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WEIGHTED ESTIMATES FOR ROUGH PARAMETRIC MARCINKIEWICZ INTEGRALS

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Abstract. We establish a weighted norm inequality for a class of rough parametric Marcinkiewicz integral operators $\mathcal{M}_{\Omega}^{\rho}$. As an application of this inequality, we obtain weighted L^p inequalities for a class of parametric Marcinkiewicz integral operators $\mathcal{M}_{\Omega,\lambda}^{*,\rho}$ and $\mathcal{M}_{\Omega,S}^{\rho}$ related to the Littlewood-Paley g_{λ}^{*}-function and the area integral S, respectively.

1. Introduction

Let $\mathbf{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For $x \in \mathbf{R}^n \backslash \{0\}$, let $x' = x/|x|$.

For a suitable C^1 function Ψ on $\mathbf{R}_+$, define the parametric Marcinkiewicz integral operator $\mathcal{M}_{\Omega,\Psi}^{\rho}$ by

$$\mathcal{M}_{\Omega,\Psi}^{\rho} f(x) = \left(\int_0^{\infty} \left| F_{\Omega,\Psi}^{\rho} f(t,x) \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{1}$$

where

$$F_{\Omega,\Psi}^{\rho} f(t,x) = \frac{1}{t^{\rho}} \int_{|u| \leq t} f(x - \Psi(|u|)u') \frac{\Omega(u')}{|u|^{n-\rho}} du,$$

$\rho = \sigma + i\tau$ $(\sigma, \tau \in \mathbf{R}$ with $\sigma > 0)$, $f \in \mathcal{S}(\mathbf{R}^n)$, the space of Schwartz functions and Ω is defined on $\mathbf{S}^{n-1}$, $\Omega \in L^1(\mathbf{S}^{n-1})$ and satisfies the vanishing condition

$$\int_{\mathbf{S}^{n-1}} \Omega(x')\, d\sigma(x') = 0. \tag{2}$$

If $\Psi(t) = t$, we shall denote $\mathcal{M}_{\Omega,\Psi}^{\rho}$ by $\mathcal{M}_{\Omega}^{\rho}$.

If $\rho = 1$, it is known that $\mathcal{M}_{\Omega}^{1}$ is the classical Marcinkiewicz integral operator introduced by E. Stein in [14]. Stein showed that if Ω is continuous and $\Omega \in \mathrm{Lip}_{\alpha}(\mathbf{S}^{n-1})$ $(0 < \alpha \leq 1)$, then $\mathcal{M}_{\Omega}^{1}$ is of type (p,p) $(1 < p \leq 2)$ and of weak type $(1,1)$. Subsequently, Benedek, Calderón, and Panzone [1] proved that $\mathcal{M}_{\Omega}^{1}$ is of type (p,p) for $p \in (1,\infty)$ if $\Omega \in C^1(\mathbf{S}^{n-1})$. Recently, Chen, Fan and Pan

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in [2] proved that $\mathcal{M}_{\Omega}^{1}$ is of type (p,p) for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ if $\Omega \in \mathbf{F}_{\alpha}(\mathbf{S}^{n-1})$ for some $\alpha > 0$. Here $\mathbf{F}_{\alpha}(\mathbf{S}^{n-1})$ denotes the space of all integrable functions Ω on $\mathbf{S}^{n-1}$ which satisfy the condition

$$\sup_{\xi \in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left(\log \frac{1}{|\xi \cdot y|}\right)^{1+\alpha} d\sigma(y) < \infty. \tag{3}$$

We point out the space $\mathbf{F}_{\alpha}(\mathbf{S}^{n-1})$ was introduced by Grafakos and Stefanov in [8] with respect to their studies of singular integrable operators. Also, it should be noted that Grafakos and Stefanov in [8] showed that

$$\bigcup_{q>1} L^{q}(\mathbf{S}^{n-1}) \subsetneqq \mathbf{F}_{\alpha}(\mathbf{S}^{n-1}) \text{ for any } \alpha > 0, \tag{4}$$

$$\bigcap_{\alpha>0} \mathbf{F}_{\alpha}(\mathbf{S}^{n-1}) \nsubseteq H^{1}(\mathbf{S}^{n-1}) \nsubseteq \bigcup_{\alpha>0} \mathbf{F}_{\alpha}(\mathbf{S}^{n-1}), \tag{5}$$

where $H^{1}(\mathbf{S}^{n-1})$ denotes the Hardy space on $\mathbf{S}^{n-1}$ in the sense of Coifman and Weiss [3]. In the meantime, the investigation of the L^p boundedness of the parametric Marcinkiewicz integral operator $\mathcal{M}_{\Omega}^{\rho}$ has also received a large amount of attention of many authors. In 1960, Hörmander [9] proved that the parametric Marcinkiewicz operator $\mathcal{M}_{\Omega}^{\rho}$ is of type (p,p) for $p \in (1,\infty)$ if $\rho > 0$ and $\Omega \in \mathrm{Lip}_{\alpha}(\mathbf{S}^{n-1})$ $(0 < \alpha \leq 1)$. In 1996, Sakamoto and Yabuta [12] studied the L^p boundedness of the parametric Marcinkiewicz integral operator $\mathcal{M}_{\Omega}^{\rho}$ if ρ is complex and proved that $\mathcal{M}_{\Omega}^{\rho}$ is of type (p,p) for $p \in (1,\infty)$ if $\mathrm{Re}(\rho) > 0$ and $\Omega \in \mathrm{Lip}_{\tau}(\mathbf{S}^{n-1})$ $(0 < \tau \leq 1)$.

On the other hand, the investigation of the weighted L^p boundedness of $\mathcal{M}_{\Omega}^{1}$ has attracted the attention of many authors. In 1990, Torchinsky and Wang in [18] proved that if $\Omega \in \mathrm{Lip}_{\tau}(\mathbf{S}^{n-1})$ $(0 < \tau \leq 1)$, then $\mathcal{M}_{\Omega}^{1}$ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$ and $\omega \in A_p$ (The Muckenhoupt's weight class, see [7] for the definition). In 1998, Sato in [13] improved the weighted L^p boundedness of Torchinsky-Wang by proving that $\mathcal{M}_{\Omega}^{1}$ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$ provided that $\Omega \in L^{\infty}(\mathbf{S}^{n-1})$ and $\omega \in A_p(\mathbf{R}^n)$. Subsequently, in 1999, Ding, Fan and Pan in [4] were able to show that $\mathcal{M}_{\Omega}^{1}$ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$ provided that $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$ and $\omega^{q'} \in A_p(\mathbf{R}^n)$. In a recent paper, Ming-Yi Lee and Chin-Cheng Lin in [11] showed that $\mathcal{M}_{\Omega}^{1}$ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$ if $\Omega \in H^1(\mathbf{S}^{n-1})$ and $\omega \in \tilde{A}_p^I(\mathbf{R}^n)$, where $\tilde{A}_p^I(\mathbf{R}^n)$ is a special class of radial weights introduced by Duoandikoetxea [6] whose definition will be recalled in Section 2.

The main purpose of this paper is to show that the weighted $L^p(\omega)$ boundedness of the parametric Marcinkiewicz operator $\mathcal{M}_{\Omega}^{\rho}$ holds under the conditions $\omega \in \tilde{A}_p^I(\mathbf{R}^n)$ and $\Omega \in \mathbf{F}_{\alpha}(\mathbf{S}^{n-1})$. In fact, we are able to prove the following more general result.

Theorem 1.1. *Let Ψ be in $C^2([0,\infty))$, convex, and an increasing function with $\Psi(0) = 0$. If $\Omega \in \mathbf{F}_{\alpha}(\mathbf{S}^{n-1})$ for some $\alpha > 0$, then there exists $C_p > 0$ such* *that*

$$\left\| \mathcal{M}^{\rho}_{\Omega,\Psi}(f) \right\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)} \tag{6}$$

for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ *and* $\omega \in \tilde{A}^I_p(\mathbf{R}_+)$.

Remark. Obviously, Theorem 1.1 represents an improvement and extension over the result of Ding, Fan and Pan [4] in the case $\omega \in \tilde{A}^I_p(\mathbf{R}_+)$. Also, Theorem 1.1 represents a substantial improvement and extension over the result (in the unweighted case) of Sakamoto and Yabuta [12] because Ω is allowed to be in the space $\mathbf{F}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$; and bearing in mind the relations

$$\mathrm{Lip}_\tau(\mathbf{S}^{n-1}) \ (0 < \tau \leq 1) \subset L^d(\mathbf{S}^{n-1}) \subset \mathbf{F}_\alpha(\mathbf{S}^{n-1}) \text{ for all } d > 1.$$

Throughout the rest of the paper the letter C will stand for a positive constant not necessarily the same one at each occurrence.

2. Some definitions

We start this section by recalling the definition of certain classes of weights and some of their basic properties.

Definition 2.1. Let $\omega(t) \geq 0$ and $\omega \in L^1_{loc}(\mathbf{R}_+)$. For $1 < p < \infty$, we say that $\omega \in A_p(\mathbf{R}_+)$ if there is a positive constant C such that for any interval $I \subset \mathbf{R}_+$,

$$\left(|I|^{-1} \int_I \omega(t)dt\right) \left(|I|^{-1} \int_I \omega(t)^{-1/(p-1)}dt\right)^{p-1} \leq C < \infty.$$

$A_1(\mathbf{R}_+)$ is the class of weights ω for which M satisfies a weak-type estimate in $L^1(\omega)$, where $M(f)$ is the Hardy-Littlewood maximal function of f.

It is known that the class $A_1(\mathbf{R}_+)$ is also characterized by all weights ω for which $M\omega(t) \leq C\omega(t)$ for a.e. $t \in \mathbf{R}_+$ and for some positive constant C.

Definition 2.2. Let $1 \leq p < \infty$. We say that $\omega \in \tilde{A}_p(\mathbf{R}_+)$ if $\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p}$, where either $\nu_i \in A_1(\mathbf{R}_+)$ is decreasing or $\nu_i^2 \in A_1(\mathbf{R}_+)$, $i = 1, 2$.

Let $A^I_p(\mathbf{R}^n)$ be the weight class defined by exchanging the cubes in the definitions of A_p for all n-dimensional intervals with sides parallel to coordinate axes (see [10]). Let $\tilde{A}^I_p = \tilde{A}_p \cap A^I_p$. If $\omega \in \tilde{A}_p$, it follows from [6] that the classical Hardy-Littlewood maximal function Mf is bounded on $L^p(\mathbf{R}^n, \omega(|x|)dx)$. Therefore, if $\omega(t) \in \tilde{A}_p(\mathbf{R}_+)$, then $\omega(|x|) \in A_p(\mathbf{R}^n)$.

By employing a similar argument as that employed in the proof of the elementary properties of A_p weight class (see for example [7]) we get the following:

Lemma 2.3. *If* $1 \leq p < \infty$, *then the weight class* $\tilde{A}^I_p(\mathbf{R}_+)$ *has the following properties:*

(*i*) $\tilde{A}^I_{p_1} \subset \tilde{A}^I_{p_2}$, *if* $1 \leq p_1 < p_2 < \infty$;

(*ii*) *For any* $\omega \in \tilde{A}^I_p$, *there exists an* $\varepsilon > 0$ *such that* $\omega^{1+\varepsilon} \in \tilde{A}^I_p$;

(iii) For any $\omega \in \tilde{A}_p^I$ and $p > 1$, there exists an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $\omega \in \tilde{A}_{p-\varepsilon}^I$.

Definition 2.4. For a suitable C^1 function Ψ on $\mathbf{R}_+$, and a suitable function Ω on $\mathbf{S}^{n-1}$ we define the family of measures $\{\sigma_t : t \in \mathbf{R}_+\}$ and the maximal operator σ^* on $\mathbf{R}^n$ by

$$\int_{\mathbf{R}^n} f d\sigma_t = \frac{1}{t^\rho} \int_{\frac{1}{2}t<|y|\leq t} f(\Psi(|y|)y') \frac{\Omega(y')}{|y|^{n-\rho}} dy,$$

and

$$\sigma^* f(x) = \sup_{t \in \mathbf{R}_+} ||\sigma_t| * f(x)|,$$

where $|\sigma_t|$ is defined in the same way as σ_t, but with Ω replaced by $|\Omega|$.

3. Main estimates

Lemma 3.1. *Assume that Ψ is in $C^2([0,\infty))$, convex, and an increasing function with $\Psi(0) = 0$. If $\Omega \in \mathbf{F}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$, then there exists a positive constant C such that for all $\xi \in \mathbf{R}^n$ we have*

$$\|\sigma_t\| \leq C; \tag{1}$$

$$\int_{2^k}^{2^{k+1}} |\hat{\sigma}_t(\xi)|^2 \frac{dt}{t} \leq C \left(\log\left|\Psi(2^{k-1})\xi\right|\right)^{-\alpha-1} \text{ if } \left|\Psi(2^{k-1})\xi\right| > 2; \tag{2}$$

$$\int_{2^k}^{2^{k+1}} |\hat{\sigma}_t(\xi)|^2 \frac{dt}{t} \leq C\left|\Psi(2^{k+1})\xi\right|. \tag{3}$$

The constant C is independent of t, ξ and $\Psi(\cdot)$.

Proof. By the definition of σ_t, one can easily see that (3.1) holds with a constant C independent of t. Next we prove (3.2). By definition,

$$\hat{\sigma}_t(\xi) = \frac{1}{t^\rho} \int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} e^{-i\Psi(s)\xi\cdot x}\Omega(x) d\sigma(x) \frac{ds}{s^{1-\rho}}.$$

By a simple change of variable and Hölder's inequality, we have

$$\int_{2^k}^{2^{k+1}} |\hat{\sigma}_t(\xi)|^2 \frac{dt}{t} \leq \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} |\Omega(x)| \left(\int_{2^k}^{2^{k+1}} |I_t(\xi,x)|^2 \frac{dt}{t}\right) d\sigma(x), \tag{4}$$

where

$$I_t(\xi,x) = \int_{1/2}^{1} e^{-i\Psi(ts)\xi\cdot x} \frac{ds}{s}.$$

Write $I_t(\xi,x)$ as

$$I_t(\xi,x) = \int_{1/2}^{1} H_t'(s) \frac{ds}{s},$$

where

$$H_t(s) = \int_{1/2}^{s} e^{-i\Psi(tw)\xi\cdot x} dw, \ 1/2 \le s \le 1.$$

By the assumptions on Ψ and using the mean value theorem we have

$$\frac{d}{dw}\left(\Psi(tw)\right) = t\Psi'(tw) \ge \frac{\Psi(tw)}{w} \ge \frac{\Psi(t/2)}{s} \text{ for } 1/2 \le w \le s \le 1.$$

Thus by van der Corput's lemma,

$$|H_t(s)| \le \left|\frac{\Psi(t/2)\xi}{s}\right|^{-1} |\xi' \cdot x|^{-1}.$$

By integration by parts, we get

$$|I_t(\xi, x)| \le C\,|\Psi(t/2)\xi|^{-1}\,|\xi'\cdot x|^{-1}$$

and hence

$$\int_{2^k}^{2^{k+1}} |I_t(\xi,x)|^2\,\frac{dt}{t} \le C\left|\Psi(2^{k-1})\xi\right|^{-2}|\xi'\cdot x|^{-2}.$$

By combining the last estimate with the trivial estimate

$$\int_{2^k}^{2^{k+1}} |I_t(\xi,x)|^2\,\frac{dt}{t} \le (\log 2)^3$$

we get

$$(5)\quad \int_{2^k}^{2^{k+1}} |I_t(\xi,x)|^2\,\frac{dt}{t} \le C\frac{\left(\log(e^{(1+\alpha)}\,|\xi'\cdot x|^{-1})\right)^{\alpha+1}}{(\log|\Psi(2^{k-1})\xi|)^{\alpha+1}} \quad \text{if } \left|\Psi(2^{k-1})\xi\right| > 2.$$

To prove (3.3), we use the cancellation condition of Ω to get

$$|\hat{\sigma}_t(\xi)| \le \int_{\mathbf{S}^{n-1}}\int_{1/2}^{2}\left|e^{-i\Psi(ts)\xi\cdot x}-1\right|\,|\Omega(x)|\,\frac{ds}{s}d\sigma(x).$$

Therefore, by the last estimate and using that Ψ is increasing we get (3.3). The lemma is proved. □

By the same argument as in [16, p. 57] we get

Lemma 3.2. *Let φ be a nonnegative, decreasing function on $[0,\infty)$ with $\int_{[0,\infty)}\varphi(t)dt = 1$. Then*

$$\left|\int_0^\infty f(x-ty')\varphi(t)dt\right| \le M_{y'}f(x),$$

where

$$M_{y'}f(x) = \sup_{R\in\mathbf{R}}\frac{1}{R}\int_0^R |f(x-sy')|\,ds$$

is the Hardy-Littlewood maximal function of f in the direction of y'.

Lemma 3.3. *Let $1 < p < \infty$ and $\omega \in \tilde{A}_p(\mathbf{R}_+)$. Assume that $\Omega \in L^1(\mathbf{S}^{n-1})$ and Ψ is in $C^2([0,\infty))$, convex, and increasing function with $\Psi(0) = 0$. Then there exists a positive constant C_p such that*

$$\|\sigma^*(f)\|_{L^p(\omega)} \le C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\omega)}. \tag{6}$$

Proof. By definition of σ_t we have

$$||\sigma_t| * f(x)| \le C \left(\int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} |\Omega(y')| \, |f(x - \Psi(s)y')| \, d\sigma(y') \frac{ds}{s} \right).$$

Thus

$$\sigma^* f(x) \le C \left(\int_{\mathbf{S}^{n-1}} |\Omega(y')| \, M_{\Psi,y'}(|f|)(x) d\sigma(y') \right), \tag{7}$$

where

$$M_{\Psi,y'} f(x) = \sup_{t \in \mathbf{R}_+} \frac{1}{t} \left| \int_0^t f(x - \Psi(s)y') ds \right|.$$

Without loss of generality, we may assume that $\Psi(s) > 0$ for all $s > 0$. By a change of variable get

$$M_{\Psi,y'} f(x) \le \sup_{t \in \mathbf{R}} \frac{1}{t} \left(\int_0^{\Psi(t)} |f(x - sy')| \frac{ds}{\Psi'(\Psi^{-1}(s))} \right).$$

Since the function $\frac{1}{t\Psi'(\Psi^{-1}(s))}$ is non-negative, decreasing and its integral over $[0, \Psi(t)]$ is equal to 1, by Lemma 3.2 we obtain

$$M_{\Psi,y'} f(x) \le M_{y'} f(x). \tag{8}$$

By (3.7)–(3.8) we get

$$\|\sigma^*(f)\|_{L^p(\omega)} \le \left(\int_{\mathbf{S}^{n-1}} |\Omega(y')| \, \|M_{y'}(|f|)\|_{L^p(\omega)} \, d\sigma(y') \right). \tag{9}$$

By (8) in [6] and since $\omega \in \tilde{A}_p(\mathbf{R}_+)$ we have

$$\|M_{y'} f\|_{L^p(\omega)} \le C \|f\|_{L^p(\omega)} \tag{10}$$

with C independent of y'. Thus, by (3.9)–(3.10) we get (3.6). This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let $1 < p < \infty$ and $\omega \in \tilde{A}_p(\mathbf{R}_+)$. Assume that $\Omega \in L^1(\mathbf{S}^{n-1})$ and Ψ is in $C^2([0,\infty))$, convex, and increasing function with $\Psi(0) = 0$. Then there exists a positive constant C_p such that*

$$\left\| \left(\sum_{k \in \mathbf{Z}} \int_{2^k}^{2^{k+1}} |\sigma_t * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\omega)} \le C_p \left\| \left(\sum_{k \in \mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(\omega)} \tag{11}$$

holds for arbitrary functions $\{g_k(\cdot)\}_{k \in \mathbf{Z}}$ on $\mathbf{R}^n$.

Proof. By a change of variable, we have

$$\left(\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}|\sigma_t*g_k|^2\frac{dt}{t}\right)^{1/2}\le\left(\sum_{k\in\mathbf{Z}}\int_1^2|\sigma_{2^kt}*g_k|^2\,dt\right)^{1/2}. \tag{12}$$

By Lemma 3.3, we get

$$\left\|\sup_{k\in\mathbf{Z}}\sup_{t\in[1,2]}|\sigma_{2^kt}*g_k|\right\|_{L^p(\omega)}\le\left\|\sigma^*(\sup_{k\in\mathbf{Z}}|g_k|)\right\|_{L^p(\omega)}\le C\left\|\sup_{k\in\mathbf{Z}}|g_k|\right\|_{L^p(\omega)}. \tag{13}$$

On the other hand, by the Riesz representation theorem, there exists a non-negative function f in $L^{p'}(\omega)$ with $\|f\|_{L^{p'}(\omega)}\le 1$ such that

$$\begin{aligned}\left\|\sum_{k\in\mathbf{Z}}\int_1^2|\sigma_{2^kt}*g_k|\,dt\right\|_{L^p(\omega)} &\le \int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_1^2|\sigma_{2^kt}*g_k(x)|\,dt\omega(x)f(x)dx\\ &\le \int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_1^2|\sigma_{2^kt}|*|g_k(x)|\,dt\omega(x)f(x)dx.\end{aligned}$$

Thus, by Fubini's theorem and Hölder's inequality we get

$$\begin{aligned}\left\|\sum_{k\in\mathbf{Z}}\int_1^2|\sigma_{2^kt}*g_k|\,dt\right\|_{L^p(\omega)} &\le \int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}|g_k(x)|\,\sigma^*((\widetilde{\omega f}))(-x)dx\\ &\le \left\|\sum_{k\in\mathbf{Z}}|g_k|\right\|_{L^p(\omega)}\left\|\sigma^*((\widetilde{\omega f}))\right\|_{L^{p'}(\omega^{1-p'})},\end{aligned}$$

where $\tilde{u}(x)=u(-x)$. Since $\omega\in\tilde{A}_p(\mathbf{R}_+)$ if and only if $\omega^{1-p'}\in\tilde{A}_{p'}(\mathbf{R}_+)$, by Lemma 3.3, we get

$$\left\|\sum_{k\in\mathbf{Z}}\int_1^2|\sigma_{2^kt}*g_k|\,dt\right\|_{L^p(\omega)}\le C_p\left\|\sum_{k\in\mathbf{Z}}|g_k|\right\|_{L^p(\omega)}. \tag{14}$$

Therefore, we can interpolate (3.13) and (3.14) (See [7, p.481], for the vector-valued interpolation) to get

$$\left\|\left(\sum_{k\in\mathbf{Z}}\int_1^2|\sigma_{2^kt}*g_k|^2\,dt\right)^{1/2}\right\|_{L^p(\omega)}\le C_p\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|_{L^p(\omega)}$$

which when combined with (3.12) yields (3.11). Thus Lemma 3.4 is proved. □

4. Proof of Theorem 1.1

For $k \in \mathbf{Z}$, let $a_k = \Psi(2^k)$. We notice that $\{a_k : k \in \mathbf{Z}\}$ is a lacunary sequence with $a_{k+1}/a_k \geq 2$. Let $\{\Lambda_k\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the interval $I_k = [a_{k+1}^{-1}, a_{k-1}^{-1}]$. To be precise, we require the following:

$$\begin{aligned} \Lambda_k &\in C^{\infty},\ 0 \leq \Lambda_k \leq 1,\ \sum_k \Lambda_k(t) = 1; \\ \operatorname{supp} \Lambda_k &\subseteq I_k;\ \left|\frac{d^s \Lambda_k(t)}{dt^s}\right| \leq \frac{C_s}{t^s}, \end{aligned}$$

where C_s is independent of the lacunary sequence $\{a_k : k \in \mathbf{Z}\}$. Let $\widehat{\Gamma_k}(\xi) = \Lambda_k(|\xi|)$.

By Minkowski's inequality we have

$$\begin{aligned} \mathcal{M}^{\rho}_{\Omega,\Psi} f(x) &= \left(\int_0^{\infty} \left|\sum_{k=0}^{\infty} 2^{-k\rho} \sigma_{2^{-k}t} * f(x)\right|^2 \frac{dt}{t}\right)^{1/2} \\ &\leq \sum_{k=0}^{\infty} 2^{-k\sigma} \left(\int_0^{\infty} |\sigma_{2^{-k}t} * f(x)|^2 \frac{dt}{t}\right)^{1/2} \\ &= \left(\frac{1}{1-2^{-\sigma}}\right) \left(\int_0^{\infty} |\sigma_t * f(x)|^2 \frac{dt}{t}\right)^{1/2}. \end{aligned}$$

Decompose

$$f * \sigma_t(x) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} (\Gamma_{k+j} * \sigma_t * f)(x) \chi_{[2^k, 2^{k+1})}(t) := \sum_{j \in \mathbf{Z}} E_j(x,t)$$

and define

$$S_j f(x) = \left(\int_0^{\infty} |E_j(x,t)|^2 \frac{dt}{t}\right)^{1/2}.$$

Then

$$\mathcal{M}^{\rho}_{\Omega,\Psi}(f) \leq \left(\frac{1}{1-2^{-\sigma}}\right) \sum_{j \in \mathbf{Z}} S_j(f)$$

holds for $f \in \mathcal{S}(\mathbf{R}^n)$.

Thus, to prove (1.6), it is enough to show that

$$\|S_j(f)\|_{L^p(\omega)} \leq C_p (1+|j|)^{-\alpha-1} \|f\|_{L^p(\omega)} \tag{1}$$

for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $\omega \in \tilde{A}^I_p(\mathbf{R}_+)$.

To prove (4.1), let us first compute the $L^2(\mathbf{R}^n)$-norm of $S_j(f)$. By using Plancherel's theorem, we have

$$\begin{aligned}
\|S_j(f)\|^2_{L^2(\mathbf{R}^n)} &= \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{2^k}^{2^{k+1}} |\Gamma_{k+j}*\sigma_t*f(x)|^2\,\frac{dt}{t}dx \\
&\leq \sum_{k\in\mathbf{Z}}\int_{\Delta_{k+j}}\left(\int_{2^k}^{2^{k+1}}|\hat{\sigma}_t(\xi)|^2\,\frac{dt}{t}\right)\left|\hat{f}(\xi)\right|^2 d\xi,
\end{aligned}$$

where

$$\Delta_k = \{\xi\in\mathbf{R}^n : |\xi|\in I_k\}.$$

By Lemma 3.1 and a straightforward computations we get

$$\begin{aligned}
\|S_j(f)\|_{L^2(\mathbf{R}^n)} &\leq C\,|j|^{-\alpha-1}\,\|f\|_{L^2(\mathbf{R}^n)} \text{ if } j\leq -1; \\
\|S_j(f)\|_{L^2(\mathbf{R}^n)} &\leq C2^{-j}\,\|f\|_{L^2(\mathbf{R}^n)} \text{ if } j\geq 0
\end{aligned}$$

and hence

$$\|S_j(f)\|_{L^2(\mathbf{R}^n)}\leq C(1+|j|)^{-\alpha-1}\,\|f\|_{L^2(\mathbf{R}^n)} \text{ for all } j\in\mathbf{Z}. \tag{2}$$

Next, let us compute the $L^p(\omega)$ boundedness of the operator S_j. For $p\in(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $\omega\in\tilde{A}^I_p(\mathbf{R}_+)$, we have

$$\begin{aligned}
\|S_j(f)\|_{L^p(\omega)} &= \left\|\left(\sum_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}}|\sigma_t*\Gamma_{k+j}*f|^2\,\frac{dt}{t}\right)^{1/2}\right\|_{L^p(\omega)} \\
&\leq C_p\left\|\left(\sum_{k\in\mathbf{Z}}|\Gamma_k*f|^2\right)^{1/2}\right\|_{L^p(\omega)} \\
&\leq C_p\,\|f\|_{L^p(\omega)}, \qquad (3)
\end{aligned}$$

where the first inequality follows by Lemma 3.4 and the last inequality follows from a well-known weighted Littlewood-Paley inequality (see [10]) because we have $\tilde{A}^I_p(\mathbf{R}_+)\subset\tilde{A}_p(\mathbf{R}_+)\subset A_p(\mathbf{R}_+)$.

By interpolating between (4.2) and (4.3) with $\omega=1$, for every $p\in(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, there is a $\theta_p>1$ such that

$$\|S_j(f)\|_{L^p(\mathbf{R}^n)}\leq C(1+|j|)^{-\theta_p}\,\|f\|_{L^p(\mathbf{R}^n)} \tag{4}$$

holds for $j\in\mathbf{Z}$. Using Stein and Weiss' interpolation theorem with change of measure, we interpolate (4.3) with (4.4) to get, for every $p\in(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $\omega\in\tilde{A}^I_p(\mathbf{R}_+)$, there is a $\eta_p>1$ such that

$$\|S_j(f)\|_{L^p(\omega)}\leq C(1+|j|)^{-\eta_p}\,\|f\|_{L^p(\omega)} \tag{5}$$

holds for $j\in\mathbf{Z}$ and hence by we get (4.1). This completes the proof of Theorem 1.1.

5. Additional results

In this section, we shall apply the result in Theorem 1.1 to get the weighted L^p boundedness for a class of parametric Marcinkiewicz operators $\mathcal{M}^{*,\rho}_{\Omega,\Psi,\lambda}$ and $\mathcal{M}^{\rho}_{\Omega,\Psi,S}$ related to the Littlewood-Paley g^*_λ-function and the area integral S, respectively. These parametric Marcinkiewicz operators are defined by

$$\begin{aligned}\mathcal{M}^{\rho}_{\Omega,\Psi,S}f(x) &= \left(\int_{\Gamma(x)} \left|F^{\rho}_{\Omega,\Psi}f(x,t)\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},\\ \mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}f(x) &= \left(\int\int_{\mathbf{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left|F^{\rho}_{\Omega,\Psi}f(x,t)\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},\end{aligned}$$

where $\lambda > 1$, $\Gamma(x) = \{(y,t) \in \mathbf{R}^{n+1}_+ : |x-y| < t\}$.

The results regarding these operators are given as follows:

Theorem 5.1. *Let Ψ be in $C^2([0,\infty))$, convex, and increasing function with $\Psi(0) = 0$. If $\Omega \in \mathbf{F}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$, there exists $C_p > 0$ such that*

$$(6)\qquad \left\|\mathcal{M}^{\rho}_{\Omega,\Psi,S}(f)\right\|_{L^p(\omega)} + \left\|\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}(f)\right\|_{L^p(\omega)} \leq \frac{C_p}{(1-2^{-\sigma})}\|f\|_{L^p(\omega)}$$

for $2 \leq p < \infty$ and $\omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+)$, where $\sigma = \mathrm{Re}\rho$.

The proof Theorem 5.1 will mainly rely on the following lemma whose proof can be obtained by using Theorem 1.1 and following the same argument employed in the proof of Theorem 5 in Torchinsky and Wang [18].

Lemma 5.2. *Let $\lambda > 1$. Then, for any nonnegative locally integrable function g, we have*

$$(7)\qquad \int_{\mathbf{R}^n} \left(\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}f(x)\right)^2 h(x)dx \leq \frac{C}{(1-2^{-\sigma})}\int_{\mathbf{R}^n} |f(x)|^2 Mh(x)dx$$

for positive constant C.

Proof of Theorem 5.1. It is easy to see that $\mathcal{M}^{\rho}_{\Omega,\Psi,S}f(x) \leq 2^{n\lambda}\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}f(x)$. Thus, we only consider the operator $\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}$. If $p = 2$, then $\omega \in \tilde{A}^I_1(\mathbf{R}_+) \subset \tilde{A}_1(\mathbf{R}_+) \subset A_1(\mathbf{R}_+)$ and hence $M\omega(x) \leq C\omega(x)$ almost everywhere. Thus by Lemma 5.2 we have

$$(8)\qquad \int_{\mathbf{R}^n} \left(\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}f(x)\right)^2 \omega(x)dx \leq \frac{C}{(1-2^{-\sigma})}\int_{\mathbf{R}^n} |f(x)|^2 \omega(x)dx,$$

which implies that $\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}$ is bounded on $L^2(\omega)$. Now, if $2 < p < \infty$, we have

$$\left\|\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}f\right\|^2_{L^p(\omega)} = \sup_h \left|\int_{\mathbf{R}^n} \left(\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}f(x)\right)^2 h(x)dx\right|,$$

where the supremum is taken over all $h(x)$ satisfying $\|h\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \le 1$. Thus, by Lemma 5.2 and using Hölder's inequality we get

$$\begin{aligned}\left\|\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda} f\right\|^2_{L^p(\omega)} &\le \frac{C}{(1-2^{-\sigma})} \sup_h \int_{\mathbf{R}^n} |f(x)|^2 Mh(x)dx \\ &\le \frac{C}{(1-2^{-\sigma})} \|f\|^2_{L^p(\omega)} \sup_h \|Mh\|_{L^{(p/2)'}(\omega^{1-(p/2)'})} \\ &\le \frac{C}{(1-2^{-\sigma})} \|f\|^2_{L^p(\omega)}\end{aligned}$$

which ends the proof of Theorem 5.1. □

We remark that Theorem 5.1 extends and improves the corresponding results in [12] *in which the authors of* [12] *proved the* $L^p(2 \le p < \infty)$ *boundedness of* $\mathcal{M}^{\rho,*}_{\Omega,\Psi,\lambda}$ *and* $\mathcal{M}^{\rho}_{\Omega,\Psi,S}$ *if* $\Psi(t) \equiv t$ *and* $\Omega \in \mathrm{Lip}_\tau(\mathbf{S}^{n-1})$ $(0 < \tau \le 1)$.

A special class of radial weights is the power weights $|x|^\gamma, \gamma \in \mathbf{R}$. It is known that $|x|^\gamma \in A_p(\mathbf{R}^n)$ if and only if $-n < \gamma < n(p-1)$. By noticing that $|x|^\gamma \in \tilde{A}^I_p(\mathbf{R}_+)$ for $\gamma \in (-1, p-1)$, and applying Theorem 1.1 we obtain the following:

Corollary 5.3. *Let* Ψ *be in* $C^2([0,\infty))$, *convex, and increasing function with* $\Psi(0) = 0$. *If* $\Omega \in \mathbf{F}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$, *and* $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, *then*

$$\left\|\mathcal{M}^{\rho}_{\Omega,\Psi}(f)\right\|_{L^p(\omega)} \le \frac{C_p}{(1-2^{-\sigma})} \|f\|_{L^p(\omega)}$$

for all $\omega(x) = |x|^\gamma$ *and* $\gamma \in (-1, p-1)$.

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