the form (1.1) is a multiplier in $S_g (R^n)$. In fact, consider the Fourier-Venar transform [10] of the function $u \in L^2 (R^n, dg)$ in the form

$$
\hat{u} (\lambda) = \lim_{n \to \infty} \left( \frac{1}{2\pi} \right)^n \int_{R^n} e^{i \sum_{i=1}^{n} \lambda_i x_i} u_n (x_1, \ldots, x_n) \ dx_1 \ldots dx_n
$$

(1.2)

where $u_n (x_1, \ldots, x_n)$ is the corresponding cylindrical function generated by $u \in L^2 (R^n, dg)$. In finite-dimensional cases, the Fourier-Venar transform $F_\omega$ is the unitary image of the Fourier transform $F$ in transforming from $L^2 (R^n, dx)$ to $L^2 (R^n, dg)$, i.e.,

$$
u_n : L^2 (R^n, dx) \ni \phi \to \pi \int_{R^n} e^{-\frac{1}{4} \sum_{i=1}^{n} x_i^2} \phi \in L^2 (R^n, dg)$$

(1.3)

So, $F_\omega = u_n F u_n^{-1}$. And so, the Fourier-Venar transform exists as a unitary operator in $L^2 (R^n, dg)$. In addition, by simply checking the relation $h_\alpha (\lambda) = i \alpha_1 h_\alpha (\lambda) ; |\alpha| = \alpha_1 + \ldots + \alpha_n$, this transformation is a unitary operator in each of the Hilbert spaces $S^{\pm m}_g$ and $\alpha^{\pm m}_g$. Thus, for an arbitrary $u \in S_g (R^n)$, we have

$$
\hat{u} (\lambda) = \rho (\lambda) e^{i \sum \lambda_i a_i}
$$

But $e^{i \sum \lambda_i a_i}$ is a multiplier function in $S_g (R^n)$, so we have the required.
Using the preceding Fourier-Venar transform, we have the following theorem:

**Theorem 1**
Every translation-invariant p.d. kernel $K \in S'_g \otimes S'_g$ admits the representation

$$
(K, \hat{u} \hat{v}) = \int_{R^n} \langle \hat{u} \hat{v} \rangle (\lambda) \, d\rho(\lambda)
$$

(1.4)

where $d\rho(\lambda) = c(\lambda) \, d\sigma(\lambda)$ is a finite measure on $R^n$ and $c(\lambda) \ll c \exp \left(1/2 \sum \frac{\lambda_j^2}{n(m_j)}\right)$

and $d\sigma(\lambda)$ is a finite measure defined on a $\sigma$-algebra of sets from $R^n$. Conversely, every measure $d\rho(\lambda)$ in the given form generates a translation-invariant p.d. generalized kernel.

**Proof:** First we construct the Hilbert space $H_k$ as the completion of the space $S_g(R^n)$ w.r.t. the quasi-scalar product $\langle u, v \rangle_k = (K, \hat{u} \hat{v})$. By $B^*$ we denote the corresponding adjoint of the operator $B : S_g \rightarrow S_g$ in $S'_g$. The p.d. kernel $K$ will be called $B$-translation-invariant if $(B^* \otimes B^*) K = K$.

The formula $(T_a u)(x) = u(a + x) \rho(x, a)$ makes sense for the representation of the group $G$ from $R^n$ in $S'_g$, for which the kernel $K$ is translation-invariant and thus $T_a$ is unitary in $H_k$, (see [6]). Therefore, the corresponding infinitesimal operators of the representation will be $K$-symmetric, i.e., if $B$ is an infinitesimal operator of the representation $T_a$, then, $(B^* \otimes 1) K = (1 \otimes B^*) K$. Moreover, these operators generate a commuting system of self-adjoint operators in $H_k$ (see [3]).

Henceforth, we shall consider $B_k$ as an infinitesimal operator of the representation $T_a$ with $k$ variables, $k = 1, 2, \ldots$

Since $K \in S'_g \otimes S'_g$, we can find $E = (E_i)_{i=1}^n$ such that $K \in S_g^{-\ell} \otimes S_g^{-\ell}$. It is clear that $H_k \supseteq S_g^{-\ell}$, moreover, the inclusion is continuous. Therefore, from the nuclearity of $S_g(R)$ we find $m = (m_j)_{j=1}^n$ such that $H_k \supseteq S_g^m(R^n)$ and the inclusion is quasi-nuclear (Hilbert-Schmidt operator).

Thus, we have the chain

$$
H_{-m,k} \supseteq H_k \supseteq S_g^m(R^n) \supseteq S_g(R^n)
$$

(1.5)

in which $H_{-m,k}$ is the dual space of $S_g^m$ w.r.t. $H_k$ and $S_g(R^n)$ is the extension of the equipment. The operators $(B_k)_{k=1}^\infty$ form a system of commuting self-adjoint operators in $H_k$ and define a differential expression $B_k$ in the form

$$
B_k = i p^{-1}(x_k) \frac{\partial}{\partial x_k} 
$$

(1.6)
where \( p(t) = \frac{1}{\sqrt{\pi}} e^{-t^2} \) is the density of a Gaussian measure.

Moreover, for every \( k \), \( D(B_k) \supset S_g(\mathbb{R}^\omega) \) where \( D(B_k) \) is the domain of definition of the operators \( B_k \).

Now, applying theorem '2' from [2] to the system of operators \((B_k)_k=1\) we have the Parseval equality in the form

\[
1 = \int_{\mathbb{R}^\omega} p(\lambda) \, d\sigma(\lambda) \quad (1.7)
\]

Here, \( d\sigma(\lambda) \) is a non-negative finite measure defined on a \( \sigma \)-algebra of cylindrical sets from \( \mathbb{R}^\omega \), \( p(\lambda) \) defines \( d\sigma(\lambda) \) almost everywhere for each \( \lambda \) and the operator valued function of which gives a non-negative quasi-nuclear operator, operates from \( S_{g}^{m} \) to \( H_{-m,k} \) for which the Hilbert norm \( \| p(\lambda) \| \leq 1 \). The integral (1.7) converges in the sense of the Hilbert norm.

So, there exists a set \( \Lambda \subset \mathbb{R}^\omega \) with total measure \( d\sigma(\Lambda) \) such that \( \mathcal{R}(p(\lambda)) \), for \( \lambda \in \Lambda \), consists of the generalized eigenvectors of the operators \( B_k \) with eigenvalues \( \lambda_k \):

\[
\langle p(\lambda)u, B_k v \rangle = \lambda_k \langle p(\lambda)u, v \rangle \quad (u \in S_{g}^{m}(\mathbb{R}^\omega), v \in S_g(\mathbb{R}^\omega)) \quad (1.8)
\]

Now, we consider the chain

\[
S_{g}^{m} \times S_{g}^{m} \supset L_2(\mathbb{R}^\omega \times \mathbb{R}^\omega, dg(x) \times dg(y)) \supset S_{g}^{m} \times S_{g}^{m} \quad (1.9)
\]

With the help of the procedure in [4, 5], the set of operators \( p(\lambda) \) defines the family of elementary kernels

\[
\Omega(\lambda) \in S_{g}^{m} \times S_{g}^{m} \quad (\| \Omega(\lambda) \|_{S_{g}^{m} \times S_{g}^{m}} \leq C < \infty). 
\]

The connection between \( p(\lambda) \) and \( \Omega(\lambda) \) is given by the equality

\[
\langle p(\lambda)u, v \rangle = (\Omega(\lambda), u \otimes v) \quad (u, v \in S_{g}^{m}(\mathbb{R}^\omega)) \quad (1.10)
\]

From the positive-definiteness of \( p(\lambda) \) and with the help of (1.10) and the inclusion \( S_{g}^{m} \supset S_{g} \), it follows that \( \Omega(\lambda) \) is p.d.,

\[
(\Omega(\lambda), u \otimes \bar{u}) \geq 0 \quad (u \in S_{g}(\mathbb{R}^\omega)) \quad (1.11)
\]

Hence, from (1.8) and the form of \( B_k \) it follows that \( \Omega(\lambda) \) satisfies the relations

\[
(\Omega(\lambda), ip^{-1}(x_k) \frac{\partial}{\partial x_k} (p(x_k)v)(x) \bar{u}(y) - \lambda_k v(x) \bar{u}(y)) = 0 \quad (1.12)
\]

\[
(\Omega(\lambda), v(x) ip^{-1}(y_k) \frac{\partial}{\partial y_k} (p(y_k)u)(y) - \lambda_k v(x) \bar{u}(y)) = 0 \quad (1.13)
\]

From (1.7) and (1.10), \( K \) has the integral representation

\[
K = \int_{\mathbb{R}^\omega} \Omega(\lambda) \, d\sigma(\lambda) \quad (1.14)
\]

which converges in the sense of the space \( S_{g}^{-m} \otimes S_{g}^{-m} \).
Now, we seek the solution of the system of equations (1.12) and (1.13) in the sense 
or generalized functions. Namely, \( \Omega(\lambda) = \lim_{n \to \infty} \Omega_n(\lambda) \) where \( \Omega_n(\lambda) \) is the corresponding 
cylindrical kernel which is obtained from the representation of the kernel \( \Omega(\lambda) \) in the 
form of a series excluding the terms which contain variables with large number \( n \) and 
its convergence takes place in \( S^m \times S^m \).

If \( u_n(x_1, \ldots, x_n) \) and \( v_n(x_1, \ldots, x_n) \) are cylindrical functions from \( S_g(R^n) \) then for 
\( u, v \in S_g(R^n) \) we have

\[
(\Omega_n(\lambda), v \otimes u) = (\Omega_n(\lambda), v_n \otimes u_n)
\]

Therefore, we have the following system of equations:

\[
(\Omega_n(\lambda), \iota p^{-1}(x_k) \frac{\partial}{\partial x_k} (p(x_k) v(x_k)) \otimes u(y) - \lambda_k v(x) u(y)) = 0
\]

\[ k = 1, 2, \ldots \]

and an analogous equation for \( u \).

Now, if

\[
\hat{\Omega}_n(\lambda) = (U_n, x \otimes U_n, y) \Omega_n(\lambda), \quad \Psi_n = U_n v_n ; \phi_n = U_n u_n
\]

where \( U_n = p(x_1) \ldots p(x_k) \), we have

\[
(\hat{\Omega}_n(\lambda), (i \frac{\partial}{\partial x_k} \Psi_n) \otimes \phi_n - \lambda_k \Psi_n \otimes \phi_n)_{L^2(R^n \times R^n, dg(x) \times dg(y))} = 0, \quad k = 1, 2, \ldots
\]

and a similar equation is obtained for \( \phi_n \). Applying to (1.16) the generalized solution for 
a system of differential equations, we find that \( \Omega_n(\lambda) \) is an ordinary function with 2n 
variables and has the form

\[
\Omega_n(\lambda, x_1, \ldots, x_n; y_1, \ldots, y_n) = \pi^n e^{i \sum \lambda_k (x_k y_k)}
\]

\[ = [n^e \lambda_{1, e} \ldots \lambda_{n, e} ] e^{i \sum \lambda_k (x_k y_k)} \]

\[ \Omega_n(\lambda, 0, \ldots, 0), \]

i.e. \( \Omega_n(\lambda) = \pi^n e^{i \sum \lambda_k (x_k - y_k)} \)

or

\[
\Omega_n(\lambda, x, y) = e^{i \sum \lambda_k (x_k - y_k)} \Omega_n(\lambda, 0, \ldots, 0)
\]

(1.17)
Namely, \((\Omega_n(\lambda), 1 \otimes 1) = (\Omega(\lambda), 1 \otimes 1)\)

\[
\begin{aligned}
&= (\int_{\mathbb{R}^n} e^{\frac{i}{2} \sum_{k=1}^{n} \lambda_k x_k^2} e^{\frac{1}{3} \sum_{k=1}^{n} \lambda_k x_k} (2\pi)^{-n/2} e^{-\frac{3}{2} \sum_{k=1}^{n} x_k^2} e^{\frac{1}{3} \sum_{k=1}^{n} y_k^2} d\lambda_1 \cdots d\lambda_n) \Omega_n(\lambda, 0, 0, \ldots, 0) \\
&= (\frac{(2\pi)^n}{(2\pi/3)^n} \exp \left\{ - \sum_{k=1}^{n} \frac{\lambda_k^2}{3} \right\} \Omega_n(\lambda, 0, 0, \ldots, 0)).
\end{aligned}
\]

But \((\Omega(\lambda), 1 \otimes 1) = c(\lambda) \geq 0\). Then evidently,

\[
\Omega_n(\lambda, 0, 0, \ldots, 0) = c(\lambda) \frac{(2\pi/3)^n}{(2\pi)^n} e^{\frac{i}{3} \sum_{k=1}^{n} \lambda_k^2}.
\]

So, we have obtained the general form for \(\Omega_n(\lambda)\);

\[
\Omega_n(\lambda) = \left( \frac{1}{\sqrt{2\pi}} \right)^n e^{\frac{i}{\sqrt{2\pi}} \sum_{k=1}^{n} \lambda_k (x_k - y_k)} e^{\frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n} \lambda_k^2} e^{\frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n} \lambda_k (x_k - y_k)} e^{\frac{2\pi}{3} n} c(\lambda) e
\]

(1.18)

Considering (1.18) we can write (1.14) in the form

\[
K = \lim_{n \to \infty} \left( \frac{1}{\sqrt{2\pi}} \right)^n e^{\frac{i}{\sqrt{2\pi}} \sum_{k=1}^{n} \lambda_k (x_k - y_k)} e^{\frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n} \lambda_k^2} e^{\frac{2\pi}{3} n} c(\lambda) d\sigma(\lambda)
\]

(1.19)
Now, using the Fournier-Venar transform and from (1.19) we have

\begin{equation}
(K, v(\mathcal{U})) = \lim_{n \to \infty} \int_{\mathbb{R}^n} \hat{u}_n(\lambda) \hat{v}_n(\lambda) e^{i\lambda \cdot \mathbf{X}} d\sigma(\lambda)
\end{equation}

\begin{equation}
= \int_{\mathbb{R}^n} \hat{u}(\lambda) \hat{v}(\lambda) d\sigma(\lambda)
\end{equation}

Finally, we have to obtain the form of the measure \(d\sigma(\lambda)\). For this purpose we use \(\|\Omega(\lambda)\|_{\mathcal{S}^{-m}} \leq c < \infty\).

According to (1.18) it is sufficient to see the norm in the elements of \(\mathcal{S}^{-m}\) with

\begin{equation}
\Omega_{n}(\lambda) = \sum_{k=1}^{n} \lambda_k x_k^2
\end{equation}

Then,

\begin{equation}
\|\Omega_{n}(\lambda)\|_{\mathcal{S}^{-m}} = \lim_{n \to \infty} \left(\frac{2\pi}{3}\right)^n \left(\frac{1}{2\pi}\right)^n \left(\frac{1}{2}\right)^n c(\lambda) e^{\omega_{n}(\lambda)}
\end{equation}

According to [9], we can expand \(\omega_{n}(\lambda)\) in the form of a series by \(\{h_{\alpha}(x)\}_{x}\)

\begin{equation}
\omega_{n}(\lambda) = \sum_{\alpha} c_{\alpha}^{(n)} h_{\alpha}(x)
\end{equation}

where

\begin{equation}
c_{\alpha}^{(n)} = \int_{\mathbb{R}^n} \omega_{n}(\lambda) h_{\alpha}(x) d\gamma(x)
\end{equation}

\begin{equation}
= \prod_{k=1}^{n} \int_{\mathbb{R}^n} e^{i\lambda_k x_k^2} h_{\alpha}(x_k) \frac{1}{\sqrt{\pi}} e^{-x_k^2} dx_k
\end{equation}

\begin{equation}
= \prod_{k=1}^{n} \int_{\mathbb{R}^n} e^{i\lambda_k x_k^2} h_{\alpha}(x_k) \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2} dx_k
\end{equation}

\begin{equation}
= \left(\frac{2\pi}{\sqrt{2\pi}}\right)^n i^{\alpha} h_{\alpha}(\lambda^{(n)}).
\end{equation}
and $\lambda^{(n)} = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots)$. Therefore, by using the definition of the space $S^{-m}_g$ we obtain

$$\|\Omega(\lambda)\|_{S^{-m}_g \otimes S^{-m}_g} = \left(\sum_{\alpha} \frac{h_{\alpha}(\lambda)}{(\alpha_i)^m_i} \frac{1}{(3\pi)^n} c(\lambda)\right) (1.22)$$

Evidently, $c(\lambda) \leq \kappa \| \delta \|_{S^{-m}_g}^2$, where $\delta_\lambda = \sum_\alpha h_\alpha(\lambda) h_\alpha(\cdot)$ is a $\delta$-function from $S'_g(R^n)$.

One can show that for $\| \delta \|_{S^{-m}_g}$ we have the inequality

$$a(m) \exp \sum_1^\infty \frac{\lambda_{k/2}^2}{m_k} \leq \| \delta \|_{S^{-m}_g} \leq \| \delta \|_{S^{-m}_g} \exp \sum_1^\infty \frac{\lambda_{k/2}^2}{n(m_k)} ,$$

where $n(m_k) \to \infty$ as $m_k \to \infty$. Now, let $m = (m_k)^{\infty}_{k=1}$ be chosen so large that $\| \delta \|_{S^{-m}_g}^2 < \infty$ and this is usually possible by preserving the construction of (1.23) and hence

$$c(\lambda) \leq c \exp \sum_1^\infty \frac{\lambda_{k/2}^2}{n(m_k)}$$

Formula (1.24) gives the expression of the measure $d\rho(\lambda)$, corresponding to translation-invariant p.d. generalized kernel, given by the set

$$R^\infty_k = \left\{ \lambda \in \mathbb{R}_n \mid \frac{\lambda_k^2}{n(m_k)} < \infty \right\}$$

Then,

$$(K, \nu \otimes u) = \int_{R^\infty_k} (u \nu)(\lambda) d\rho(\lambda)$$

Conversely, every kernel in the (1.25) is a translation-invariant p.d. kernel on $S'_g \otimes S'_g$ (see [1]). It can be shown that the measure $d\rho(\lambda) = c(\lambda) d\sigma(\lambda)$ generates a kernel $K \in S'_g \otimes S'_g$.

2. The Case of $K \in \sigma'_{g} \otimes \sigma'_{g}$

Let us consider the kernel $K \in \sigma'_{g} \otimes \sigma'_{g}$ and then following the preceding subsection we obtain

$$\|\Omega(\lambda)\|_{\sigma^{-m}_g \otimes \sigma^{-m}_g} = c(\lambda) e^{1/2 \sum \lambda_i^2 / m_k}$$

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and therefore,

\[ c(\lambda) \leq \sum_{k=1}^{\infty} \frac{\lambda_k^2/2}{m_k} \]  \hspace{1cm} (2.1)

and hence we can prove the following theorem:

**Theorem 2**

Every translation-invariant p.d. kernel \( K e^g \otimes e^g \) admits the representation

\[ (K, v \otimes \bar{u}) = \int \Lambda^\wedge \hat{u}(\lambda) \hat{v}(\lambda) \, d\rho(\lambda) \]  \hspace{1cm} (2.2)

where \( d\rho(\lambda) = c(\lambda) \, d\sigma(\lambda) \), \( c(\lambda) \) satisfies (2.1).

From Theorem 1 and Theorem 2 we have:

**Theorem 3**

Every Translation-invariant p.d. kernel \( K e^g \otimes e^g \) is contained in \( S^g_+ \otimes S^g_+ \).

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