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Numerical Conformal Mapping onto the Entire Complex Plane Bounded with Finite Straight Slit and Logarithmic Spiral Slits

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Abstract. This paper presents a fast boundary integral equation method with for computing conformal mappings of multiply connected regions. We consider the canonical region consists of the entire complex plane bounded by a finite straight slit on the line Im $\omega = 0$ and finite logarithmic spiral slits. Some numerical examples are given to show the effectiveness of the proposed method.

1. Introduction

Conformal mapping is a special mapping that transform a region onto another region while preserving the angle between curves in the sense of magnitude and direction. Because of this unique characteristic, the idea of conformal mapping have been applied in several real life problems as discussed in [1, 2]. By means of conformal maps, problem from a complicated region can be transformed into some standardized region where it can be solved easily. Despite its uniqueness, exact conformal maps are known for few cases only. For others, researchers have to overcome this limitation using numerical approximation. Trefethen [3] has discussed several method for computing the conformal maps based on expansion methods, iterative methods and integral equation methods. Any simply connected region can be mapped to a unit disk. Koebe [4] have listed thirty nine type of canonical regions for the multiply connected region. A canonical region is known for a region that has simpler geometry and can be uniquely determined by specifying the conformal moduli. From these thirty nine canonical region, Koebe [4] have cataloged it into five categories. For the numerical computation of the conformal mapping onto these categories, see [5, 6, 7, 8, 9, 10, 11]. Beside these thirty nine canonical regions, circular region which is a region bounded by multiple circle is also another important canonical region[12, 13].

In this paper, we present a numerical method for the conformal mapping that maps the original region G onto the entire complex plane bounded with straight slit on the line Im $\omega = 0$ and logarithmic spiral slits (see Fig. 1).
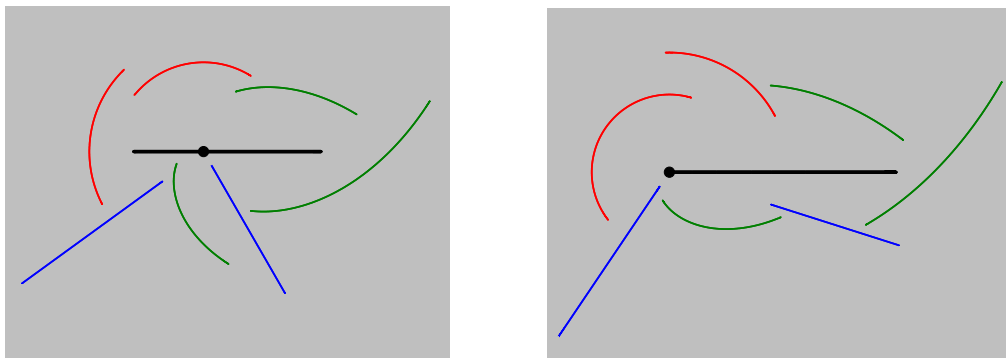
Figure 1: The canonical regions Ω_1 (left) and Ω_2 (right)

2. Auxiliary material

Let G be a bounded multiply connected region of connectivity $m + 1$. The boundary G consists of $m + 1$ smooth Jordan curves Γ_j, $j = 0, 1, 2, \ldots, m$, i.e., $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$ in the extended complex plane. The orientation of Γ is such that G is always on the left of Γ. The curve Γ_j is parameterized by 2π-periodic twice continuously differentiable complex function $\eta_j(t)$ with non-vanishing first derivative, i.e.,

$$\eta_j'(t) = \frac{d\eta_j(t)}{dt} \neq 0, \ t \in J_j = [0, 2\pi], \ j = 0, 1, \ldots, m.$$

Let the total parameter domain J be the disjoint union of $m + 1$ intervals $J_0, J_1, \ldots, J_m$. We define a parameterization η of the whole boundary Γ on J by

$$\eta(t) = \begin{cases} \eta_0(t), & t \in J_0 = [0, 2\pi], \\ \vdots \\ \eta_m(t), & t \in J_m = [0, 2\pi]. \end{cases} \tag{1}$$

We assume that Γ_0 map onto the finite straight slit on the line Im $\omega = 0$ while Γ_j, $j = 1, 2, \ldots, m$ will be mapped onto the logarithmic spiral slits with prescribed angles θ_j, $j = 1, 2, \ldots, m$. The mapping function $\omega(z)$ will be determined by computing two unknown real functions on J, a function $S(t)$ and a piecewise constant real function $R(t)$. Let H be the space of all real Hölder continuous 2π-periodic functions and L be the subspace of H which contains the piecewise real constant functions $R(t)$. Let $A_1(t)$ and $A_2(t)$ be complex continuously differentiable 2π-periodic functions for all $t \in J$. For $p = 1, 2$, the adjoint function $\tilde{A}_p$ of A_p is defined by [14]

$$A_1(t) = e^{i(\pi/2 - \theta)}, \qquad A_2(t) = e^{-i(\pi/2 - \theta)}, \qquad \tilde{A}_p(t) = \frac{\eta'(t)}{A_p(t)}.$$

The generalized Neumann kernel $\tilde{N}_p(s, t)$ and the kernel $\tilde{M}_p(s, t)$ formed with $\tilde{A}_p$ are defined by

$$\tilde{N}_p(s, t) = \frac{1}{\pi} \text{Im} \left(\frac{\tilde{A}_p(s)}{\tilde{A}_p(t)} \frac{\eta'(t)}{\eta(t) - \eta(s)} \right), \qquad \tilde{M}_p(s, t) = \frac{1}{\pi} \text{Re} \left(\frac{\tilde{A}_p(s)}{\tilde{A}_p(t)} \frac{\eta'(t)}{\eta(t) - \eta(s)} \right).$$

Then,

$$\tilde{N}_p(s, t) = -N_p^*(s, t) \text{ and } \tilde{M}_p(s, t) = -M_p^*(s, t), \tag{2}$$

where $N_p^*(s, t) = N_p(t, s)$ is the adjoint kernel of the generalized Neumann kernel $N_p(s, t)$. We

define the Fredholm integral operators $\mathbf{N}^*{}_p$ by

$$\mathbf{N}_p^* \upsilon(t) \;=\; \int_J N_p^*(t,s)\upsilon(s)ds, \qquad t \in J.$$

The eigenfunctions of N corresponding to the eigenvalue $\lambda = -1$ are $\left\{\chi^{[1]}, \chi^{[2]}, \ldots, \chi^{[m]}\right\}$, where

$$\chi^{[j]}(\xi) = \begin{cases} 1, & \xi \in \Gamma_j, \\ 0, & \text{otherwise}, \end{cases} \qquad j = 1, 2, \ldots, m.$$

We also define an integral operator $\mathbf{J}$ by

$$\mathbf{J}\mu(s) := \int_J \frac{1}{2\pi} \sum_{j=0}^{m} \chi^{[j]}(s)\chi^{[j]}(t)\mu(t)dt. \tag{3}$$

Let $\varphi(t)$ be the derivative of the unknown function $S(t)$ which shall be calculated by using the following theorem given in [15].

Theorem 1 *Let $\upsilon, \varphi, \psi, \phi \in H$, $f(z)$ be analytic in G $g(z)$ be analytic in G^- with $g(\infty) = 0$ such that the boundary values of the functions f and g are given by*

$$\tilde{A}_p(t)f(\eta(t)) + \tilde{A}_p(t)g(\eta(t)) = \upsilon + \mathrm{i}\varphi, \tag{4}$$

where the integral $\mathbf{J}$ is a given function defined as

$$\mathbf{J}\varphi = \tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_m). \tag{5}$$

Let also the boundary values of the function g satisfy

$$\tilde{A}_p(t)g(\eta(t)) = \psi + \mathrm{i}\phi. \tag{6}$$

Then the function φ is the unique solution of the integral equation

$$(\mathbf{I} + \mathbf{N}_p^* + \mathbf{J})\varphi = \mathbf{M}^*\upsilon + 2\phi + \tilde{h}. \tag{7}$$

For $j = 0, 1, \ldots, m$, the functions $S_j(t)$ can be written as a summation of φ and ν_j,

$$S_j(t) = \int \varphi(t)dt + \nu_j = \rho_j(t) + \nu_j, \quad t \in J_j, \tag{8}$$

where ν_j are undetermined real constants and shall be calculated by Theorem 2[15]. The derivative of the the unknown function $S(t)$ i.e. $\varphi(t)$ is 2π-periodic. Thus, the function $\varphi(t)$ can be represented by a Fourier series

$$\varphi(t) = a_0^{[j]} + \sum_{k=1}^{\infty} a_k^{[j]} \cos kt + \sum_{k=1}^{\infty} b_k^{[j]} \sin kt, \quad t \in J_j. \tag{9}$$

Hence the functions $\rho_k(t)$ can be calculated by the Fourier series representation

$$\rho_j(t) = a_0^{[j]}t + \sum_{k=1}^{\infty} \frac{a_k^{[j]}}{k} \sin kt - \sum_{k=1}^{\infty} \frac{b_k^{[j]}}{k} \cos kt, \quad t \in J_j. \tag{10}$$

Theorem 2 *Let $\gamma, \mu \in H$ and $h, \nu \in L$ such that*

$$A_p f = \gamma + h + \mathrm{i}[\mu + \nu] \tag{11}$$

are boundary values of a function $f(z)$ analytic in G. Then the functions $h = (h_0, h_1, \ldots, h_m)$ and $\nu = (\nu_0, \nu_1, \ldots, \nu_m)$ have each element given by

$$h = \sum_{k=0}^{m} \left(\gamma, \vartheta^{[k]} \right) \chi^{[k]}, \tag{12}$$

$$\nu = \sum_{k=0}^{m} \left(\mu, \vartheta^{[k]} \right) \chi^{[k]}, \tag{13}$$

where $\vartheta^{[k]}$ is the unique solution of the integral equation

$$(\mathbf{I} + \mathbf{N}_p^* + \mathbf{J})\vartheta^{[k]} = -\chi^{[k]}, \quad k = 1, 2, \ldots, m. \tag{14}$$

By obtaining $\rho(t)$ and $\nu(t)$, we can have $S(t)$ by (8).

3. Computing the mapping function

Let the function Φ be defined by

$$\Phi(z) = \frac{1}{2} \left(\frac{z - \alpha}{1 - \overline{\alpha} z} + \frac{1 - \overline{\alpha} z}{z - \alpha} \right) + d, \ d = \left\{ \begin{array}{ll} 0, & \Omega_1, \\ 1, & \Omega_2. \end{array} \right.$$

Let also the two functions $g_1(z)$ and $g_2(z)$ be defined by:

$$g_1(z) = \left\{ \begin{array}{ll} \Phi(z), & t \in J_0, \\ 1, & t \in J_j, j = 1, 2, \ldots, m, \end{array} \right. \qquad g_2(z) = \left\{ \begin{array}{ll} 1, & t \in J_0, \\ \Phi(z), & t \in J_j, j = 1, 2, \ldots, m. \end{array} \right.$$

Then the boundary values of the mapping function $\omega(\eta(t))$ satisfy

$$e^{\mathrm{i}(\pi/2 - \theta_j)} \log \left(\frac{\omega(\eta(t))}{g_1} \right) = -R_j + \mathrm{i}S(t). \tag{15}$$

where $R_j = 0, R_1, \cdots, R_m$ and $\theta_j = 0, \theta_1, \theta_2, \ldots, \theta_m$. The mapping function $w = \omega(z)$ can be written as

$$\omega(z) = \Phi(z) e^{(z - \alpha) f(z) + \mathrm{i}h_0} \tag{16}$$

where $f(z)$ is an analytic function in G, h_0 is undetermined real constant.

Then the boundary values of the function $f(z)$ satisfies

$$A_1(t) f(\eta(t)) = [h(t) - \gamma(t)] + \mathrm{i}[\rho(t) - \hat{\mu}(t) + \nu(t)] \tag{17}$$

where

$$h(t) = h_0 \cos \theta(t) - R(t), \tag{18}$$

$$\nu(t) = \hat{\nu}(t) - h_0 \sin \theta(t), \tag{19}$$

$$\gamma(t) + \mathrm{i}\mu(t) = e^{\mathrm{i}(\pi/2 - \theta)} \log g_2. \tag{20}$$

By differentiating (17) with respect to t, we have

$$\tilde{A}_2(t)F(\eta(t)) + \tilde{A}_2 G(\eta(t)) = \mathrm{i}\varphi(t),$$

where

$$F(\eta(t)) = (\eta(t)-\alpha)f(\eta(t)) + (\eta(t)-\alpha)^2 f'(\eta(t)), \qquad G(\eta(t)) = (\eta(t)-\alpha)\frac{g_2'(\eta(t))}{g_2(\eta(t))}.$$

From Theorem 1, we have the following

$$(\mathbf{I} + \mathbf{N}_2^* + \mathbf{J})\varphi(t)(t) = 2\mathrm{Im}(\tilde{A}_2 G(\eta(t))).$$

The mapping function can be calculated from (16)

$$\omega(\eta(t)) = g_1(\eta(t))e^{e^{-\mathrm{i}(\pi/2-\theta_j)}(-R_j+\mathrm{i}S(t))}. \tag{21}$$

4. Numerical examples

We consider two examples which are region with five connectivity and region with thirteen connectivity. The integral equations are discretized by Nyström method with the trapezoidal rule to obtain a system of linear algebraic equations. The resulting system is solved by a fast convergent iterative method as described in [16]. The test regions and their resulting canonical regions are presented in the Fig. 2 and Fig. 3.
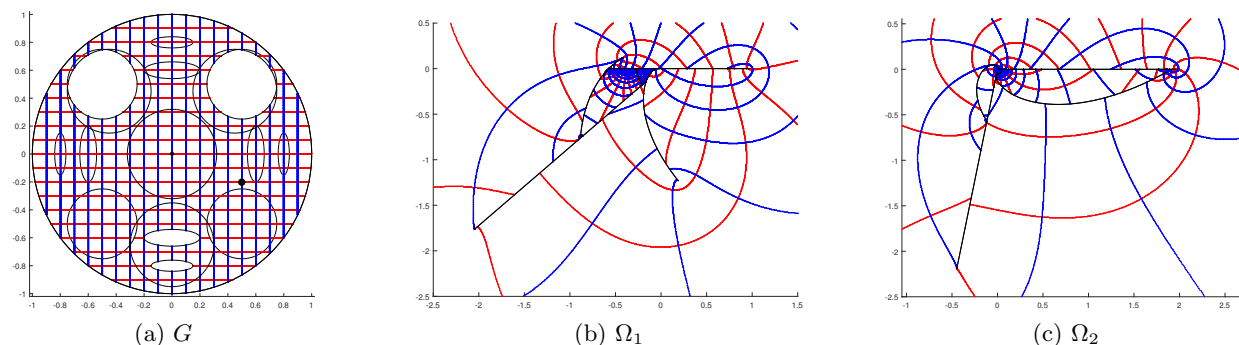

(a) G (b) Ω_1 (c) Ω_2

Figure 2: The original image G and its image Ω_1 (center) Ω_2 (right) for Example 1.

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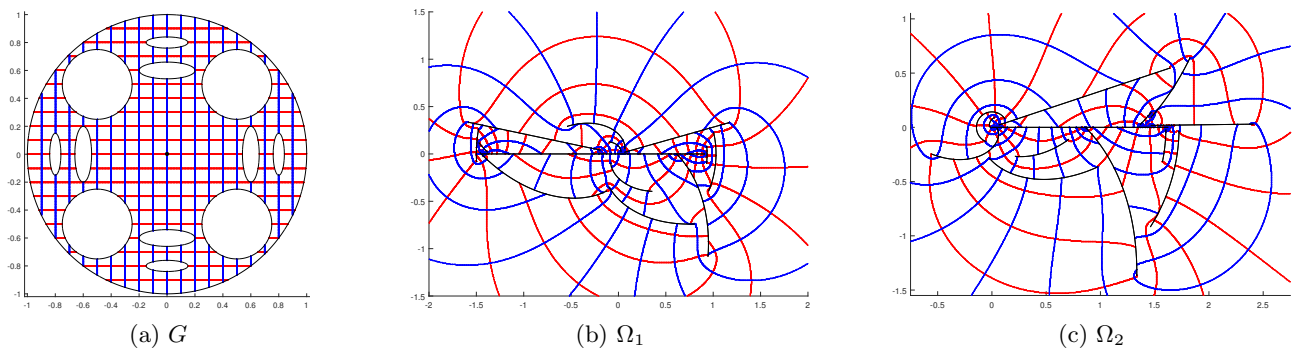[11] Yunus A, Murid A H M and Nasser M M S 2014 *Bull. Malays. Math. Sci. Soc.* **2** 1–24

(a) G (b) Ω_1 (c) Ω_2

Figure 3: The original image G and its image Ω_1 (center) Ω_2 (right) for Example 2.

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