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An algorithm approach to maximal monotone operators and pseudo-contractions

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Abstract

The purpose of this article is to find the minimum norm solution of maximal monotone operators and strict pseudo-contractions in Hilbert spaces. A parallel algorithm is constructed. Some analysis techniques are used to show the convergence of the presented algorithm.

Keywords: Maximal monotone operator, strict pseudo-contractions, zero point, fixed point, minimum-norm.
2010 MSC: 49J40, 47J20, 47H09, 65J15.

1. Introduction

1.1. Problem statement

Let $\mathcal{H}$ be a real Hilbert space. Its inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $\mathcal{C}_i (i = 1, \cdots, N)$ be the nonempty closed convex subset of $\mathcal{H}$. Suppose the intersection of $\{\mathcal{C}_i\}_{i=1}^N$ denoted by $\mathcal{D}$ is nonempty, i.e., $\mathcal{D} = \bigcap_{i=1}^N \mathcal{C}_i \neq \emptyset$. The prototype of the problem of image recovery can be stated as follows. The original (unknown) image $x^\dagger$ is known a priori to belong $\mathcal{D}$; given only the metric projections $P_{\mathcal{C}_i}$ recover $x^\dagger$ by an iterative scheme. This problem is referred to as the convex feasibility problem (CFP), see for instance [1, 4, 5, 7, 10, 18]. One effective approach for solving (CFP) is algorithmic iteration. In this article, our purpose is to find the common fixed points or/and zero points of two nonlinear mappings by using algorithmic approach. Next, we recall some existing results in the literature.

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1.2. Existing results

Let $T = \alpha_0 I + \sum_{n=1}^{N} \alpha_n[\beta_n P_{\mathcal{C}_n} + (1-\beta_n)I]$ where $P_{\mathcal{C}_n}(n = 1, \cdots, N)$ is the metric projection of $\mathcal{H}$ onto $\mathcal{C}_n$, $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=1}^{N} \alpha_n = 1$ and $\{\beta_n\} \subset (0, 2)$.

Iteration 1.(Picard's iteration) Initialization $x_0 \in \mathcal{H}$ and iterative step

$$x_{n+1} = Tx_n, \forall n \in \mathbb{N}. \tag{1.1}$$

Crombez [8] proved that the sequence $\{x_n\}$ generated by (1.1) converges weakly to an element of $(\mathcal{D} =) \bigcap_{i=1}^{N} \mathcal{C}_i$.

Let $(\mathcal{H} \supset)\mathcal{C} \neq \emptyset$ be a closed convex set. Let $S, T : \mathcal{C} \to \mathcal{C}$ be two nonlinear mappings. We use $F(S)$ and $F(T)$ to denote the set of fixed points of S and T, respectively. In [15], Takahashi and Tamura proved that the following Das and Debata's iteration converges weakly to $x^\dagger \in (\mathcal{D} =)F(S) \cap F(T)$.

Iteration 2.(Das and Debata's iteration [9]) Initialization $x_1 \in \mathcal{C}$ and iterative step

$$x_{n+1} = \beta_n x_n + (1-\beta_n)S(\alpha_n x_n + (1-\alpha_n)Tx_n), \forall n \in \mathbb{N}, \tag{1.2}$$

where S and T are two nonexpansive mappings and $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \in (0, 1)$ are two iterative parameters.

Let $A : \mathcal{C} \to \mathcal{H}$ be an inverse strongly monotone mapping with coefficient $\alpha > 0$. Let $B : \mathcal{H} \to \mathcal{H}$ be a maximal monotone mapping. we denote by $(A + B)^{-1}0$ zero points of $A + B$, by $J_\lambda^B = (I + \lambda B)^{-1}$ the resolvent of B for $\lambda > 0$. For finding $x^\dagger \in (\mathcal{D} =)F(S) \cap (A + B)^{-1}0$, Takahashi, Takahashi and Toyoda [14] constructed the following iteration.

Iteration 3. Initialization $x_1 \in \mathcal{C}$ and iterative step

$$x_{n+1} = \beta_n x_n + (1-\beta_n)S(\alpha_n x + (1-\alpha_n)J_{\lambda_n}^B(x_n - \lambda_n Ax_n)), \forall n \in \mathbb{N}, \tag{1.3}$$

where the parameters $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$.

Subsequently, Takahashi, Takahashi and Toyoda proved the strong convergence of the sequence $\{x_n\}$ to $x^\dagger \in (\mathcal{D} =)F(S) \cap (A + B)^{-1}0$ provided the domain of B is included in $\mathcal{C}$ and the parameters satisfy the conditions:

(*i*) $0 < a \leq \lambda_n \leq b < 2\alpha$ and $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$;
(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_n \alpha_n = \infty$;
(*iii*) $0 < c \leq \beta_n \leq d < 1$.

One purpose of this article is to extend the above algorithm to a general case in which S is a strict pseudo-contraction.

1.3. Minimization problem

At the same time, in the practical problem, it is always needed to find minimum norm solution. A typical example is the least-squares solution to the constrained linear inverse problem ([12]) which is stated below.

Example 1.1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. For given $b \in \mathcal{H}_2$, finding $x^\dagger \in \mathcal{C}$ such that

$$Ax^\dagger = b. \tag{1.4}$$

Note that (1.4) can be reduced to solve the minimization problem of finding a point $x^\dagger \in \Gamma$ such that

$$x^\dagger = \arg\min_{x\in\Gamma} \|x\|. \tag{1.5}$$

It is clear that (1.5) is equivalent to $x^\dagger = P_\Gamma(0)$. This indicates that one can use projection technique to find the minimum norm solution. In this respect, there are a large number references in which the projection technique is applied to find the minimum norm solution of some nonlinear operators, see e.g., [2, 3, 19, 20, 21].

Remark 1.2. We observe that the above algorithm (1.3) can also find the minimum-norm solution $x^\dagger \in (\mathcal{D} =)F(S) \cap (A+B)^{-1}0$ provided $0 \in C$. However, if $0 \notin C$, then this algorithm (1.3) does not work to find the minimum-norm solution.

A natural problem arise in the mind if we can find the minimum norm solution without using the projection technique. This is our another purpose of this article. We will devote to find the minimum norm solution $x^\dagger \in (\mathcal{D} =)F(S) \cap (A+B)^{-1}0$ where S is a strict pseudo-contraction. We suggest the following algorithm: for initialization $x_0 \in \mathcal{C}$, let the sequence $\{x_n\}$ be generated by

$$u_{n+1} = \sigma_n u_n + \varsigma_n S u_n + \delta_n J^B_{\mu_n}((1-\xi_n)u_n - \mu_n A u_n), \forall n \geq 0.$$

We will show the above algorithm converges strongly to $x^\dagger = P_{F(S)\cap(A+B)^{-1}0}(0)$ which is the minimum-norm element in $F(S) \cap (A+B)^{-1}0$.

2. Preliminaries

2.1. Notations

Throughout this paper, we assume that $\mathcal{H}$ is a real Hilbert space equipped up its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $(\mathcal{H} \supset)\mathcal{C} \neq \emptyset$ be a closed convex set.

• A self-mapping Ψ on $\mathcal{C}$ is said to be nonexpansive if

$$\|\Psi u - \Psi u^\dagger\| \leq \|u - u^\dagger\|,$$

for all $u, u^\dagger \in \mathcal{C}$.

• A mapping $\Psi : \mathcal{C} \to \mathcal{C}$ is said to be strictly pseudo-contractive if

$$\|\Psi u - \Psi u^\dagger\|^2 \leq \|u - u^\dagger\|^2 + \kappa\|(I-\Psi)u - (I-\Psi)u^\dagger\|^2, \tag{2.1}$$

for all $u, u^\dagger \in \mathcal{C}$ and for some constant $\kappa \in [0, 1)$. In this case, we always say that Ψ is a κ-strict pseudo-contraction.

Remark 2.1. It is obvious that (2.1) equals

$$\langle \Psi u - \Psi u^\dagger, u - u^\dagger \rangle \leq \|u - u^\dagger\|^2 - \frac{1-\kappa}{2}\|(I-\Psi)u - (I-\Psi)u^\dagger\|^2, \tag{2.2}$$

for all $u, u^\dagger \in \mathcal{C}$.

• A single-valued mapping $G : \mathcal{C} \to \mathcal{H}$ is said to be inverse strongly monotone if

$$\langle Gu - Gu^\dagger, u - u^\dagger \rangle \geq \zeta\|Gu - Gu^\dagger\|^2 \tag{2.3}$$

for some $\zeta > 0$ and for all $u, u^\dagger \in C$. Subsequently, we call G is ζ-inverse strongly monotone.

Remark 2.2. From (2.3), we deduce that $\|Gu - Gu^\dagger\|\|u - u^\dagger\| \geq \langle Gu - Gu^\dagger, u - u^\dagger \rangle \geq \zeta\|Gu - Gu^\dagger\|^2$. Thus, $\|Gu - Gu^\dagger\| \leq 1/\zeta\|u - u^\dagger\|$ for all $u, u^\dagger \in \mathcal{C}$. That is, G is $1/\zeta$-Lipschitz continuous.

Let $W : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued mapping. We denote by $dom(W)$ the effective domain of W, i.e., $dom(W) = \{u \in \mathcal{H} : Wu \neq \emptyset\}$.

• W is said to be monotone if

$$\langle x - x^\dagger, u - u^\dagger \rangle \geq 0$$

for all $x, x^\dagger \in dom(W)$, $u \in Wx$ and $u^\dagger \in Wx^\dagger$.

A monotone mapping W on $\mathcal{H}$ is said to be maximal iff its graph is not strictly contained in the graph of any other monotone mapping on $\mathcal{H}$. We use $W^{-1}0$ to denote the set of zero points of W, that is, $W^{-1}(0) = \{u \in \mathcal{H} : 0 \in Wu\}$, see [6] and [11].

Let W be a maximal monotone mapping on $\mathcal{H}$ and let λ be a positive constant. Now we know that $(I + \lambda W)^{-1}$, the resolvent of W is a single-valued mapping from $\mathcal{H}$ onto $dom(W)$ which is denoted by J_λ^W. That is, $J_\lambda^W = (I + \lambda W)^{-1}$.

2.2. Lemmas

Next, we collect several useful lemmas which will be cited in the next section.

Lemma 2.3. *Properties of the resolvent J_λ^W are listed as follows.*

(i) *(Firmly-nonexpansive)* $\|J_\lambda^W u - J_\lambda^W u^\dagger\|^2 \le \langle J_\lambda^W u - J_\lambda^W u^\dagger, u - u^\dagger\rangle$, $\forall u, u^\dagger \in \mathcal{C}$.

(ii) $F(J_\lambda^W) = W^{-1}0$, $\forall \lambda > 0$.

(iii) *(Resolvent identity)* $\forall \lambda > 0$ *and* $\forall \mu > 0$, *we have the following identity*

$$J_\lambda^W x^\dagger = J_\mu^W\Big(\frac{\mu}{\lambda}x^\dagger + (1 - \frac{\mu}{\lambda})J_\lambda^W x^\dagger\Big), \forall x^\dagger \in \mathcal{H}. \tag{2.4}$$

Lemma 2.4 ([22])**.** *Let $\mathcal{H}$ be a real Hilbert space. Let $(\mathcal{H} \supset)\mathcal{C} \neq \emptyset$ be a closed convex set. Let $R : \mathcal{C} \to \mathcal{H}$ be a ρ-strict pseudo-contraction. Set $U = \gamma I + (1-\gamma)R$, $\forall \gamma \in (0, 1)$. Then, $F(U) = F(R)$ and U is nonexpansive when $\gamma \in [\rho, 1)$.*

Lemma 2.5 ([16])**.** *Let $\mathcal{H}$ be a real Hilbert space. Let $(\mathcal{H} \supset)\mathcal{C} \neq \emptyset$ be a closed convex set. Let $U : \mathcal{C} \to \mathcal{H}$ be an inverse strongly monotone mapping with coefficient $\alpha > 0$. Then, we have*

$$\|(I - \varsigma U)x - (I - \varsigma U)x^\dagger\|^2 \le \|x - x^\dagger\|^2 + \varsigma(\varsigma - 2\alpha)\|Ux - Ux^\dagger\|^2, \forall x, x^\dagger \in \mathcal{C}.$$

Especially, $I - \varsigma U$ is nonexpansive when $0 \le \varsigma \le 2\alpha$.

Lemma 2.6 ([22])**.** *Let $\mathcal{H}$ be a real Hilbert space. Let $(\mathcal{H} \supset)\mathcal{C} \neq \emptyset$ be a closed convex set. Let $R : \mathcal{C} \to \mathcal{C}$ be a λ-strict pseudo-contraction. Then $I - R$ is demi-closed at 0, i.e.,*

$$\begin{cases} x_n \rightharpoonup x \in \mathcal{C} \\ x_n - Rx_n \to 0 \end{cases} \Longrightarrow x \in F(R).$$

Lemma 2.7 ([13])**.** *Let X be a Banach space. Let $\{u_n\} \subset X$ and $\{v_n\} \subset X$ be two bounded sequences satisfying $u_{n+1} = (1 - \delta_n)v_n + \delta_n u_n, \forall n \ge 0$ where $\{\delta_n\} \subset (\varpi_1, \varpi_2) \subset (0, 1)$ is a real sequence. Then,*

$\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|u_{n+1} - u_n\|) \le 0$ *implies that* $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Lemma 2.8 ([17])**.** *Let $\{\varrho_n\} \subset [0, +\infty)$, $\{\vartheta_n\} \subset (0, 1)$ and $\{\eta_n\}$ be three real number sequences. Suppose $\{\varrho_n\}$, $\{\vartheta_n\}$ and $\{\eta_n\}$ satisfy the following three conditions*

(i) $\varrho_{n+1} \le (1 - \vartheta_n)\varrho_n + \eta_n\vartheta_n$,

(ii) $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(iii) $\limsup_{n\to\infty} \eta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\eta_n\vartheta_n| < \infty$.

Then $\lim_{n\to\infty} \varrho_n = 0$.

3. Algorithm and Convergence

Let $\mathcal{H}$ be a real Hilbert space. Let $(\mathcal{H} \supset)\mathcal{C} \neq \emptyset$ be a closed convex set. Let $A : \mathcal{C} \to \mathcal{H}$ be an inverse strongly monotone mapping with coefficient $\alpha > 0$ and let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone mapping with its resolvent $J_\mu^B = (I + \mu B)^{-1}$. Let $S : \mathcal{C} \to \mathcal{C}$ be a strict pseudo-contraction with coefficient $\kappa \in [0, 1)$.

Our objective is to find $x^\dagger \in F(S) \cap (A + B)^{-1}$ such that its norm is minimal in $F(S) \cap (A + B)^{-1}$. Our means is to use algorithmic approach. Now, we first introduce our algorithm.

Algorithm 3.1. *Initialization $x_0 \in \mathcal{C}$ and iterative step*

$$u_{n+1} = \sigma_n u_n + \varsigma_n S u_n + \delta_n J_{\mu_n}^B((1 - \xi_n)u_n - \mu_n A u_n), \forall n \geq 0, \tag{3.1}$$

where $\{\mu_n\} \subset (0, 2\alpha)$ and $\{\xi_n\}$, $\{\sigma_n\}$, $\{\varsigma_n\}$ and $\{\delta_n\}$ are four real number sequences in $(0, 1)$.

Theorem 3.2. *Suppose $F(S) \cap (A + B)^{-1}0 \neq \emptyset$ and $dom(B) \subset \mathcal{C}$. Assume that the following restrictions are satisfied.*

(i) $\sigma_n + \varsigma_n + \delta_n = 1$ *for all* $n \geq 0$,
(ii) $\lim_{n\to\infty} \xi_n = 0$ *and* $\sum_n \xi_n = \infty$,
(iii) $\sigma_n \in [c, d] \subset (\kappa, 1)$ *and* $0 < \liminf_{n\to\infty} \varsigma_n \leq \limsup_{n\to\infty} \varsigma_n < 1 - \kappa$,
(iv) $a(1 - \xi_n) \leq \mu_n \leq b(1 - \xi_n)$ *where* $[a, b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$,
(v) $\lim_{n\to\infty}(\frac{\varsigma_{n+1}}{1-\kappa-\sigma_{n+1}+\kappa\delta_{n+1}} - \frac{\varsigma_n}{1-\kappa-\sigma_n+\kappa\delta_n}) = 0$ *and* $\lim_{n\to\infty}(\frac{\delta_{n+1}}{1-\kappa-\sigma_{n+1}+\kappa\delta_{n+1}} - \frac{\delta_n}{1-\kappa-\sigma_n+\kappa\delta_n}) = 0$.

Then the sequence $\{u_n\}$ generated by (3.1) converges strongly to $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$ which is the minimum norm element in $F(S) \cap (A + B)^{-1}0$.

Proof. Let any $u^\dagger \in F(S) \cap (A + B)^{-1}0$. Then, we have $u^\dagger = J_{\mu_n}^B(u^\dagger - \mu_n A u^\dagger) = J_{\mu_n}^B\Big(\xi_n u^\dagger + (1 - \xi_n)(u^\dagger - \mu_n A u^\dagger/(1 - \xi_n))\Big)$ for all $n \geq 0$. Since J_μ^B is nonexpansive for all $\mu > 0$, we deduce

$$\begin{aligned}
&\|J_{\mu_n}^B((1 - \xi_n)u_n - \mu_n A u_n) - u^\dagger\| \\
&= \|J_{\mu_n}^B((1 - \xi_n)(u_n - \mu_n A u_n/(1 - \xi_n))) - J_{\mu_n}^B(\xi_n u^\dagger + (1 - \xi_n)(u^\dagger - \mu_n A u^\dagger/(1 - \xi_n)))\| \\
&\leq \|((1 - \xi_n)(u_n - \mu_n A u_n/(1 - \xi_n))) - (\xi_n u^\dagger + (1 - \xi_n)(u^\dagger - \mu_n A u^\dagger/(1 - \xi_n)))\| \\
&= \|(1 - \xi_n)((u_n - \mu_n A u_n/(1 - \xi_n)) - (u^\dagger - \mu_n A u^\dagger/(1 - \xi_n))) + \xi_n(-u^\dagger)\|.
\end{aligned} \tag{3.2}$$

Noting that the norm $\|\cdot\|$ is convex and the mapping A is α-inverse strongly monotone, we obtain

$$\begin{aligned}
&\|(1 - \xi_n)((u_n - \mu_n A u_n/(1 - \xi_n)) - (u^\dagger - \mu_n A u^\dagger/(1 - \xi_n))) + \xi_n(-u^\dagger)\|^2 \\
&\leq (1 - \xi_n)\|(u_n - \mu_n A u_n/(1 - \xi_n)) - (u^\dagger - \mu_n A u^\dagger/(1 - \xi_n))\|^2 + \xi_n\|u^\dagger\|^2 \\
&= (1 - \xi_n)\|(u_n - u^\dagger) - \mu_n(A u_n - A u^\dagger)/(1 - \xi_n)\|^2 + \xi_n\|u^\dagger\|^2 \\
&= (1 - \xi_n)\Big(\|u_n - u^\dagger\|^2 - \frac{2\mu_n}{1 - \xi_n}\langle A u_n - A u^\dagger, u_n - u^\dagger\rangle + \frac{\mu_n^2}{(1 - \xi_n)^2}\|A u_n - A u^\dagger\|^2\Big) \\
&\quad + \xi_n\|u^\dagger\|^2 \\
&\leq (1 - \xi_n)\Big(\|u_n - u^\dagger\|^2 - \frac{2\alpha\mu_n}{1 - \xi_n}\|A u_n - A u^\dagger\|^2 + \frac{\mu_n^2}{(1 - \xi_n)^2}\|A u_n - A u^\dagger\|^2\Big) + \xi_n\|u^\dagger\|^2 \\
&= (1 - \xi_n)\Big(\|u_n - u^\dagger\|^2 + \frac{\mu_n}{(1 - \xi_n)^2}(\mu_n - 2(1 - \xi_n)\alpha)\|A u_n - A u^\dagger\|^2\Big) + \xi_n\|u^\dagger\|^2.
\end{aligned} \tag{3.3}$$

By condition (iv), we derive that $\mu_n - 2(1 - \xi_n)\alpha \leq 0$ for all $n \geq 0$. Thus, from (3.2) and (3.3), we obtain

$$\begin{aligned}
&\|J_{\mu_n}^B((1 - \xi_n)u_n - \mu_n A u_n) - u^\dagger\|^2 \\
&\leq (1 - \xi_n)\Big(\|u_n - u^\dagger\|^2 + \frac{\mu_n}{(1 - \xi_n)^2}(\mu_n - 2(1 - \xi_n)\alpha)\|A u_n - A u^\dagger\|^2\Big) + \xi_n\|u^\dagger\|^2 \\
&\leq (1 - \xi_n)\|u_n - u^\dagger\|^2 + \xi_n\|u^\dagger\|^2.
\end{aligned} \tag{3.4}$$

Applying (2.1) and (2.2), we obtain

$$\begin{aligned}\|\sigma_n(u_n-u^\dagger)+\varsigma_n(Su_n-u^\dagger)\|^2 &= \sigma_n^2\|u_n-u^\dagger\|^2+\varsigma_n^2\|Su_n-u^\dagger\|^2+2\sigma_n\varsigma_n\langle Su_n-u^\dagger,u_n-u^\dagger\rangle \\ &\le \sigma_n^2\|u_n-u^\dagger\|^2+\varsigma_n^2[\|u_n-u^\dagger\|^2+\kappa\|u_n-Su_n\|^2] \\ &\quad +2\sigma_n\varsigma_n[\|u_n-u^\dagger\|^2-\frac{1-\kappa}{2}\|u_n-Su_n\|^2] \\ &= (\sigma_n+\varsigma_n)^2\|u_n-u^\dagger\|^2+[\varsigma_n^2\kappa-(1-\kappa)\sigma_n\varsigma_n]\|u_n-Su_n\|^2 \\ &= (\sigma_n+\varsigma_n)^2\|u_n-u^\dagger\|^2+\varsigma_n[(\sigma_n+\varsigma_n)\kappa-\sigma_n]\|u_n-Su_n\|^2 \\ &\le (\sigma_n+\varsigma_n)^2\|u_n-u^\dagger\|^2,\end{aligned} \tag{3.5}$$

which implies that

$$\|\sigma_n(u_n-u^\dagger)+\varsigma_n(Su_n-u^\dagger)\|\le(\sigma_n+\varsigma_n)\|u_n-u^\dagger\|. \tag{3.6}$$

Note that $I-\mu_nA/(1-\xi_n)$ is nonexpansive by Lemma 2.5. From (3.2), we have

$$\begin{aligned}&\|J_{\mu_n}^B((1-\xi_n)u_n-\mu_nAu_n)-u^\dagger\| \\ &\le\|(1-\xi_n)((u_n-\mu_nAu_n/(1-\xi_n))-(u^\dagger-\mu_nAu^\dagger/(1-\xi_n)))+\xi_n(-u^\dagger)\| \\ &\le(1-\xi_n)\|(u_n-\mu_nAu_n/(1-\xi_n))-(u^\dagger-\mu_nAu^\dagger/(1-\xi_n))\|+\xi_n\|u^\dagger\| \\ &\le(1-\xi_n)\|u_n-u^\dagger\|+\xi_n\|u^\dagger\|.\end{aligned} \tag{3.7}$$

By (3.1) and (3.7), we get

$$\begin{aligned}\|u_{n+1}-u^\dagger\| &= \|\sigma_n(u_n-u^\dagger)+\varsigma_n(Su_n-u^\dagger)+\delta_n(J_{\mu_n}^B((1-\xi_n)u_n-\mu_nAu_n)-u^\dagger)\| \\ &\le\|\sigma_n(u_n-u^\dagger)+\varsigma_n(Su_n-u^\dagger)\|+\delta_n\|J_{\mu_n}^B((1-\xi_n)u_n-\mu_nAu_n)-u^\dagger\| \\ &\le(\sigma_n+\varsigma_n)\|u_n-u^\dagger\|+\delta_n(1-\xi_n)\|u_n-u^\dagger\|+\xi_n\|u^\dagger\| \\ &=(1-\delta_n\xi_n)\|u_n-u^\dagger\|+\delta_n\xi_n\|u^\dagger\| \\ &\le\max\{\|u_n-u^\dagger\|,\|u^\dagger\|\}.\end{aligned}$$

By induction, we have

$$\|u_{n+1}-u^\dagger\|\le\max\{\|x_0-u^\dagger\|,\|u^\dagger\|\}.$$

So, $\{u_n\}$ is bounded. We also deduce that $\{Au_n\}$ is bounded according to the Lipschitzian continuity of A. Set $x_n=(1-\xi_n)u_n-\mu_nAu_n$ and $y_n=J_{\mu_n}^Bx_n$ for all $n\ge0$. It is easy to see that $\{x_n\}$, $\{J_{\mu_n}^Bu_n\}$, $\{Su_n\}$ and $\{y_n\}$ are all bounded.

(3.1) can be rewritten as

$$u_{n+1}=\left(\sigma_n-\frac{\kappa\varsigma_n}{1-\kappa}\right)u_n+\frac{\varsigma_n}{1-\kappa}\left(\kappa u_n+(1-\kappa)Su_n\right)+\delta_ny_n,\ \text{for all } n\ge0. \tag{3.8}$$

Observe that

$$(\sigma_n-\frac{\kappa\varsigma_n}{1-\kappa})+\frac{\varsigma_n}{1-\kappa}+\delta_n=1$$

and

$$0<\liminf_{n\to\infty}(\sigma_n-\frac{\kappa\varsigma_n}{1-\kappa})\le\limsup_{n\to\infty}(\sigma_n-\frac{\kappa\varsigma_n}{1-\kappa})<1.$$

Set $u_{n+1}=(\sigma_n-\frac{\kappa\varsigma_n}{1-\kappa})u_n+(1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa})z_n$ for all $n\ge0$. It follows that

$$z_{n+1}-z_n=\frac{u_{n+2}-(\sigma_{n+1}-\frac{\kappa\varsigma_{n+1}}{1-\kappa})u_{n+1}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}}-\frac{u_{n+1}-(\sigma_n-\frac{\kappa\varsigma_n}{1-\kappa})u_n}{1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa}}$$

$$\begin{aligned}
&= \frac{\frac{\varsigma_{n+1}}{1-\kappa}(\kappa u_{n+1}+(1-\kappa)Su_{n+1})+\delta_{n+1}y_{n+1}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} \\
&\quad - \frac{\frac{\varsigma_n}{1-\kappa}(\kappa u_n+(1-\kappa)Su_n)+\delta_n y_n}{1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa}} \\
&= \frac{\delta_{n+1}(y_{n+1}-y_n)}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} + \left(\frac{\delta_{n+1}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\delta_n}{1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa}}\right)y_n \\
&\quad + \frac{\frac{\varsigma_{n+1}}{1-\kappa}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}}\Big(\kappa(u_{n+1}-u_n)+(1-\kappa)(Su_{n+1}-Su_n)\Big) \\
&\quad + \left(\frac{\frac{\varsigma_{n+1}}{1-\kappa}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\frac{\varsigma_n}{1-\kappa}}{1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa}}\right)(\kappa u_n+(1-\kappa)Su_n).
\end{aligned}$$

Applying Lemma 2.4, we deduce that $\kappa I+(1-\kappa)S$ is nonexpansive. Hence, we get

$$\begin{aligned}
\|z_{n+1}-z_n\| \le& \frac{\delta_{n+1}\|y_{n+1}-y_n\|}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} + \left|\frac{\delta_{n+1}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\delta_n}{1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa}}\right|\|y_n\| \\
&+ \frac{\frac{\varsigma_{n+1}}{1-\kappa}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}}\|\kappa(u_{n+1}-u_n)+(1-\kappa)(Su_{n+1}-Su_n)\| \\
&+ \left|\frac{\frac{\varsigma_{n+1}}{1-\kappa}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\frac{\varsigma_n}{1-\kappa}}{1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa}}\right|\|\kappa u_n+(1-\kappa)Su_n\| \\
\le& \frac{\delta_{n+1}\|y_{n+1}-y_n\|}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} + \left|\frac{\delta_{n+1}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\delta_n}{1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa}}\right|\|y_n\| \qquad (3.9)\\
&+ \left|\frac{\frac{\varsigma_{n+1}}{1-\kappa}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\frac{\varsigma_n}{1-\kappa}}{1-\sigma_n+\frac{\kappa\varsigma_n}{1-\kappa}}\right|\|\kappa u_n+(1-\kappa)Su_n\| \\
&+ \frac{\frac{\varsigma_{n+1}}{1-\kappa}}{1-\sigma_{n+1}+\frac{\kappa\varsigma_{n+1}}{1-\kappa}}\|u_{n+1}-u_n\|.
\end{aligned}$$

Notice that

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|J^B_{\mu_{n+1}}x_{n+1} - J^B_{\mu_n}x_n\| \\
&\le \|J^B_{\mu_{n+1}}x_{n+1} - J^B_{\mu_{n+1}}x_n\| + \|J^B_{\mu_{n+1}}x_n - J^B_{\mu_n}x_n\| \\
&\le \|x_{n+1}-x_n\| + \|J^B_{\mu_{n+1}}x_n - J^B_{\mu_n}x_n\| \\
&= \|(I-\mu_{n+1}A)u_{n+1} - (I-\mu_{n+1}A)u_n + (\mu_n-\mu_{n+1})Au_n + \xi_n u_n - \xi_{n+1}u_{n+1}\| \\
&\quad + \|J^B_{\mu_{n+1}}x_n - J^B_{\mu_n}x_n\| \\
&\le \|(I-\mu_{n+1}A)u_{n+1} - (I-\mu_{n+1}A)u_n\| + |\mu_{n+1}-\mu_n|\|Au_n\| \\
&\quad + \xi_n\|u_n\| + \xi_{n+1}\|u_{n+1}\| + \|J^B_{\mu_{n+1}}x_n - J^B_{\mu_n}x_n\|.
\end{aligned}$$

By the assumption, $\mu_{n+1}\in(0,2\alpha)$. Thus $I-\mu_{n+1}A$ is nonexpansive. So, we have

$$\|(I-\mu_{n+1}A)u_{n+1} - (I-\mu_{n+1}A)u_n\| \le \|u_{n+1}-u_n\|.$$

Applying the property of the resolvent (2.4), we have

$$J^B_{\mu_{n+1}}x_n = J^B_{\mu_n}\Big(\frac{\mu_n}{\mu_{n+1}}x_n + (1-\frac{\mu_n}{\mu_{n+1}})J^B_{\mu_{n+1}}x_n\Big).$$

It follows that

$$\|J^B_{\mu_{n+1}}x_n - J^B_{\mu_n}x_n\| = \Big\|J^B_{\mu_n}\Big(\frac{\mu_n}{\mu_{n+1}}x_n + (1-\frac{\mu_n}{\mu_{n+1}})J^B_{\mu_{n+1}}x_n\Big) - J^B_{\mu_n}x_n\Big\|$$

$$\begin{aligned}&\leq \Big\| \Big(\frac{\mu_n}{\mu_{n+1}} x_n + (1 - \frac{\mu_n}{\mu_{n+1}}) J^B_{\mu_{n+1}} x_n \Big) - x_n \Big\| \\ &\leq \frac{|\mu_{n+1} - \mu_n|}{\mu_{n+1}} \|x_n - J^B_{\mu_{n+1}} x_n\|.\end{aligned}$$

So,

$$\begin{aligned}\|y_{n+1} - y_n\| \leq &\|u_{n+1} - u_n\| + |\mu_{n+1} - \mu_n| \|Au_n\| + \xi_n \|u_n\| + \xi_{n+1} \|u_{n+1}\| \\ &+ \frac{|\mu_{n+1} - \mu_n|}{\mu_{n+1}} \|x_n - J^B_{\mu_{n+1}} x_n\|.\end{aligned}$$

Substituting the last inequality into (3.9) to deduce

$$\begin{aligned}\|z_{n+1} - z_n\| \leq &\frac{\delta_{n+1} + \frac{\varsigma_{n+1}}{1-\kappa}}{1 - \sigma_{n+1} + \frac{\kappa\varsigma_{n+1}}{1-\kappa}} \|u_{n+1} - u_n\| + \left| \frac{\delta_{n+1}}{1 - \sigma_{n+1} + \frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\delta_n}{1 - \sigma_n + \frac{\kappa\varsigma_n}{1-\kappa}} \right| \|y_n\| \\ &+ \left| \frac{\frac{\varsigma_{n+1}}{1-\kappa}}{1 - \sigma_{n+1} + \frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\frac{\varsigma_n}{1-\kappa}}{1 - \sigma_n + \frac{\kappa\varsigma_n}{1-\kappa}} \right| \|\kappa u_n + (1-\kappa) S u_n\| \\ &+ |\mu_{n+1} - \mu_n| \|Au_n\| + \xi_n \|u_n\| + \xi_{n+1} \|u_{n+1}\| + \frac{|\mu_{n+1} - \mu_n|}{\mu_{n+1}} \|x_n - J^B_{\mu_{n+1}} x_n\|.\end{aligned} \tag{3.10}$$

Since

$$0 < \frac{\delta_{n+1} + \frac{\varsigma_{n+1}}{1-\kappa}}{1 - \sigma_{n+1} + \frac{\kappa\varsigma_{n+1}}{1-\kappa}} < 1,$$

$$\frac{\delta_{n+1}}{1 - \sigma_{n+1} + \frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\delta_n}{1 - \sigma_n + \frac{\kappa\varsigma_n}{1-\kappa}} = \frac{(1-\kappa)\delta_{n+1}}{1 - \kappa - \sigma_{n+1} + \kappa\delta_{n+1}} - \frac{(1-\kappa)\delta_n}{1 - \kappa - \sigma_n + \kappa\delta_n} \to 0,$$

$$\frac{\frac{\varsigma_{n+1}}{1-\kappa}}{1 - \sigma_{n+1} + \frac{\kappa\varsigma_{n+1}}{1-\kappa}} - \frac{\frac{\varsigma_n}{1-\kappa}}{1 - \sigma_n + \frac{\kappa\varsigma_n}{1-\kappa}} = \frac{(1-\kappa)\varsigma_{n+1}}{1 - \kappa - \sigma_{n+1} + \kappa\delta_{n+1}} - \frac{(1-\kappa)\varsigma_n}{1 - \kappa - \sigma_n + \kappa\delta_n} \to 0$$

(by condition (v)), $\xi_n \to 0$, $\mu_{n+1} - \mu_n \to 0$ and $\liminf_{n\to\infty} \mu_n > 0$, we obtain

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|u_{n+1} - u_n\|) \leq 0.$$

This together with Lemma 2.7 imply that

$$\lim_{n\to\infty} \|z_n - u_n\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = \lim_{n\to\infty} (1 - \sigma_n + \frac{\kappa\varsigma_n}{1-\kappa}) \|z_n - u_n\| = 0.$$

From (3.3) and (3.8), we have

$$\begin{aligned}\|u_{n+1} - u^\dagger\|^2 \leq &(\sigma_n - \frac{\kappa\varsigma_n}{1-\kappa}) \|u_n - u^\dagger\|^2 + \frac{\varsigma_n}{1-\kappa} \|\kappa u_n + (1-\kappa) S u_n - u^\dagger\|^2 \\ &+ \delta_n \|J^B_{\mu_n} x_n - u^\dagger\|^2 \\ \leq &(\sigma_n + \varsigma_n) \|u_n - u^\dagger\|^2 + \delta_n \|J^B_{\mu_n} x_n - u^\dagger\|^2 \\ \leq &\delta_n \Big\{ (1 - \xi_n) \Big(\frac{\mu_n}{(1-\xi_n)^2} (\mu_n - 2(1-\xi_n)\alpha) \|Au_n - Au^\dagger\|^2 \Big) \\ &+ \|u_n - u^\dagger\|^2 + \xi_n \|u^\dagger\|^2 \Big\} + (\sigma_n + \varsigma_n) \|u_n - u^\dagger\|^2\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&= (1-\delta_n\xi_n)\|u_n - u^\dagger\|^2 + \frac{\delta_n\mu_n}{(1-\xi_n)^2}(\mu_n - 2(1-\xi_n)\alpha)\|Au_n - Au^\dagger\|^2 \\
&\quad + \delta_n\xi_n\|u^\dagger\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{\delta_n\mu_n}{(1-\xi_n)^2}(2(1-\xi_n)\alpha - \mu_n)\|Au_n - Au^\dagger\|^2 \\
&\leq \|u_n - u^\dagger\|^2 - \|u_{n+1} - u^\dagger\|^2 + \delta_n\xi_n\|u^\dagger\|^2 \\
&\leq (\|u_n - u^\dagger\| - \|u_{n+1} - u^\dagger\|)\|u_{n+1} - u_n\| + \delta_n\xi_n\|u^\dagger\|^2.
\end{aligned}
$$

Since $\lim_{n\to\infty}\xi_n = 0$, $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$ and $\liminf_{n\to\infty}\frac{\delta_n\mu_n}{(1-\xi_n)^2}(2(1-\xi_n)\alpha - \mu_n) > 0$, we deduce

$$
\lim_{n\to\infty}\|Au_n - Au^\dagger\| = 0. \tag{3.12}
$$

Using Lemma 2.3, we have

$$
\begin{aligned}
\|J^B_{\mu_n}x_n - u^\dagger\|^2 &= \|J^B_{\mu_n}x_n - J^B_{\mu_n}(u^\dagger - \mu_n Au^\dagger)\|^2 \\
&\leq \langle x_n - (u^\dagger - \mu_n Au^\dagger), J^B_{\mu_n}x_n - u^\dagger\rangle \\
&= \frac{1}{2}\Big(\|x_n - (u^\dagger - \mu_n Au^\dagger)\|^2 + \|J^B_{\mu_n}x_n - u^\dagger\|^2 \\
&\quad - \|(1-\xi_n)u_n - \mu_n(Au_n - \mu_n Au^\dagger) - J^B_{\mu_n}x_n\|^2\Big) \\
&\leq \frac{1}{2}\Big((1-\xi_n)\|u_n - u^\dagger\|^2 + \xi_n\|u^\dagger\|^2 + \|J^B_{\mu_n}x_n - u^\dagger\|^2 \\
&\quad - \|(1-\xi_n)u_n - J^B_{\mu_n}x_n - \mu_n(Au_n - \mu_n Au^\dagger)\|^2\Big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|J^B_{\mu_n}x_n - u^\dagger\|^2 &\leq (1-\xi_n)\|u_n - u^\dagger\|^2 + \xi_n\|u^\dagger\|^2 \\
&\quad - \|(1-\xi_n)u_n - J^B_{\mu_n}x_n - \mu_n(Au_n - \mu_n Au^\dagger)\|^2 \\
&= (1-\xi_n)\|u_n - u^\dagger\|^2 + \xi_n\|u^\dagger\|^2 - \|(1-\xi_n)u_n - J^B_{\mu_n}x_n\|^2 \\
&\quad + 2\mu_n\langle(1-\xi_n)u_n - J^B_{\mu_n}x_n, Au_n - Au^\dagger\rangle - \mu_n^2\|Au_n - Au^\dagger\|^2 \\
&\leq (1-\xi_n)\|u_n - u^\dagger\|^2 + \xi_n\|u^\dagger\|^2 - \|(1-\xi_n)u_n - J^B_{\mu_n}x_n\|^2 \\
&\quad + 2\mu_n\|(1-\xi_n)u_n - J^B_{\mu_n}x_n\|\|Au_n - Au^\dagger\|.
\end{aligned}
$$

This together with (3.11) imply that

$$
\begin{aligned}
\|u_{n+1} - u^\dagger\|^2 &\leq (\sigma_n + \varsigma_n)\|u_n - u^\dagger\|^2 + \delta_n(1-\xi_n)\|u_n - u^\dagger\|^2 + \delta_n\xi_n\|u^\dagger\|^2 \\
&\quad - \delta_n\|(1-\xi_n)u_n - J^B_{\mu_n}x_n\|^2 \\
&\quad + 2\mu_n\delta_n\|(1-\xi_n)u_n - J^B_{\mu_n}x_n\|\|Au_n - Au^\dagger\| \\
&= (1-\delta_n\xi_n)\|u_n - u^\dagger\|^2 + \delta_n\xi_n\|u^\dagger\|^2 - \delta_n\|(1-\xi_n)u_n - J^B_{\mu_n}x_n\|^2 \\
&\quad + 2\mu_n\delta_n\|(1-\xi_n)u_n - J^B_{\mu_n}x_n\|\|Au_n - Au^\dagger\|.
\end{aligned}
$$

Hence,

$$
\delta_n\|(1-\xi_n)u_n - J^B_{\mu_n}x_n\|^2 \leq \|u_n - u^\dagger\|^2 - \|u_{n+1} - u^\dagger\|^2 - \delta_n\xi_n\|u_n - u^\dagger\|^2 + \delta_n\xi_n\|u^\dagger\|^2
$$

$$\begin{aligned}&+2\mu_n\delta_n\|(1-\xi_n)u_n-J^B_{\mu_n}x_n\|\|Au_n-Au^\dagger\|\\ \le&(\|u_n-u^\dagger\|+\|u_{n+1}-u^\dagger\|)\|u_{n+1}-u_n\|+\delta_n\xi_n\|u^\dagger\|^2\\ &+2\mu_n\delta_n\|(1-\xi_n)u_n-J^B_{\mu_n}x_n\|\|Au_n-Au^\dagger\|.\end{aligned}$$

By assumptions (i) and (iii), we get $\limsup_{n\to\infty}\delta_n<1$. Since $\|u_{n+1}-u_n\|\to 0$, $\xi_n\to 0$ and $\|Au_n-Au^\dagger\|\to 0$, we deduce

$$\lim_{n\to\infty}\|(1-\xi_n)u_n-J^B_{\mu_n}x_n\|=0.$$

This implies that

$$\lim_{n\to\infty}\|u_n-y_n\|=\lim_{n\to\infty}\|u_n-J^B_{\mu_n}((1-\xi_n)u_n-\mu_nAu_n)\|=0. \tag{3.13}$$

Observe that

$$\begin{aligned}\|u_n-Su_n\|&\le\|u_{n+1}-u_n\|+\|u_{n+1}-Su_n\|\\ &\le\|u_{n+1}-u_n\|+\sigma_n\|u_n-Su_n\|+\delta_n\|y_n-u_n\|.\end{aligned}$$

Then,

$$\begin{aligned}\|u_n-Su_n\|&\le\frac{1}{1-\sigma_n}(\|u_{n+1}-u_n\|+\delta_n\|y_n-u_n\|)\\ &\to 0.\end{aligned} \tag{3.14}$$

Since $F(S)\cap(A+B)^{-1}0$ is convex, $P_{F(S)\cap(A+B)^{-1}0}(0)$ exists and is unique which is denoted by $\tilde{x}$, i.e., $\tilde{x}=P_{F(S)\cap(A+B)^{-1}0}(0)$.

Set $v_n=u_n-\frac{\mu_n}{1-\xi_n}(Au_n-A\tilde{x})$ for all $n\ge 0$. In (3.12), we choose $u^\dagger=\tilde{x}$. From (3.12), we get $\|Au_n-A\tilde{x}\|\to 0$. Next, we first prove $\limsup_{n\to\infty}\langle\tilde{x},v_n-\tilde{x}\rangle\ge 0$. Let $\{v_{n_i}\}$ be a subsequence of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle\tilde{x},v_n-\tilde{x}\rangle=\lim_{i\to\infty}\langle\tilde{x},v_{n_i}-\tilde{x}\rangle.$$

Since $\{u_n\}$ is bounded and $\|Au_n-A\tilde{x}\|\to 0$, we deduce that $\{v_n\}$ is bounded. Thus, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ such that $v_{n_{i_j}}\rightharpoonup w\in\mathcal{C}$. It is easy to check that $\{u_{n_{i_j}}\}$ and $\{y_{n_{i_j}}\}$ also converge weakly to w. From (3.14), we have

$$\lim_{j\to\infty}\|u_{n_{i_j}}-Su_{n_{i_j}}\|=0. \tag{3.15}$$

Applying Lemma 2.6 to (3.15), we deduce $w\in F(S)$.

Let $v\in Bu$. Then, we have

$$(1-\xi_n)u_n-\mu_nAu_n\in(I+\mu_nB)y_n\Rightarrow\frac{1-\xi_n}{\mu_n}u_n-Au_n-\frac{y_n}{\mu_n}\in By_n.$$

Since B is monotone, we have, for $(u,v)\in B$,

$$\begin{aligned}&\left\langle\frac{1-\xi_n}{\mu_n}u_n-Au_n-\frac{y_n}{\mu_n}-v,y_n-u\right\rangle\ge 0\\ \Rightarrow&\langle(1-\xi_n)u_n-\mu_nAu_n-y_n-\mu_nv,y_n-u\rangle\ge 0\\ \Rightarrow&\langle Au_n+v,y_n-u\rangle\le\frac{1}{\mu_n}\langle u_n-y_n,y_n-u\rangle-\frac{\xi_n}{\mu_n}\langle u_n,y_n-u\rangle\\ \Rightarrow&\langle Aw+v,y_n-u\rangle\le\frac{1}{\mu_n}\langle u_n-y_n,y_n-u\rangle-\frac{\xi_n}{\mu_n}\langle u_n,y_n-u\rangle+\langle Aw-Au_n,y_n-u\rangle\\ \Rightarrow&\langle Aw+v,y_n-u\rangle\le\frac{1}{\mu_n}\|u_n-y_n\|\|y_n-u\|+\frac{\xi_n}{\mu_n}\|u_n\|\|y_n-u\|+\|Aw-Au_n\|\|y_n-u\|.\end{aligned}$$

It follows that

$$\begin{aligned}\langle Aw+v, w-u\rangle \leq & \frac{1}{\mu_n}\|u_n-y_n\|\|y_n-u\|+\frac{\xi_n}{\mu_n}\|u_n\|\|y_n-u\| \\ & +\|Aw-Au_n\|\|y_n-u\|+\langle Aw+v, w-y_n\rangle. \end{aligned} \tag{3.16}$$

Since Au_n is strongly convergent and u_n is weakly convergent to w, we have $\lim_{n\to\infty}\langle u_n-w, Au_n-Aw\rangle=0$. By the inverse strong monotonicity of A, we have

$$\alpha\|Au_n-Aw\|^2\leq\langle u_n-w, Au_n-Aw\rangle\to 0.$$

So, $Au_n\to Aw$. Hence, from (3.16), we derive

$$\langle Aw+v, w-u\rangle\leq 0.$$

By the maximal monotonicity of B, we obtain immediately that $-Aw\in Bw$. Therefore, $0\in(A+B)w$. Hence, we have $w\in F(S)\cap(A+B)^{-1}0$. Noting that $\tilde{x}=P_{F(S)\cap(A+B)^{-1}0}(0)$, we get $\langle\tilde{x}, w-\tilde{x}\rangle\geq 0$. Therefore,

$$\limsup_{n\to\infty}\langle\tilde{x}, v_n-\tilde{x}\rangle=\lim_{j\to\infty}\langle\tilde{x}, v_{n_{i_j}}-\tilde{x}\rangle=\langle\tilde{x}, w-\tilde{x}\rangle\geq 0.$$

From (3.11), we have

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 \leq & (\sigma_n+\varsigma_n)\|u_n-\tilde{x}\|^2+\delta_n\|J^B_{\mu_n}x_n-\tilde{x}\|^2 \\ \leq & \delta_n\left\|(1-\xi_n)\left((u_n-\frac{\mu_n}{1-\xi_n}Au_n)-(\tilde{x}-\frac{\mu_n}{1-\xi_n}A\tilde{x})\right)-\xi_n\tilde{x}\right\|^2 \\ & +(\sigma_n+\varsigma_n)\|u_n-\tilde{x}\|^2 \\ = & \delta_n\Bigg((1-\xi_n)^2\left\|(u_n-\frac{\mu_n}{1-\xi_n}Au_n)-(\tilde{x}-\frac{\mu_n}{1-\xi_n}A\tilde{x})\right\|^2 \\ & -2\xi_n(1-\xi_n)\left\langle\tilde{x}, (u_n-\frac{\mu_n}{1-\xi_n}Au_n)-(\tilde{x}-\frac{\mu_n}{1-\xi_n}A\tilde{x})\right\rangle+\xi_n^2\|\tilde{x}\|^2\Bigg) \\ & +(\sigma_n+\varsigma_n)\|u_n-\tilde{x}\|^2 \\ \leq & (\sigma_n+\varsigma_n)\|u_n-\tilde{x}\|^2+\delta_n\Bigg((1-\xi_n)^2\|u_n-\tilde{x}\|^2 \\ & -2\xi_n(1-\xi_n)\left\langle\tilde{x}, u_n-\frac{\mu_n}{1-\xi_n}(Au_n-A\tilde{x})-\tilde{x}\right\rangle+\xi_n^2\|\tilde{x}\|^2\Bigg) \\ \leq & (1-\delta_n\xi_n)\|u_n-\tilde{x}\|^2+\delta_n\xi_n[-2(1-\xi_n)\langle\tilde{x}, v_n-\tilde{x}\rangle+\xi_n\|\tilde{x}\|^2]. \end{aligned} \tag{3.17}$$

It is easy to check that $\sum_n(1-\sigma_n)\xi_n=\infty$ and $\limsup_{n\to\infty}(-2(1-\xi_n)\langle\tilde{x}, v_n-\tilde{x}\rangle+\xi_n\|\tilde{x}\|^2)\leq 0$. Applying Lemma 2.8 to (3.17), we conclude that $u_n\to\tilde{x}$. This completes the proof. □

Corollary 3.3. *Let $\mathcal{H}$ be a real Hilbert space. Let $(\mathcal{H}\supset)\mathcal{C}\neq\emptyset$ be a closed convex set. Let $A:\mathcal{C}\to\mathcal{H}$ be an inverse strongly monotone mapping with coefficient $\alpha>0$ and let $B:\mathcal{H}\to 2^{\mathcal{H}}$ be a maximal monotone mapping such that $dom(B)\subset\mathcal{C}$. Let $S:\mathcal{C}\to\mathcal{C}$ be a nonexpansive mapping. Suppose $F(S)\cap(A+B)^{-1}0\neq\emptyset$. For given $x_0\in\mathcal{C}$, let the sequence $\{u_n\}\subset C$ be generated by*

$$u_{n+1}=\sigma_nu_n+\varsigma_nSu_n+\delta_nJ^B_{\mu_n}((1-\xi_n)u_n-\mu_nAu_n), \forall n\geq 0, \tag{3.18}$$

where $\{\mu_n\}\subset(0,2\alpha)$, $\{\xi_n\}\subset(0,1)$, $\{\sigma_n\}\subset(0,1)$, $\{\varsigma_n\}\subset(0,1)$ and $\{\delta_n\}\subset(0,1)$ satisfy the following conditions

(*i*) *$\sigma_n+\varsigma_n+\delta_n=1$ for all $n\geq 0$;*

(ii) $\lim_{n\to\infty} \xi_n = 0$ *and* $\sum_n \xi_n = \infty$;

(iii) $\sigma_n \in [c, d] \subset (0, 1)$ *and* $0 < \liminf_{n\to\infty} \varsigma_n \le \limsup_{n\to\infty} \varsigma_n < 1$;

(iv) $a(1 - \xi_n) \le \mu_n \le b(1 - \xi_n)$ *where* $[a, b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$;

(v) $\lim_{n\to\infty}(\frac{\varsigma_{n+1}}{1-\sigma_{n+1}} - \frac{\varsigma_n}{1-\sigma_n}) = 0$ *and* $\lim_{n\to\infty}(\frac{\delta_{n+1}}{1-\sigma_{n+1}} - \frac{\delta_n}{1-\sigma_n}) = 0$.

Then the sequence $\{u_n\}$ *generated by* (3.18) *converges strongly to* $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$.

Corollary 3.4. *Let* $\mathcal{H}$ *be a real Hilbert space. Let* $(\mathcal{H} \supset)\mathcal{C} \neq \emptyset$ *be a closed convex set. Let* $A : \mathcal{C} \to \mathcal{H}$ *be an inverse strongly monotone mapping with coefficient* $\alpha > 0$ *and let* $B : \mathcal{H} \to 2^{\mathcal{H}}$ *be a maximal monotone mapping such that* $dom(B) \subset \mathcal{C}$. *Let* $J^B_\mu = (I + \mu B)^{-1}$ *be the resolvent of* B *for* μ. *Suppose* $(A+B)^{-1}0 \neq \emptyset$. *For given* $x_0 \in \mathcal{C}$, *let the sequence* $\{u_n\} \subset C$ *be generated by*

$$u_{n+1} = \sigma_n u_n + (1 - \sigma_n) J^B_{\mu_n}((1 - \xi_n)u_n - \mu_n A u_n), \forall n \ge 0, \tag{3.19}$$

where $\{\mu_n\} \subset (0, 2\alpha)$, $\{\xi_n\} \subset (0, 1)$ *and* $\{\sigma_n\} \subset (0, 1)$ *satisfy the following conditions*

(i) $\lim_{n\to\infty} \xi_n = 0$ *and* $\sum_n \xi_n = \infty$;

(ii) $\sigma_n \in [c, d] \subset (0, 1)$;

(iii) $a(1 - \xi_n) \le \mu_n \le b(1 - \xi_n)$ *where* $[a, b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$.

Then $\{u_n\}$ *generated by* (3.19) *converges strongly to* $\tilde{x} = P_{(A+B)^{-1}0}(0)$.

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References

[1] H. H. Bauschke, J. M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Rev., **38** (1996), 367–426. 1.1

[2] L. C. Ceng, Q. H. Ansari, J. C. Yao, *Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem*, Nonlinear Anal., **75** (2012), 2116–2125. 1.3

[3] L. C. Ceng, S. M. Guu, J. C. Yao, *Hybrid methods with regularization for minimization problems and asymptotically strict pseudocontractive mappings in the intermediate sense*, J. Global Optim., **60** (2014), 617–634. 1.3

[4] Y. Censor, *Parallel application of block-iterative methods in medical imaging and radiation therapy*, Math. Programming, **42** (1988), 307–325. 1.1

[5] S. S. Chang, J. Kim, Y. J. Cho, J. Sim, *Weak and strong convergence theorems of solutions to split feasibility problem for nonspreading type mapping in Hilbert spaces*, Fixed Point Theory Appl., **2014** (2014), 12 pages. 1.1

[6] Y. J. Cho, Narin Petrot, *Regularization method for Noor's variational inequality problem induced by a hemicontinuous monotone operator*, Fixed Point Theory Appl., **2012** (2012), 13 pages. 2.1

[7] P. L. Combettes, *Hilbertian convex feasibility problem: Convergence of projection methods*, Appl. Math. Optim., **35** (1997), 311–330. 1.1

[8] G. Crombez, *Image recovery by convex combinations of projections*, J. Math. Anal. Appl., **155** (1991), 413–419. 1.2

[9] G. Das, J. P. Debata, *Fixed points of quasi-nonexpansive mappings*, Indian J. Pure Appl. Math., **17** (1986), 1263–1269. 1.2

[10] A. Hamdi, Y. C. Liou, Y. Yao, C. Luo, *The common solutions of the split feasibility problems and fixed point problems*, J. Inequal. Appl., **2015** (2015), 17 pages. 1.1

[11] H. Pathak, Y. J. Cho, *Strong convergence of a proximal-type algorithm for an occasionally pseudomonotone operator in Banach spaces*, Fixed Point Theory Appl., **2012** (2012), 13 pages. 2.1

[12] A. Sabharwal, L. C. Potter, *Convexly constrained linear inverse problems: iterative least-squares and regularization*, IEEE Trans. Signal Process., **46** (1998), 2345–2352. 1.3

[13] T. Suzuki, *Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces*, Fixed Point Theory Appl., **2005** (2005), 103–123. 2.7

[14] S. Takahashi, W. Takahashi, M. Toyoda, *Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces*, J. Optim. Theory Appl., **147** (2010), 27–41. 1.2

[15] W. Takahashi, T. Tamura, *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Anal., **5** (1998), 45–56. 1.2

[16] W. Takahashi, M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl., **118** (2003), 417–428. 2.5

[17] H. K. Xu, *Iterative Algorithms for Nonlinear Operators*, J. London Math. Soc., **2** (2002), 240–256. 2.8

[18] Y. Yao, R. P. Agarwal, M. Postolache, Y. C. Liou, *Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem*, Fixed Point Theory Appl., **2014** (2014), 14 pages. 1.1

[19] Y. Yao, R. Chen, H. K. Xu, *Schemes for finding minimum-norm solutions of variational inequalities*, Nonlinear Anal., **72** (2010), 3447–3456. 1.3

[20] Y. Yao, G. Marino, H. K. Xu, Y. C. Liou, *Construction of minimum-norm fixed points of pseudocontractions in Hilbert spaces*, J. Inequal. Appl., **2014** (2014), 14 pages. 1.3

[21] Y. Yao, H. K. Xu, *Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications*, Optim., **60** (2011), 645–658. 1.3

[22] H. Y. Zhou, *Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **69** (2008), 456–462. 2.4, 2.6