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Reproducing Formula and Algorithm of a Family of Biorthonormal Wavelets

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صيغة وخوارزمية لتوليد عائلة مويجات ثنائية التعامد المعير

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فى هذا البحث نقدم عائلة من المويجات ثنائية التعامد المعير، المولدة بواسطة مويجة متعامدة معيرة من نوع MRA (التحليل متعدد التفكيك) والتى لها وسيطان اختياريان، ومن ثم نناقش طرائق جديدة لإيجاد النواة، كما نناقش خوارزمية "ماليت" لإعادة البناء والتفكيك، وذلك بالنسبة لعائلة من المويجات المتعامدة المعيرة.

Keywords: *Multi-resolution analysis, Orthonormal wavelets, Family of biorthonormal wavelets.*

ABSTRACT

In this paper we present a family of biorthonormal wavelets generated by a MRA orthonormal wavelet, which possesses two arbitrary parameters. Then we discuss new expressions of reproducing kernel as well as the corresponding Mallat's reconstruction and decomposition algorithm with respect to the family of biorthonormal wavelets.

1 Introduction

All over this paper, we denote the Fourier transform of $f(t)$ by $\widehat{f}(\omega)$, $\sum\limits_n := \sum\limits_{n=-\infty}^{\infty}$, and

$$F_{m,n}(t) := 2^{\frac{m}{2}} F(2^m t - n), \ \ m, n \in Z.$$

$L^p(R)$ $(p \geq 1)$ denotes the family of functions $f(t)$ satisfying $\int\limits_{-\infty}^{\infty} |f(t)|^p dt < \infty$.
In $L^2(R)$, the inner product and the norm are defined by

$$(f, g) = \int\limits_{-\infty}^{\infty} f(t)\overline{g(t)}dt, \ \ \| f \| = \left(\int\limits_{-\infty}^{\infty} |f(t)|^2 dt \right)^{\frac{1}{2}},$$

and the Parseval identity [1]

$$(f(t), g(t)) = (\widehat{f}(\omega), \widehat{g}(\omega)), \quad f, g \in L^2(R)$$

holds. $L^2_{2\pi}$ denotes the family of functions $f(t)$ satisfying

$$\int\limits_{-\pi}^{\pi} |f(t)|^2 dt < \infty \ \text{ and } f(t + 2\pi) = f(t).$$

In the recent ten years, wavelet analysis has drawn attention widely from both mathematicians and engineers. Wavelet analysis presents a kind of new bases for representing functions (signals). Y. Meyer [2] constructed the first orthonormal wavelet whose dyadic dilations and integer translations constitute an orthonormal basis for $L^2(R)$. Later on, in 1989, S. Mallat [3] introduced the important concept of multiresolution analysis (for simplicity, MRA) of $L^2(R)$ which is the best approach for constructing the orthonormal wavelets. Here we introduce briefly the theory of MRA [3,4,5] which will be used heavily in this paper.

Definition. Let $\{V_m\}_{m\in Z}$ be a sequence of the closed subspaces of $L^2(R)$ satisfying the following:

(i) $V_m \subset V_{m+1}, \ \ m \in Z; \ \ \overline{\bigcup\limits_{m\in Z} V_m} = L^2(R); \ \ \bigcap\limits_{m\in Z} V_m = \{0\}.$

(ii) $f(t) \in V_m \ \leftrightarrow \ f(2t) \in V_{m+1}, \ \ m \in Z.$

(iii) There exists a function $\varphi(t) \in V_0$ such that $\{\varphi(t - n)\}_{n\in Z}$ is an orthonormal basis of V_0.
Then $\{V_m\}$ is called a multiresolution analysis (MRA), and $\varphi(t)$ is called an orthogonal scaling function.

It is clear by this concept that there exists a sequence $\{\alpha_n\} : \sum\limits_n |\alpha_n|^2 < \infty$, such that

$$\frac{1}{2}\varphi(\frac{t}{2}) = \sum_n \alpha_n \varphi(t - n) \ \ (L^2(R)). \tag{1}$$

Equation (1) is called the bi-scaling equation and the coefficient $\{\alpha_n\}$ is called the impulse response. Taking Fourier transform of (1), it follows that

$$\widehat{\varphi}(2\omega) = m(\omega)\widehat{\varphi}(\omega), \tag{2}$$

where $m(\omega)$ is called the transfer function

$$m(\omega) = \sum_n \alpha_n e^{-in\omega} \ (L^2_{2\pi}), \tag{3}$$

and $m(\omega)$ satisfies the identity

$$| \, m(\omega) \, |^2 + | \, m(\omega+\pi) \, |^2 = 1, \ a.e. \ \omega \in R. \tag{4}$$

Theorem A. Let the function $\psi(t)$ satisfy the condition

$$\widehat{\psi}(\omega) = e^{-i\frac{\omega}{2}} \, \overline{m(\frac{\omega}{2}+\pi)} \, \widehat{\varphi}(\frac{\omega}{2}). \tag{5}$$

Then $\{\psi_{m,n}(t)\}_{m,n\in Z}$ is an orthonormal basis for $L^2(R)$.

The function $\psi(t)$ is called a MRA orthonormal wavelet. The formula (5) is a basic formula for finding an orthonormal wavelet.

From this, one obtains the orthogonal decomposition for $L^2(R)$ as follows.

Let $W_m = \overline{span}\{\psi_{m,n}(t), \ n \in Z\}$ (where "$\overline{span}$" is a closure of linear combination). Then $\{\psi_{m,n}(t)\}_{n\in Z}$ is an orthonormal basis of W_m, and $\{W_m\}_{m\in Z}$ possesses the following properties:

$$W_m \perp W_{m+1}, \quad f(t) \in W_m \leftrightarrow f(2t) \in W_{m+1},$$

and

$$V_m \bigoplus W_m = V_{m+1} \ (\bigoplus \text{denotes orthogonal sum}), \quad L^2(R) = \bigoplus_m W_m. \tag{6}$$

However, for decomposition and reconstruction of the functions (signals), the orthogonality of bases is not very important, so one turn to discuss the bi-orthonormal wavelets [4].

Let $\{g_n(t)\}$ be a sequence of $L^2(R)$ and $V = \overline{span}\{g_n(t), n \in Z\}$. If there is a pair of positive constants A, B such that

$$A \sum_n | \, c_n \, |^2 \leq \| \sum_n c_n \, g_n(t) \, \|^2 \leq B \sum_n | \, c_n \, |^2$$

for all $\sum_n | \, c_n \, |^2 < \infty$, then $\{g_n(t)\}$ is called a Riesz basis of the subspace V of $L^2(R)$[5].

All Riesz bases can be obtained as the images of the orthonormal bases under the bounded linear operators. Riesz bases are the next best bases to the orthonormal bases.

Let $g(t) \in L^2(R)$, $g_n(t) = g(t-n)$ and $V = \overline{span}\{g(t-n), n \in Z\}$. It is well known [5] that the sufficient and necessary condition for the fact that $\{g(t-n)\}_{n\in Z}$ is a Riesz basis of the subspace V of $L^2(R)$ is that there exists a pair of positive constants A, B such that $A \leq \sum_n | \, \widehat{g}(\omega + 2n\pi) \, |^2 \leq B, \ a.e. \ \omega \in R.$

Theorem B. Let $\{g_n(t)\}$ and $\{h_n(t)\}$ be two Riesz bases of a subspace $U \subset L^2(R)$, and $(g_m, h_n) = \delta_{m,n}$. Then $f(t) = \sum_n (f, g_n) h_n(t) = \sum_n (f, h_n) g_n(t)$ for any $f \in U$.

Definition. Let $\psi^{(1)}(t),\ \psi^{(2)}(t) \in L^2(R)$. If both $\{\psi^{(1)}_{m,n}(t)\}$ and $\{\psi^{(2)}_{m,n}(t)\}$ are Riesz bases of $L^2(R)$, and

$$(\psi^{(1)}_{m,n}(t), \psi^{(2)}_{m',n'}(t)) = \delta_{m,m'}\delta_{n,n'}\ (m, m', n, n' \in Z),$$

then $\{\psi^{(1)}(t), \psi^{(2)}(t)\}$ is called a biorthonormal wavelet [5]of $L^2(R)$.

In addition, it is well known[4,5] that the sufficient and necessary condition for the fact that the $\{f(t-n)\}_{n\in Z}$ is an orthonormal system of $L^2(R)$ is that

$$\sum_n \mid \widehat{f}(\omega + 2n\pi) \mid^2 = \frac{1}{2\pi},\ a.e.\ \omega \in R. \tag{7}$$

For a given MRA, we have known that there exists always an MRA orthonormal wavelet $\psi(t)$. In this paper, based on $\psi(t)$, we construct a family of bi-orthonormal wavelets $\{\Psi^{(1,l,r)}(t), \Psi^{(2,l,r)}(t)\}$ with two chosen arbitrarily parameters l, r. Further, with help of them we give new expresses of reproducing kernel for orthogonal complementary space W_m, and discuss the corresponding Mallat' algorithm with respect to them.

2 Main Results

In this paper, first we give the following two theorems.

Theorem 1. Let $\{V_m\}$ be a MRA of $L^2(R)$, $\varphi(t)$ the corresponding orthogonal scaling function and $\psi(t)$ the MRA orthonormal wavelet based on $\varphi(t)$. For $l \in Z, \mid r \mid < \frac{1}{2}$ $(r \in R)$, set

$$\Psi^{(1,l,r)}(t) = \psi(t) + r(\psi(t+l) - \psi(t-l)),\quad \Psi^{(2,l,r)}(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k C_n^k r^n \psi(t + nl - 2kl), \tag{8}$$

where $C_n^m = \frac{n!}{m!(m-n)!}$. Then $\{\Psi^{(1,l,r)}(t), \Psi^{(2,l,r)}(t)\}$ is a family of biorthonormal wavelets of $L^2(R)$.

Next, we give new expressions of reproducing kernel and new reproducing formulas of W_m.

Let

$$r_m^{(j,k)}(x,t) = 2^m r_0^{(j,k)}(2^m x, 2^m t),\ j \neq k\ (j,k = 1,2),$$

where $r_0^{(j,k)}(x,t) := \sum\limits_{-\infty}^{\infty} \Psi^{(j,l,r)}(t-n)\overline{\Psi^{(k,l,r)}}(x-n)$.

Theorem 2. Under the assumptions of Theorem 1, suppose further that

(i) The MRA orthonormal wavelets $\psi(t) = O((1+ \mid t \mid)^{-(4+\epsilon)})$ $(\epsilon > 0)$.

(ii) $(\widehat{\psi}(\omega))^{(i)} \in L^1(R)$ $(i = 0, 1, 2, 3)$.

Then, for any $h(t) \in W_m$, the following two equalities hold

$$h(t) = \int_{-\infty}^{\infty} h(x) r_m^{(j,k)}(x,t)dx,\quad j \neq k\ (j,k = 1,2).$$

The above two equalities are new reproducing formulas of W_m, and $r_m^{(j,k)}(x,t)$ are new expressions of reproducing kernel.

Then, we discuss the corresponding Mallat's reconstruction and decomposition algorithm.

3 Proof of Theorem 1

In order to prove Theorem 1 , we need two Lemmas.

Lemma 1. Let $W_m^{(j)} = \overline{span}\{\Psi_{m,n}^{(j,l,r)}(t),\ n \in Z\}$ $(j = 1, 2)$. Then $\{\Psi_{m,n}^{(j,l,r)}(t)\}_{n\in Z}$ is a Riesz basis of the space $W_m^{(j)}$ $(j = 1, 2)$ and $W_m^{(j)} \subset V_m$ $(j = 1, 2)$.

Proof. First we prove that $\{\Psi_{m,n}^{(j,l,r)}\}_{n\in Z}$ is a Riesz basis of the space $W_m^{(j)}$.

Since $\Psi_{m,n}^{(j,l,r)}(t) = 2^{\frac{m}{2}}\Psi^{(j,l,r)}(2^m t - n)$, their Fourier transforms are

$$\widehat{\Psi}_{m,n}^{(j,l,r)}(\omega) = 2^{-\frac{m}{2}}\widehat{\Psi}^{(j,l,r)}(\frac{\omega}{2^m})\, e^{-i\frac{n}{2^m}\omega},\ j = 1, 2. \tag{9}$$

For the case $j = 1$, we consider a sum $\sum\limits_{\nu} \mid \widehat{\Psi}_{m,n}^{(1,l,r)}(\omega + 2\nu\pi) \mid^2$.

On the one hand, by (9), we have

$$\sum_{\nu} \mid \widehat{\Psi}_{m,n}^{(1,l,r)}(\omega + 2\nu\pi) \mid^2 = 2^{-m}\sum_{\nu} \mid \widehat{\Psi}^{(1,l,r)}(\frac{\omega + 2\nu\pi}{2^m}) \mid^2 .$$

Taking $\nu = 2^m k,\ 2^m k + 1,\ 2^m k + 2,\ ...,\ 2^m k + 2^m - 1,\ k \in Z$ we can split the integers $\nu \in Z$ into 2^m groups , corresponding to this, the sum on right-hand side of the above equality can be split into 2^m sum , that is, it can be rewritten by rearrangement of the terms in the form

$$\sum_{\nu} \mid \widehat{\Psi}_{m,n}^{(1,l,r)}(\omega + 2\nu\pi) \mid^2 = 2^{-m}\{\sum_{k} \mid \widehat{\Psi}^{(1,l,r)}(\frac{\omega}{2^m} + 2k\pi) \mid^2 + \sum_{k} \mid \widehat{\Psi}^{(1,l,r)}(\frac{\omega + 2\pi}{2^m} + 2k\pi) \mid^2 + ...$$

$$+ \sum_{k} \widehat{\Psi}^{(1,l,r)}(\frac{\omega + 2(2^m - 1)\pi}{2^m} + 2k\pi) \mid^2\}. \tag{10}$$

On the other hand , taking Fourier transform in the first formula of (8), we have

$$\widehat{\Psi}^{(1,l,r)}(\omega) = \widehat{\psi}(\omega)(1 + 2ir\sin(l\,\omega)). \tag{11}$$

Again, by (5),

$$\widehat{\Psi}^{(1,l,r)}(\omega) = e^{-i\frac{\omega}{2}}\overline{m(\frac{\omega}{2} + \pi)}\widehat{\varphi}(\frac{\omega}{2})(1 + 2ir\sin(l\,\omega)), \tag{12}$$

and so

$$S_1(\omega) := \sum_{k} \mid \widehat{\Psi}^{(1,l,r)}(\omega + 2k\pi) \mid^2 = \mid 1 + 2ir\sin(l\,\omega) \mid^2 s_1(\omega), \tag{13}$$

where

$$s_1(\omega) := \sum_{k} \mid m(\frac{\omega}{2} + k\pi + \pi)\widehat{\varphi}(\frac{\omega}{2} + k\pi) \mid^2 . \tag{14}$$

Splitting the odd and even terms of the sum on the right-hand side of (14) and noticing that $m(\omega)$ is a periodic function with $2\pi-$period, we have

$$s_1(\omega) = \sum_{\nu} \mid m(\frac{\omega}{2} + 2\nu\pi + \pi)\widehat{\varphi}(\frac{\omega}{2} + 2\nu\pi) \mid^2 + \sum_{\nu} \mid m(\frac{\omega}{2} + 2\nu\pi)\widehat{\varphi}(\frac{\omega}{2} + 2\nu\pi - \pi) \mid^2$$

$$=| m(\frac{\omega}{2}+\pi) |^2 \sum_\nu | \widehat{\varphi}(\frac{\omega}{2}+2\nu\pi) |^2 + | m(\frac{\omega}{2}) |^2 \sum_\nu | \widehat{\varphi}(\frac{\omega}{2}+2\nu\pi-\pi) |^2 .$$

Since $\varphi(t)$ is an orthogonal scaling function , by its definition and (7) we see that

$$\sum_\nu | \widehat{\varphi}(\frac{\omega}{2}+2\nu\pi) = \frac{1}{2\pi}, \qquad \sum_\nu | \widehat{\varphi}(\frac{\omega-2\pi}{2}+2\nu\pi) = \frac{1}{2\pi}, \ a.e.\ \omega \in R.$$

From this and (4),

$$s_1(\omega) = \frac{1}{2\pi},\ a.e.\ \omega \in R.$$

Again, by (13), we get for almost everywhere $\omega \in R$,

$$S_1(\omega) = \sum_k | \widehat{\Psi}^{(1,l,r)}(\omega+2k\pi) |^2 = \frac{1}{2\pi}|1+2ir\sin(l\omega)|^2.$$

Combining this with (10), we obtain for almost everywhere $\omega \in R$,

$$\sum_\nu |\widehat{\Psi}^{(1,l,r)}_{m,n}(\omega+2\nu\pi)|^2 = \frac{2^{-m}}{2\pi}\{|1+2ir\sin(l\frac{\omega}{2^m})|^2 + |1+2ir\sin(l\frac{\omega+2\pi}{2^m})|^2 + ...$$

$$+|1+2ir\sin(l\frac{\omega+2(2^m-1)\pi}{2^m})|^2\}.$$

Again, in view of $1-2 \mid r \mid \leq \mid 1+2ir\sin(ly) \mid \leq 1+2 \mid r \mid$, we obtain

$$\frac{(1-2\mid r\mid)^2}{2\pi} \leq \sum_\nu | \widehat{\Psi}^{(1,l,r)}_{m,n}(\omega+2\nu\pi) |^2 \leq \frac{(1+2\mid r\mid)^2}{2\pi}, a.e. \omega \in R.$$

According to the sufficient and necessary condition for the fact that it is a Riesz basis , we know that $\{\Psi^{(1,l,r)}_{m,n}(t)\}_{n\in Z}$ is a Riesz basis of the space $W^{(1)}_m$.

Similarly, taking Fourier transform in the second formula of (8), and using the binomial formula $(a+b)^n = \sum_{k=0}^{n} C^k_n a^k b^{n-k}$ and the summation formula of geometrical series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, we get by (5)

$$\begin{aligned}\widehat{\Psi}^{(2,l,r)}(\omega) &= \sum_{n=0}^{\infty} \widehat{\psi}(\omega) \sum_{k=0}^{n} C^k_n (re^{il\omega})^{n-k}(-re^{-il\omega})^k \\ &= \widehat{\psi}(\omega)(1-2ir\sin(l\omega))^{-1} \\ &= e^{-i\frac{\omega}{2}}\overline{m(\tfrac{\omega}{2}+\pi)}\,\widehat{\varphi}(\tfrac{\omega}{2})(1-2ir\sin(l\omega))^{-1}.\end{aligned} \tag{15}$$

Again, replacing (12) by (15), and then going along the above derived line , we can also prove that $\{\Psi^{(2,l,r)}_{m,n}(t)\}_{n\in Z}$ is a Riesz basis of the space $W^{(2)}_m$.

Next, we prove that $W^{(j)}_m \subset V_{m+1},\ j=1,2.$

Set

$$m^{(1)}(\omega) = e^{-i\omega}\overline{m(\omega+\pi)}\,(1+2ir\sin(2l\omega)), \tag{16}$$

$$m^{(2)}(\omega) = e^{-i\omega}\overline{m(\omega+\pi)}\,(1-2ir\sin(2l\omega))^{-1}, \tag{17}$$

where $m(\omega)$ is stated in (3).

Since $m(\omega) \in L^2_{2\pi}$, it is clear that $m^{(j)}(\omega) \in L^2_{2\pi},\ j = 1, 2$. So we can expand them as follows:

$$m^{(j)}(\omega) = \sum_n \beta_n^{(j)} e^{in\omega}\ \ (L^2_{2\pi}),\ \ j = 1, 2.$$

From this and (16), (17), we see by (12) and (15) that

$$\widehat{\Psi}^{(j,l,r)}(\omega) = \sum_n \beta_n^{(j)} e^{in\frac{\omega}{2}}\, \widehat{\varphi}(\frac{\omega}{2})\ \ (L^2(R)),\ \ j = 1, 2.$$

Taking inverse Fourier transform of the above formula , we have

$$\Psi^{(j,l,r)}(t) = \sum_n 2\beta_n^{(j)}\, \varphi(2t + n)\ \ (L^2(R)),\ \ j = 1, 2. \tag{18}$$

Further,

$$\begin{aligned}\Psi^{(j,l,r)}_{m,n}(t) = 2^{\frac{m}{2}}\Psi^{(j,l,r)}(2^m t - n) &= \sum_k 2^{\frac{m+2}{2}}\beta^{(j)}_{2n-k}\varphi(2^{m+1}t - k)\\ &= \sum_k \sqrt{2}\beta^{(j)}_{2n-k}\varphi_{m+1,k}(t)\ \ (L^2(R))\ \ (n \in Z), j = 1, 2,\end{aligned}$$

Because $\varphi_{m+1,k}(t) \in V_{m+1}$ $(k \in Z)$, the above equalities show that

$$\Psi^{(j,l,r)}_{m,n}(t) \in V_{m+1}\ \ (n \in Z), j = 1, 2.$$

By the definition of $W_m^{(j)}$, we know that $W_m^{(j)} \subset V_{m+1},\ j = 1, 2$.

So the proof of Lemma 1 is completed.

Lemma 2. $W_m^{(j)} = W_m$ $(j = 1, 2)$.

Proof. For the case j =1, let

$$h^{(1)}(\omega) = -\frac{m^{(1)}(\omega) + m^{(1)}(\omega + \pi)}{1 + 2ir\sin(2l\omega)},\quad h^{(2)}(\omega) = \frac{m(\omega) + m(\omega + \pi)}{1 + 2ir\sin(2l\omega)},$$

$$h^{(3)}(\omega) = e^{i\omega}\frac{m^{(1)}(\omega) - m^{(1)}(\omega + \pi)}{1 + 2ir\sin(2l\omega)},\quad h^{(4)}(\omega) = -e^{i\omega}\frac{m(\omega) - m(\omega + \pi)}{1 + 2ir\sin(2l\omega)}.$$

By (4) and (16), we get

$$m(\omega)m^{(1)}(\omega + \pi) - m(\omega + \pi)m^{(1)}(\omega) = -e^{-i\omega}(1 + 2ir\sin(2l\omega)) \neq 0,\ \ a.e.\ \omega \in R$$

and so

$$\begin{aligned} m(\omega)h^{(1)}(\omega) + m^{(1)}(\omega)h^{(2)}(\omega) &= e^{-i\omega},\\ m(\omega)h^{(3)}(\omega) + m^{(1)}(\omega)h^{(4)}(\omega) &= 1,\ \ a.e.\ \omega \in R. \end{aligned} \tag{19}$$

On the other hand, noticing that both $m(\omega)$ and $m^{(1)}(\omega)$ are periodic functions with $2\pi-$period and $l \in Z$, it is clear that $h^{(j)}(\omega)$ $(j = 1, ..., 4)$ are periodic functions with $\pi-$period, and so we can expand them into Fourier series as follows:

$$h^{(j)}(\omega) = \sum_n \gamma^{(j)}_{2n} e^{2in\omega}\ \ (j = 1, ..., 4).$$

From this and (19),

$$m(\omega)\sum_n \gamma_{2n}^{(1)} e^{2in\omega} + m^{(1)}(\omega)\sum_n \gamma_{2n}^{(2)} e^{2in\omega} = e^{-i\omega}, \quad a.e.\ \omega \in R,$$

$$m(\omega)\sum_n \gamma_{2n}^{(3)} e^{2in\omega} + m^{(1)}(\omega)\sum_n \gamma_{2n}^{(4)} e^{2in\omega} = 1, \quad a.e.\ \omega \in R.$$

Replacing ω by $\frac{\omega}{2^{m+1}}$ in the above two equalities , and then multiplying by

$$\widehat{\varphi}(\frac{\omega}{2^{m+1}})e^{-ik\frac{\omega}{2^m}} \ (k \in Z),$$

we obtain by (2) for almost everywhere $\omega \in R$

$$\widehat{\varphi}(\frac{\omega}{2^m})\sum_n \gamma_{2n}^{(1)} e^{-i(k-n)2^{-m}\omega} + m^{(1)}(\frac{\omega}{2^{m+1}})\widehat{\varphi}(\frac{\omega}{2^{m+1}})\sum_n \gamma_{2n}^{(2)} e^{-i(k-n)2^{-m}\omega} =$$

$$\widehat{\varphi}(\frac{\omega}{2^{m+1}})e^{-i(k+\frac{1}{2})2^{-m}\omega},$$

$$\widehat{\varphi}(\frac{\omega}{2^m})\sum_n \gamma_{2n}^{(3)} e^{-i(k-n)2^{-m}\omega} + m^{(1)}(\frac{\omega}{2^{m+1}})\widehat{\varphi}(\frac{\omega}{2^{m+1}})\sum_n \gamma_{2n}^{(4)} e^{-i(k-n)2^{-m}\omega} = \widehat{\varphi}(\frac{\omega}{2^{m+1}})e^{-ik2^{-m}\omega}.$$

But, by (12) and (15-17), we know that

$$\Psi^{(j,l,r)}(\omega) = m^{(j)}(\frac{\omega}{2})\,\widehat{\varphi}(\frac{\omega}{2}), \ j = 1, 2. \tag{20}$$

So, for almost everywhere $\omega \in R$, we have

$$\widehat{\varphi}(\frac{\omega}{2^m})\sum_n \gamma_{2n}^{(1)} e^{-i(k-n)2^{-m}\omega} + \widehat{\Psi}^{(1,l,r)}(\frac{\omega}{2^m})\sum_n \gamma_{2n}^{(2)} e^{-i(k-n)2^{-m}\omega} = \widehat{\varphi}(\frac{\omega}{2^{m+1}})e^{-i(k+\frac{1}{2})2^{-m}\omega},$$

$$\widehat{\varphi}(\frac{\omega}{2^m})\sum_n \gamma_{2n}^{(3)} e^{-i(k-n)2^{-m}\omega} + \widehat{\Psi}^{(1,l,r)}(\frac{\omega}{2^m})\sum_n \gamma_{2n}^{(4)} e^{-i(k-n)2^{-m}\omega} = \widehat{\varphi}(\frac{\omega}{2^{m+1}})e^{-ik2^{-m}\omega}.$$

Again, taking inverse Fourier transform of the above two equalities , we obtain for $k \in Z$,

$$\sum_n \gamma_{2k-2n}^{(1)}\varphi_{m,n}(t) + \sum_n \gamma_{2k-2n}^{(2)}\Psi_{m,n}^{(1,l,r)}(t) = \sqrt{2}\varphi_{m+1,2k+1}(t),$$

$$\sum_n \gamma_{2k-2n}^{(3)}\varphi_{m,n}(t) + \sum_n \gamma_{2k-2n}^{(4)}\Psi_{m,n}^{(1,l,r)}(t) = \sqrt{2}\varphi_{m+1,2k}(t).$$

Since $\{\varphi_{m+1,n}\}_{n\in Z}$ is an orthonormal basis of V_{m+1}, the above two equalities mean that

$$V_{m+1} \subset V_m + W_m^{(1)}. \tag{21}$$

By the definition of MRA , we see that $V_m \subset V_{m+1}$. Again, by Lemma 1 we see that $W_m^{(1)} \subset V_{m+1}$, so

$$V_m + W_m^{(1)} \subset V_{m+1}.$$

From this and (21) ,

$$V_{m+1} = V_m + W_m^{(1)}. \tag{22}$$

Similar to the derivation process of (22), we can also obtain

$$V_{m+1} = V_m + W_m^{(2)}. \tag{23}$$

Below, we shall prove that $V_m \perp W_m^{(j)}$ $(j = 1, 2)$.
Using the Parseval identity and the definition of inner product,we get

$$(\,\Psi_{m,k}^{(j,l,r)}(t)\,,\,\varphi_{m,n}(t)\,) = (\,\widehat{\Psi}_{m,k}^{(j,l,r)}(\omega)\,,\,\widehat{\varphi}_{m,n}(\omega)\,) = \int_{-\infty}^{\infty} \widehat{\Psi}_{m,k}^{(j,l,r)}(\omega)\overline{\widehat{\varphi}_{m,n}(\omega)}\,d\omega$$

Again, by (9) and $\widehat{\varphi}_{m,n}(\omega) = 2^{-\frac{m}{2}}\widehat{\varphi}(\frac{\omega}{2^m})e^{-i\frac{n}{2^m}\omega}$ we get

$$(\,\Psi_{m,k}^{(j,l,r)}(t)\,,\,\varphi_{m,n}(t)\,) = 2^{-m}\int_{-\infty}^{\infty} \widehat{\Psi}^{(j,l,r)}(2^{-m}\omega)\overline{\widehat{\varphi}(2^{-m}\omega)}\,e^{i(n-k)2^{-m}\omega}d\omega$$

$$= \int_{-\infty}^{\infty} \overline{\widehat{\varphi}(\omega)}\,\widehat{\Psi}^{(j,l,r)}(\omega)e^{i(n-k)\omega}d\omega =: S_2. \tag{24}$$

But, by (2) and (20), the last term of the above formula (24) equals

$$\begin{aligned} S_2 &= \int_{-\infty}^{\infty} m^{(j)}(\tfrac{\omega}{2})\overline{m(\tfrac{\omega}{2})}\mid \widehat{\varphi}(\tfrac{\omega}{2})\mid^2 e^{i(n-k)\omega}d\,\omega \\ &= \sum_{\nu}\int_{2\nu\pi}^{(2\nu+2)\pi} m^{(j)}(\tfrac{\omega}{2})\overline{m(\tfrac{\omega}{2})}\mid \widehat{\varphi}(\tfrac{\omega}{2})\mid^2 e^{i(n-k)\omega}d\,\omega\ . \\ &= \int_0^{2\pi} e^{i(n-k)\omega}s_2(\omega)d\,\omega,\ \ k,n\in Z, \end{aligned}$$

where

$$s_2(\omega) = \sum_{\nu} m^{(j)}(\frac{\omega}{2}+\nu\pi)\overline{m(\frac{\omega}{2}+\nu\pi)}\ \mid \widehat{\varphi}(\frac{\omega}{2}+\nu\pi)\mid^2.$$

Replacing $\mid m(\frac{\omega}{2}+\nu\pi+\pi)\mid^2$ by $m^{(j)}(\frac{\omega}{2}+\nu\pi)\overline{m(\frac{\omega}{2}+\nu\pi)}$ in (14) and passing a direct calculation which is similar to $s_1(\omega)$, we can obtain by (16) and (17) that $s_2(\omega) = 0$. Combining this with (24), we get

$$(\,\Psi_{m,k}^{(j,l,r)}(t)\,,\,\varphi_{m,n}(t)\,) = 0,\ \ k,n\in Z,\ j = 1,2.$$

Since $\{\Psi_{m,k}^{(j,l,r)}(t)\}_{k\in Z}$ is a Riesz basis of $W_m^{(j)}$ and $\{\varphi_{m,n}(t)\}_{n\in Z}$ is an orthonormal basis of V_m. The above formulas show that $W_m^{(j)} \perp V_m,\ j = 1,2$.

Again, by (22) and (23) we know that

$$V_{m+1} = V_m \bigoplus W_m^{(j)},\ \ j = 1,2.$$

Finally, by (6), we obtain

$$W_m^{(j)} = W_m,\ \ j = 1,2.$$

This completes the proof of Lemma 2.

Proof of Theorem 1. By Lemmas 1 and 2, and $L^2(R) = \bigoplus_m W_m$ of (6), we can conclude using the definition of Riesz basis that $\{\Psi_{m,n}^{(j,l,r)}(t)\}_{m,n\in Z}$ $(j = 1, 2)$ are two different Riesz bases of $L^2(R)$.

In order to prove that $\{\Psi^{(1,l,r)}(t), \Psi^{(2,l,r)}(t)\}$ is a family of biorthonormal wavelets of $L^2(R)$, by the definition, we need only to prove that the equality $(\Psi_{m,n}^{(1,l,r)}, \Psi_{m',n'}^{(2,l,r)}) = \delta_{m,m'}\delta_{n,n'}$ holds.

Case 1: $m = m'$. By (11), (15) and (5), we have

$$\widehat{\Psi}^{(1,l,r)}(\omega)\overline{\widehat{\Psi}^{(2,l,r)}}(\omega) = \mid \widehat{\psi}(\omega) \mid^2 . \tag{25}$$

Using the Parseval identity and the definition of inner product, we obtain by (9) and (25)

$$(\Psi_{m,n}^{(1,l,r)}(t), \Psi_{m,n'}^{(2,l,r)}(t)) = (\widehat{\Psi}_{m,n}^{(1,l,r)}(\omega), \widehat{\Psi}_{m,n'}^{(2,l,r)}(\omega)) = \int_{-\infty}^{\infty} \mid \widehat{\psi}(\omega) \mid^2 e^{-i(n-n')\omega} d\omega := S_3. \tag{26}$$

By the definition of the subspace W_0, we know that $\{\psi(t-n)\}_{n\in Z}$ is an orthonormal basis of W_0, so we get by (7)

$$\sum_\nu \mid \widehat{\psi}(\omega + 2\nu\pi) \mid^2 = \frac{1}{2\pi},$$

and the integral S_3 in (26) is calculated as follows:

$$S_3 = \sum_\nu \int_{2\nu\pi}^{2(\nu+1)\pi} \mid \widehat{\psi}(\omega) \mid^2 e^{-i(n-n')\omega} d\omega = \int_0^{2\pi} \sum_\nu \mid \widehat{\psi}(\omega + 2\nu\pi) \mid^2 e^{-i(n-n')\omega} d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-i(n-n')\omega} d\omega = \delta_{n,n'}.$$

Combining this with (26), we get

$$(\Psi_{m,n}^{(1,l,r)}(t), \Psi_{m,n'}^{(2,l,r)}(t)) = \delta_{n,n'}.$$

Case 2: $m \neq m'$. In view of $W_m \bot W_{m'}$, noticing that

$$\Psi_{m,n}^{(1,l,r)}(t) \in W_m^{(1)} = W_m, \quad \Psi_{m',n'}^{(2,l,r)}(t) \in W_{m'}^{(2)} = W_{m'}, \; n, n' \in Z,$$

it is clear that $(\Psi_{m,n}^{(1,l,r)}(t), \Psi_{m',n'}^{(2,l,r)}(t)) = 0$, and hence the proof of Theorem 1 is completed.

4 Proof of Theorem 2

To prove Theorem 2, we need the following two lemmas.

Lemma 3. Under the condition of Theorem 2, then

$$\Psi^{(j,l,r)}(t-n)\overline{\Psi^{(k,l,r)}}(x-n) = O((1+ \mid x-t \mid)^{-\frac{3}{2}}(1+ \mid x-n \mid)^{-\frac{3}{2}}), \quad j \neq k \, (j, k = 1, 2).$$

Proof. On the one hand, by the condition (ii) and noticing that the functions

$$1 + 2ir\sin(2l\omega), \quad (1 - 2ir\sin(2l\omega))^{-1}, \quad |r| < \frac{1}{2}, \ l \in Z$$

and their derivatives are all bounded with respect to ω, we know by (11) and (15) that

$$(\widehat{\Psi}^{(j,l,r)}(\omega))^{(i)} \in L^1(R) \ (i = 0, 1, 2, 3).$$

On the other hand, by the condition (i), we know that $\psi(t) \in L^1(R)$ and $t^3\psi(t) \in L^1(R)$. Further, $(\widehat{\psi}(\omega))'''$ is continuous on R. So we obtain from (11) and (15) that $(\widehat{\Psi}^{(j,l,r)}(\omega))'''$ is continuous on R.

Combining the above two results, using the property of differentiability of Fourier transform [1], we get that the Fourier transform of $(\widehat{\Psi}^{(j,l,r)})'''$ is equal to $(it)^3$ multiplied by the Fourier transform of $\widehat{\Psi}^{(j,l,r)}(\omega)$. But the Fourier transform of $\widehat{\Psi}^{(j,l,r)}(\omega)$ is just equal to $\Psi^{(j,l,r)}(-t)$, and so the Fourier transform of $(\widehat{\Psi}^{(j,l,r)})'''$ is equal to $(it)^3\Psi^{(j,l,r)}(-t)$.

Since $(\widehat{\Psi}^{(j,l,r)}(\omega))''' \in L^1(R)$, by Riemman-Lebesgue Lemma [1], Fourier transform of $(\widehat{\Psi}^{(j,l,r)}(\omega))'''$ tends to zero when $|t| \to \infty$. Again because it is continuous on R, the Fourier transform of $(\widehat{\Psi}^{(j,l,r)}(\omega))'''$ is bounded on the whole real axis R.

From this, we know that

$$\Psi^{(j,l,r)}(t) = O((1+ \mid t \mid)^{-3}), \quad j = 1, 2.$$

Again, using the inequality

$$(1+ \mid a \mid)(1+ \mid b \mid) \geq 1+ \mid a - b \mid,$$

we obtain the conclusion of Lemma 3 immediately.

By Lemma 3, the following lemma is clearly true.

Lemma 4. Under the conditions of Theorem 2, let

$$K_N^{(j,k)}(x,t) = \sum_{-N}^{N} \Psi^{(j,l,r)}(t-n)\overline{\Psi^{(k,l,r)}(x-n)}, \quad j \neq k \ (j,k = 1,2). \tag{27}$$

Then

(i) $K_N^{(j,k)}(x,t) = O((1+ \mid x - t \mid)^{-\frac{3}{2}})$

(ii) the series

$$r_0^{(j,k)}(x,t) := \sum_{-\infty}^{\infty} \Psi^{(j,l,r)}(t-n)\overline{\Psi^{(k,l,r)}}(x-n), \quad j \neq k \ (j,k = 1,2) \tag{28}$$

converges uniformly on every bounded closed interval in R for fixed $t \in R$ with respect to x.

Proof of Theorem 2. Let $h(t) \in W_m$. By the property of $\{W_m\}$, $\widetilde{h}(t) := h(\frac{t}{2^m}) \in W_0$. Again by Lemma 1 and Lemma 2, we know that $\{\Psi^{(j,l,r)}(t-n)\}, j = 1, 2$ are two Riesz bases of W_0 and $(\Psi^{(1,l,r)}(t-m), \Psi^{(2,l,r)}(t-n)) = \delta_{m,n}$. By Theorem B we find

$$\widetilde{h}(t) = \sum_n c_n^{(k,0)}\Psi^{(j,l,r)}(t-n) \, (L^2(R)), \ j \neq k \ (j,k = 1,2),$$

where

$$c_n^{(k,0)} = \int_{-\infty}^{\infty} \widetilde{h}(x)\overline{\Psi^{(k,l,r)}(x-n)}dx,\ k=1,2. \tag{29}$$

Let $S_N^{(j,l,r)}(t) = \sum_{-N}^{N} c_n^{(k,0)}\Psi^{(j,l,r)}(t-n)$. By (29), we can rewrite them in the form

$$S_N^{(j,l,r)}(t) = \int_{-\infty}^{\infty} \widetilde{h}(x)K_N^{(j,k)}(x,t)dx \quad j \neq k\ (j=1,2),$$

where $K_N^{(j,k)}(x,t)$ is given by (27). Consider the deviations:

$$S_N^{(j,l,r)}(t) - \int_{-\infty}^{\infty} \widetilde{h}(x)r_0^{(j,k)}(x,t)dx = \int_{-\infty}^{\infty} \widetilde{h}(x)\{K_N^{(j,k)}(x,t) - r_0^{(j,k)}(x,t)\}dx.$$

Because $(1+y)^{-\frac{1}{2}} \to 0\ \ (y \to \infty)$, for given $\eta > 0$, there exists M_0 such that $(1+M_0)^{-\frac{1}{2}} < \eta$. So, fixing M_0, we split the integral on the right-hand side of the above formula into two parts

$$s_N^{(j,l,r)}(t) - \int_{-\infty}^{\infty} \widetilde{h}(x)r_0^{(j,k)}(x,t)dx =: I_1(t) + I_2(t), \tag{30}$$

where

$$I_1(t) = \int_{|x-t|\leq M_0} \widetilde{h}(x)\{K_N^{(j,k)}(x,t) - r_0^{(j,k)}(x,t)\}dx,$$

$$I_2(t) = \int_{|x-t|> M_0} \widetilde{h}(x)\{K_N^{(j,k)}(x,t) - r_0^{(j,k)}(x,t)\}dx.$$

First, we estimate $I_2(t)$.

By Lemma 4 and (28), we know easily that $r_0^{(j,k)}(x,t) = O((1+\mid x-t \mid)^{-\frac{3}{2}})$. Using Cauchy's integral inequality, we get

$$I_2(t) = O((1+M_0)^{-\frac{1}{2}})\{\int_{-\infty}^{\infty} \mid \widetilde{h}(x) \mid^2 dx \cdot \int_{-\infty}^{\infty} (1+\mid x-t \mid)^{-2}dx\}^{\frac{1}{2}}.$$

and hence

$$I_2(t) = O((1+M_0)^{-\frac{1}{2}})\sqrt{2} \parallel \widetilde{h} \parallel. \tag{31}$$

Next, we estimate $I_1(t)$.

By Lemma 4, we know that for fixed t, $K_N^{(j,k)}(x,t)$ converges uniformly to $r_0^{(j,k)}(x,t)$ on $|x-t| \leq M_0$. So, for given $\eta > 0$,

$$|K_N^{(j,k)}(x,t) - r_0^{(j,k)}(x,t)| < \eta, \quad |x-t| \leq M_0,$$

when N is large enough. Using Cauchy's inequality, we get

$$|I_1(t)| \leq \eta \{ \int\limits_{|x-t|\leq M_0} dx \cdot \int\limits_{-\infty}^{\infty} | \widetilde{h} |^2 dx\}^{\frac{1}{2}} \leq \eta\sqrt{2M_0} \parallel \widetilde{h} \parallel .$$

From this and (30-31), we obtain $S_N^{(j,l,r)}(t) \to \int\limits_{-\infty}^{\infty} \widetilde{h}(x)r_0^{(j,k)}(x,t)dx,\ N \to \infty.$

On the other hand, since $S_N^{(j,l,r)}(t) \to \widetilde{h}(t)\,(L^2(R))$, by Riesz'Theorem, we know that there exists a subsequence $\{N_k\}$ such that $S_{N_k}^{(j,l,r)}(t) \to \widetilde{h}(t)$.

Thus, by the uniqueness of limits, we obtain $\widetilde{h}(t) = \int\limits_{-\infty}^{\infty} \widetilde{h}(x)r_0^{(j,k)}(x,t)dx,\ j \neq k\ (j,k = 1,2).$ Again, by the definition $\widetilde{h}(t) = h(\frac{t}{2^m})$, they are rewritten in the form

$$h(\frac{t}{2^m}) = \int\limits_{-\infty}^{\infty} h(\frac{x}{2^m})r_0^{(j,k)}(x,t)dx = \int\limits_{-\infty}^{\infty} h(x)2^m r_0^{(j,k)}(2^m x,t)dx.$$

Replacing $\frac{t}{2^m}$ by t, and using (28) we obtain

$$h(t) = \int\limits_{-\infty}^{\infty} h(x)2^m r_0^{(j,k)}(2^m x, 2^m t)dx = \int\limits_{-\infty}^{\infty} h(x)r_m^{(j,k)}(x,t)dx,$$

where $r_m^{(j,k)}(x,t) = 2^m r_0^{(j,k)}(2^m x, 2^m t),\ j \neq k\ (j,k = 1,2)$, and so we complete the proof of Theorem 2.

5 Algorithm

In this section, we give the corresponding Mallat's reconstruction and decomposition algorithm with respect to the family of biorthogonal wavelets $\{\Psi^{(1,l,r)}(t),\ \Psi^{(2,l,r)}(t)\}$ based on the MRA orthonormal wavelet $\psi(t)$.

5.1 Reconstruction Algorithm

Since $\{\varphi_{m+1,n}(t)\}_{n\in Z}$ is an orthogonal basis of V_{m+1}, for each $f(t) \in V_{m+1}$,

$$f(t) = \sum_n c_n^{m+1}\varphi_{m+1,n}(t),$$

where $c_n^{m+1} = (f(t), \varphi_{m+1,n}(t)), n \in Z$.

By the concept of MRA, we see $\{\varphi_{m,n}(t)\}_{n\in Z}$ is an orthonormal basis of V_m. By Lemmas 1 and 2, we see that $\{\Psi_{m,n}^{(j,l,r)}(t)\}_{n\in Z},\ j = 1,2$ are two Riesz bases of W_m. Again, because $V_{m+1} = V_m \bigoplus W_m$, we know that $\{\varphi_{m,n}(t), \Psi_{m,n'}^{(j,l,r)}(t)\}_{n,n'\in Z},\ j = 1,2$ are two bases of the space V_{m+1}. So, for $f(t) \in V_{m+1}$, we have

$$f(t) = \sum_n c_n^m \varphi_{m,n}(t) + \sum_n d_n^{(k,m)}\Psi_{m,n}^{(j,l,r)}(t), \quad k \neq j,\ k,j = 1,2,$$

where $c_n^m = (f(t), \varphi_{m,n}(t)), \quad d_n^{(k,m)} = (f(t), \Psi_{m,n}^{(k,l,r)}(t)) \quad k = 1, 2.$
Hence

$$\sum_n c_n^{m+1} \varphi_{m+1,n}(t) = \sum_n c_n^m \varphi_{m,n}(t) + \sum_n d_n^{(k,m)} \Psi_{m,n}^{(j,l,r)}(t), \quad k \neq j\,(k, j = 1, 2). \tag{32}$$

But by (1) and (18), we know that

$$\varphi_{m,n}(t) = \sum_\nu p_{\nu-2n} \varphi_{m+1,\nu}(t) \quad (p_n = \sqrt{2}\alpha_n). \tag{33}$$

$$\Psi_{m,n}^{(j,l,r)}(t) = \sum_\nu q_{\nu-2n}^{(j)} \varphi_{m+1,\nu}(t) \quad (q_n^{(j)} = \sqrt{2}\beta_{-n}^{(j)})\,(j = 1, 2). \tag{34}$$

Combining (33) and (34) with (32), and then interchanging the order of the summations, we have

$$\sum_\nu c_\nu^{m+1} \varphi_{m+1,\nu}(t) = \sum_\nu \sum_n (c_n^m p_{\nu-2n} + d_n^{(k,m)} q_{\nu-2n}^{(j)}) \varphi_{m+1,\nu}(t), \quad k \neq j\,(k, j = 1, 2).$$

So by equating the coefficients, we obtain by (33) and (34) that the corresponding Mallat's reconstruction algorithm is as follows:

$$c_\nu^{m+1} = \sqrt{2} \sum_n (c_n^m \alpha_{\nu-2n} + d_n^{(k,m)} \beta_{2n-\nu}^{(j)}) \quad k \neq j\,(k, j = 1, 2),$$

where $c_n^m = (f(t), \varphi_{m,n}(t))$, $d_n^{(k,m)} = (f(t), \Psi_{m,n}^{(k,l,r)}(t))$, and α_n, $\beta_n^{(j)}$ are given by (1) and (18).

5.2 Decomposition Algorithm

Multiplying both sides of (32) by the factors $\overline{\Psi_{m,n'}^{(k,l,r)}(t)}\,(k = 1, 2)$ and $\overline{\varphi_{m,n'}(t)}$ respectively, and then integrating both sides over $(-\infty, \infty)$, by the definition of inner product, we obtain the following two equalities for $j \neq k\,(j, k = 1, 2)$

$$\sum_n c_n^{m+1} (\varphi_{m+1,n}(t), \Psi_{m,n'}^{(k,l,r)}(t)) = \sum_n c_n^m (\varphi_{m,n}(t), \Psi_{m,n'}^{(k,l,r)}(t)) + \sum_n d_n^{(k,m)} (\Psi_{m,n}^{(j,l,r)}(t), \Psi_{m,n'}^{(k,l,r)}(t)) \tag{35}$$

$$\sum_n c_n^{m+1} (\varphi_{m+1,n}(t), \varphi_{m,n'}(t)) = \sum_n c_n^m (\varphi_{m,n}(t), \varphi_{m,n'}(t)) + \sum_n d_n^{(k,m)} (\Psi_{m,n}^{(j,l,r)}(t), \varphi_{m,n'}(t)). \tag{36}$$

Since $\{\Psi^{(1,l,r)}(t), \Psi^{(2,l,r)}(t)\}$ is a biorthonomal wavelet of $L^2(R)$, by the definition we know that

$$(\Psi_{m,n}^{(j,l,r)}(t), \Psi_{m,n'}^{(k,l,r)}(t)) = \delta_{n,n'}, \;\; j \neq k\,(j, k = 1, 2).$$

Again, since $\{\varphi_{m,n}(t)\}_{n \in Z}$ is an orthonormal basis of V_m, we see that

$$(\varphi_{m,n}(t), \varphi_{m,n'}(t)) = \delta_{n,n'}. \tag{37}$$

Since $\varphi_{m,n}(t) \in V_m$, $\Psi_{m,n'}^{(j,l,r)}(t) \in W_m$ and $V_m \perp W_m$, we have

$$(\Psi_{m,n'}^{(j,l,r)}(t), \varphi_{m,n}(t)) = 0 \;(j = 1, 2).$$

Combining these results, we can rewrite (35) and (36) in the form

$$d_{n'}^{(k,m)} = \sum_n c_n^{m+1}(\varphi_{m+1,n}, \Psi_{m,n'}^{(k,l,r)}), \quad (k=1,2). \tag{38}$$

$$c_{n'}^{(k,m)} = \sum_n c_n^{m+1}(\varphi_{m+1,n}, \varphi_{m,n'}). \tag{39}$$

However, by (33) and (34) as well as the properties of inner product we get by (37)

$$(\varphi_{m+1,n}, \Psi_{m,n'}^{(k,l,r)}) = \sum_\nu \overline{q_{\nu-2n'}^{(j)}}(\varphi_{m+1,n}, \varphi_{m+1,\nu}) = \overline{q_{n-2n'}^{(k)}} \quad (k=1,2).$$

$$(\varphi_{m+1,n}, \varphi_{m,n'}) = \sum_\nu \overline{p_{\nu-2n'}}(\varphi_{m+1,n}, \varphi_{m+1,\nu}) = \overline{p_{n-2n'}}.$$

From this and (38-39), we obtain that the corresponding Mallat's decomposition algorithm is as follows:

$$d_\nu^{(k,m)} = \sqrt{2}\sum_n c_n^{m+1}\overline{\beta_{2\nu-n}^{(k)}} \quad (k=1,2).$$

$$c_\nu^m = \sqrt{2}\sum_n c_n^{m+1}\overline{\alpha_{n-2\nu}},$$

where $c_n^m = (f(t), \varphi_{m,n}(t))$, $d_n^{(k,m)} = (f(t), \Psi_{m,n}^{(k,l,r)}(t))$, and α_n, $\beta_n^{(k)}$ are given by (1) and (18).

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