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Modified Profile Likelihood Estimation in the Lomax Distribution

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Abstract In this paper, we consider improving maximum likelihood inference for the scale parameter of the Lomax distribution. The improvement is based on using modifications to the maximum likelihood estimator based on the Barndorff-Nielsen modification of the profile likelihood function. We apply these modifications to obtain improved estimators for the scale parameter of the Lomax distribution in the presence of a nuisance shape parameter. Due to the complicated expression for the Barndorff-Nielsen's modification, several approximations to this modification are considered in this paper, including the modification based on the empirical covariances and the approximation based on using suitably derived approximate ancillary statistics. We obtained the approximations for the Lomax profile likelihood function and the corresponding modified maximum likelihood estimators. They are not available in simple closed forms and can be obtained numerically as roots of some complicated likelihood equations. Comparisons between maximum profile likelihood estimator and modified profile likelihood estimators in terms of their biases and mean squared errors were carried out using simulation techniques. We found that the approximation based on the empirical covariances to have the best performance according to the criteria used. Therefore we recommend to use this modified version of the maximum likelihood estimator for the Lomax scale parameter, especially for small sample sizes with heavy censoring, which is quite common in industrial life testing experiments and reliability studies. An example based on real data is given to illustrate the methods considered in this paper.

Keywords Modified Maximum Profile Likelihood Method, Lomax Distribution, Barndorff-Nielsen's Adjustment Method

1. Introduction

The Lomax distribution, commonly referred to as the "Pareto type II" distribution, was developed to model business failure data (Lomax[1]). The Lomax model belongs to the declining failure rate family in the lifetime distribution context, see Chahkandi and Ganjali [2]. The Lomax distribution has been proposed as a heavy tailed distribution by Bryson[3] to replace the Exponential, Weibull, and Gamma distributions. The Lomax distribution is crucial for the analysis of lifetime data sets in various fields including business, medical sciences, and engineering (Johnson et al. [4]). More examples can be found in Corbellini et al. [5], Ghitany et al. [6] and Holland et al. [7].

The pdf and cdf of the underlying Lomax lifetime distribution are given respectively by

$$f(y,\theta,\beta)=\frac{\beta\theta}{(1+\beta y)^{\theta+1}}\ ,\ \ y>0,\beta>0,\theta>0 \qquad (1)$$

$$F(y,\theta,\beta)=1-\frac{1}{(1+\beta y)^{\theta}}\ ,\ y>0,\beta>0,\theta>0 \qquad (2)$$

where θ and β are the shape parameter and the scale parameter, respectively. Many authors have addressed Lomax model inferences from Bayesian, E-Bayesian, and maximum likelihood estimation perspectives in the literature. For instance, E-Bayesian estimation was used by

Okasha [8] to compute estimates of the unknown parameters in addition to estimating related survival time characteristics like the hazard and reliability functions. Baklizi et al. [9] used likelihood (least square & weighted least square) and Bayesian inference for parameter estimation in this model under progressively censoring data. Howlader and Hossain [10] considered Bayesian estimation of the Lomax distribution's survival function. Cramer and Schmiedt [11] used type-II censored competing risks data from this model to calculate maximum likelihood estimates for the distribution parameters. Al-Zahrani and Al-Sobhi [12] used Bayesian and maximum likelihood estimation to estimate the parameters based on general progressive censored data. Moreover, Mahmoud et al. [13] compared the maximum likelihood (ML) and Bayes techniques for this model.

The goal of this paper is to extend the work of the previous authors on maximum likelihood estimation to include modified estimation of the scale parameter in the presence of a nuisance shape parameter based on several approximations for Brandorff-Nielsen's modified profile likelihood function. Furthermore, the maximum profile likelihood estimator and maximum modified profile likelihood estimators for the scale parameter were compared using simulation according to their biases and mean squares errors.

The organization of this paper is as follows. The profile likelihood function and its properties are discussed in Section 2. Several approximations to Barndorff-Nielsen's adjustment are presented in Section 3. In Section 4, the adjustments are derived for inference on the Lomax scale parameter under type II censoring data. Section 5 presents the findings of a simulation study designed to investigate and compare the performance of estimators that are derived from the profile likelihood function and adjusted profile likelihood functions. In Section 6, numerical examples using real data are presented. Section 7 concludes the paper.

2. Profile Likelihood Function

We consider a model parametrized by a parameter (θ, β), where β denotes the parameter of interest and θ is a nuisance parameter. The larger the nuisance parameter's dimension, the higher its potential impact on the inference results for the parameter of interest. As a result, replacing θ with the restricted maximum likelihood estimator $\hat{\theta}_\beta$ is a simple approach of removing the effect of the nuisance parameter on inference. Let $L(\theta, \beta)$ be the likelihood function and let $l(\theta, \beta) = log\big(L(\theta, \beta)\big)$, where log is the natural logarithm, then $l_p(\beta) = l(\theta, \beta)|_{\theta=\hat{\theta}_\beta} = l\big(\hat{\theta}_\beta, \beta\big)$ is called the profile log-likelihood function and the maximum profile likelihood estimator of β , under this approach , is represented as $\hat{\beta}_p$. However, because $l_p(\beta)$ does not attempt to approximate a true conditional or marginal likelihood function, the profile likelihood function is not a real likelihood function and thus lacks some of the favorable characteristics of a true likelihood function. This is because, by keeping the nuisance parameter at its point estimate, we are ignoring the uncertainty that comes with such estimation to some extent. Details on profile and modified profile likelihood functions can be found in Severini [14].

3. Modified Profile Likelihoods

There are several modifications to the profile likelihood function proposed. They are all designed to reduce the effect of the nuisance parameter on inference about the parameter of interest. We will discuss some of them in the following subsections.

3.1. Barndorff-Nielsen's Modified Profile Likelihood Function

Barndorff-Nielsen [15] developed a modification that, if it exists, approximates the marginal or conditional likelihood function for the parameter of interest. He proposed a formula for calculating the approximate conditional density of the maximum likelihood method given an ancillary statistic "a". He called this formula the p* equation. Several authors have utilized modified profile likelihood functions for inference including Yang and Xie [16] and Ferrari et al. [17]. The approach used in this paper follows closely the approach of Ferrari et al. [17] for the Weibull shape parameter. The modified profile log-likelihood function of Barndorff-Nielsen is

$$l_{BN}(\beta) = l_p(\beta) - \log\left|\frac{\partial\hat{\theta}_\beta}{\partial\hat{\theta}}\right| - \frac{1}{2}\log\left|j_{\theta\theta}\big(\hat{\theta}_\beta, \beta\big)\right|, \quad (3)$$

where $j_{\theta\theta}\big(\hat{\theta}_\beta, \beta\big) = -\frac{\partial^2 l(\hat{\theta}_\beta,\beta)}{\partial\theta^2}$ and $\frac{\partial\,\hat{\theta}_\beta}{\partial\hat{\theta}}$ is a partial derivative matrix of $\hat{\theta}_\beta$ with respect to $\hat{\theta}$. The most challenging part of computing the $l_{BN}(\beta)$ is in finding $\left|\frac{\partial\hat{\theta}_\beta}{\partial\hat{\theta}}\right|$. There is another equivalent modification for $l_{BN}(\beta)$ that avoid this term. It requires a sample space derivative of the log-likelihood function, as well as an ancillary statistic "a" such that $\big(\hat{\theta}, \hat{\beta}, a\big)$ is a minimal sufficient statistic, see [15].

The next three approximation approaches avoid the difficulties of evaluating the sample space derivatives emanating from this Barndorff-Nielsen's approach.

3.2. Population Covariance Approximation of the Modified Profile Likelihood Function

Severini [18] presented the following approximation for Barndorff- Nielsen's modified profile likelihood function:

$$\bar{l}_{BN}(\beta) = l_p(\beta) + \frac{1}{2}\log\left|j_{\theta\theta}(\hat{\theta}_\beta, \beta)\right| - \log\left|\mathrm{I}_\theta\big(\hat{\theta}_\beta, \beta\,; \hat{\theta}, \hat{\beta}\big)\right|, \quad (4)$$

where,

$$\mathrm{I}_{\theta}(\theta,\beta\,;\theta_0,\beta_0)=E_{(\theta_0,\beta_0)}\{l_{\theta}(\theta,\beta)l_{\theta}(\theta_0,\beta_0)^T\} \quad (5)$$

with $l_{\theta}(\theta,\beta)=\frac{\partial l(\theta,\beta)}{\partial\theta}$. Here, $\hat{\theta}_{\beta}$ is the restricted maximum likelihood estimator. $\hat{\theta}$ and $\hat{\beta}$ are the maximum likelihood estimators of θ and β, respectively. θ_0 and β_0 is a value of the parameter that is different from θ and β, respectively. $I_{\theta}(\theta,\beta\,;\theta_0,\beta_0)$ is independent of the ancillary statistic "a" and $I_{\theta}(\theta,\beta\,;\theta_0,\beta_0)$ represents the covariance between $l_{\theta}(\theta,\beta)$ and $l_{\theta}(\theta_0,\beta_0)$ .The corresponding modified maximum profile likelihood estimator (MMPLE) is represented as $\hat{\hat{\beta}}_{BN}$.

3.3. Empirical Covariances Approximation of the Modified Profile Likelihood Function

According to Severini [19], the empirical covariances approximation, presented below, is useful when calculating expected values of log likelihood derivative products is difficult. This approximation is as follows:

$$\check{l}_{BN}\ (\beta)=l_p(\beta)+$$
$$+\frac{1}{2}\log\left|j_{\theta\theta}(\hat{\theta}_{\beta},\beta)\right|-\log|\check{I}_{\theta}(\hat{\theta}_{\beta},\beta;\hat{\Theta},\hat{\beta})|, \quad (6)$$

where,

$$\check{\mathrm{I}}_{\Theta}\ (\hat{\theta}_{\beta},\beta;\hat{\Theta},\hat{\beta})=\sum_{j=1}^{n} l_{\theta}^{(j)}(\hat{\theta}_{\beta},\beta)\ l_{\theta}^{(j)}(\hat{\Theta}\ ,\hat{\beta}\ )^T. \quad (7)$$

Here, $l_{\theta}^{(j)}$ is the score function of the j^{th} observation, and the corresponding modified maximum profile likelihood estimator (MMPLE) under this approximation is represented as $\check{\hat{\beta}}_{BN}$.

3.4. An Approximation based on an Ancillary Statistic

Fraser and Reid [20] and Fraser et al. [21] presented an approximation, which is given by

$$\tilde{l}_{BN}\ (\beta)=l_p(\beta)+$$
$$+\frac{1}{2}\log\left|j_{\theta\theta}(\hat{\theta}_{\beta},\beta)\right|-\log\left|l_{\theta;y}(\hat{\theta}_{\beta},\beta)\hat{V}_{\theta}\right|, \quad (8)$$

where,

$$l_{\theta;y}(\theta,\beta)=\frac{\partial l_{\theta}(\theta,\beta)}{\partial y^T} \quad (9)$$

Here, $\partial l_{\theta}(\theta,\beta)$ is the score function for, $y^T=(y_1,\dots\dots,y_n)$ and

$$\hat{V}_{\Theta}=\left(-\frac{\partial F(y_1;\hat{\Theta},\hat{\beta})/\partial\hat{\Theta}}{f_1(y_1;\hat{\Theta},\hat{\beta})},\dots\dots\dots,-\frac{\partial F(y_n;\hat{\Theta},\hat{\beta})/\partial\hat{\Theta}}{f_n(y_n;\hat{\Theta},\hat{\beta})}\right), \quad (10)$$

where $f_j(y;\theta,\beta)$ and $F_j(y;\theta,\beta)$ being the probability density function and the cumulative distribution function of y_j, respectively, and $\hat{V}_{\Theta}$ is the approximate ancillary statistic. The corresponding modified maximum profile likelihood estimator (MMPLE) under this approximation is represented as $\tilde{\hat{\beta}}_{BN}$.

4. Modified Profile Likelihoods for the Lomax Scale Parameter

Let $y_{(1)},\dots\dots\dots,y_{(r)}$ be the smallest order statistics from a sample of size n of a Lomax distribution $L(\Theta,\beta)$. Here, the number of failure time (Say r) is fixed and prespecified and the time of study T is random. Therefore, Observation ceases (stopped) after the r^{th} failure $(r< n)$. The likelihood function for the $(\theta\ ,\beta)$ parameters is given by

$$L(\Theta,\beta)=\prod_{j=1}^{r} f(y_{(j)};\Theta,\beta)\prod_{j=r+1}^{n} S(y_{(r)};\Theta,\beta)$$
$$=\left[S(y_{(r)};\Theta,\beta)\right]^{n-r}\prod_{j=1}^{r} f\left(y_{(j)};\Theta,\beta\right)$$
$$=\left[(1+\beta y_{(r)})^{-\Theta}\right]^{n-r}\prod_{j=1}^{r}\left[\beta\Theta\left(1+\beta y_{(j)}\right)^{-(\Theta+1)}\right]$$
$$=\left[\left(1+\beta y_{(r)}\right)^{-\Theta}\right]^{n-r}\cdot\beta^r\Theta^r\cdot\prod_{j=1}^{r}\left(1+\beta y_{(j)}\right)^{-(\Theta+1)} \quad (11)$$

Therefore, the log-likelihood function is given by,

$$l(\Theta,\beta)=r\log\beta+r\log\Theta-\sum_{j=1}^{r}\log\left(1+\beta y_{(j)}\right)-$$
$$\Theta\left[\sum_{j=1}^{r}\log(1+\beta y_{(j)})+(n-\ r)\log(1+\beta y_{(r)})\right] \quad (12)$$

The log-likelihood function's first derivative with respect to θ is given by

$$\frac{\partial l}{\partial\Theta}=\frac{r}{\Theta}-$$
$$-\left[\sum_{j=1}^{r}\log\left(1+\beta y_{(j)}\right)+(n-r)\log\left(1+\beta y_{(r)}\right)\right] \quad (13)$$

The root of this equation in θ for a fixed value of β is

$$\hat{\Theta}_{\beta}=\frac{r}{\sum_{j=1}^{r}\log\left(1+\beta y_{(j)}\right)+(n-r)\log(1+\beta y_{(r)})} \quad (14)$$

This root represents the restricted MLE of θ for a given β.

Note that under no censoring $(r=n)$, this estimator reduces to the one given before in Uncensored data.

Substituting $\hat{\theta}_{\beta}$ in the log-likelihood equation we obtain the profile log-likelihood function

$$l_p(\beta)=l\left(\hat{\Theta}_{\beta},\beta\right)=$$
$$=r\ log\,\beta+r\ log\left(\hat{\Theta}_{\beta}\right)\ -\sum_{j=1}^{r} log(1+\beta y_{(j)})-\left(\hat{\Theta}_{\beta}\right)\cdot$$
$$\left[\sum_{j=1}^{r} log(1+\beta y_{(j)})+(n-\ r)\ log(1+\beta y_{(r)})\right] \quad (15)$$

It follows,

$$l_p(\beta) = l(\widehat{\Theta}_\beta, \beta)$$
$$= r\log\beta + r\log\left(\frac{r}{\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})}\right) - \sum_{j=1}^{r}\log(1+\beta y_{(j)}) - \left(\frac{r}{\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})}\right) \cdot \left[\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})\right] \quad (16)$$

The MLE of β, which is the maximum profile likelihood estimator of β is the solution of the following equation

$$\frac{\partial l_p(\beta)}{\partial\beta} = \frac{r}{\beta} - -\sum_{j=1}^{r}\frac{y_{(j)}}{1+\beta y_{(j)}} - \hat{\theta}_\beta\sum_{j=1}^{r}\frac{y_{(j)}}{1+\beta y_{(j)}} - \frac{\hat{\theta}_\beta(n-r)y_{(r)}}{1+\beta y_{(r)}} = 0 \quad (17)$$

Which is equivalent to the following equation,

$$\frac{r}{\beta} - \sum_{j=1}^{r}\frac{y_{(j)}}{1+\beta y_{(j)}} - \left(\frac{r}{\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})}\right)\sum_{j=1}^{r}\frac{y_{(j)}}{1+\beta y_{(j)}} - \frac{\left(\frac{r}{\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})}\right)(n-r)y_{(r)}}{1+\beta y_{(r)}} = 0$$

The MLE $\hat{\beta}$ can't be obtained analytically and we need to find it numerically by applying some iterative methods like the Newton-Raphson method or direct optimization techniques.

Calculating $j_{\theta\theta}(\hat{\theta}_\beta, \beta)$ from the observed Fisher information matrix $j(\theta,\beta) = -\left(\frac{\partial^2}{\partial\theta^2} l(\theta,\beta)\right)_{\theta=\hat{\theta}_\beta}$ which is obtained from the log-likelihood function for Lomax distribution evaluated at $(\hat{\theta}_\beta, \beta)$ we obtain

$$j_{\Theta\Theta}(\widehat{\Theta}_\beta, \beta) = \frac{\left(\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})\right)^2}{r} \quad (19)$$

Now, we will consider some approximations to the modified profile likelihood for Lomax parameter β using Barndorff-Nielsen's Method that are described in section 3.

Equation (4) is not possible to derive in type II censoring because y_j are order statistics (Not iid). Therefore, we make use of the empirical covariances.

Using (7), it follows that

$$l_\theta^{(j)}(\beta, \hat{\theta}_\beta) = \frac{r}{\widehat{\Theta}_\beta} - \log(1+\beta y_{(j)}) - (n-)\log(1+\beta y_{(r)})$$

and

$$l_\theta^{(j)}(\hat{\beta}, \hat{\theta}) = \frac{r}{\widehat{\Theta}} - \log(1+\hat{\beta} y_{(j)}) - (n-r)\log(1+\hat{\beta} y_{(r)})$$

Then,

$$\check{l}_\theta(\beta, \hat{\theta}_\beta; \hat{\beta}, \hat{\theta}) = \sum_{j=1}^{r}\left[\left(\frac{r}{\widehat{\Theta}_\beta} - \log(1+\beta y_{(j)}) - (n-r)\log(1+\beta y_{(r)})\right)\left(\frac{r}{\widehat{\Theta}} - \log(1+\hat{\beta} y_{(j)}) - (n-r)\log(1+\hat{\beta} y_{(r)})\right)\right]$$

From (6,7,16 & 19), we obtain

$$\check{l}_{BN}(\beta) = r\log\beta + r\log[\widehat{\Theta}_\beta] - \sum_{j=1}^{r}\log(1+\beta y_{(j)}) - (\widehat{\Theta}_\beta) \cdot \left[\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})\right] + \frac{1}{2}\log\left[\frac{\left(\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})\right)^2}{r}\right] - \log\left[\sum_{j=1}^{r}\left[\left(\frac{r}{\widehat{\Theta}_\beta} - \log(1+\beta y_{(j)}) - (n-r)\log(1+\beta y_{(r)})\right)\left(\frac{r}{\widehat{\Theta}} - \log(1+\hat{\beta} y_{(j)}) - (n-r)\log(1+\hat{\beta} y_{(r)})\right)\right]\right]$$

where,

$$\widehat{\Theta}_\beta = \frac{r}{\sum_{j=1}^{r}\log(1+\beta y_{(j)}) + (n-r)\log(1+\beta y_{(r)})}$$

and where $\hat{\theta}$ and $\hat{\beta}$ are the maximum likelihood estimators of θ and β. The corresponding estimator is $\hat{\check{\beta}}_{BN}$. There is no closed form expression for the MLE $\hat{\check{\beta}}_{BN}$ and we need to find it numerically by applying some iterative methods to solve the likelihood equation and compute the estimate $\hat{\check{\beta}}_{BN}$.

We also obtain, an approximation based on ancillary statistics. Using (9) it follows that

$$\partial l_\theta(\hat{\theta}_\beta, \beta) = \frac{r}{\hat{\theta}_\beta} - \left(\sum_{j=1}^{r}\log[1+\beta y_{(j)}] + (n-r)\log[1+\beta y_{(r)}]\right)$$

Therefore,

$$l_{\theta;y}(\hat{\theta}_\beta,\beta)=\frac{\partial l_\theta(\hat{\theta}_\beta,\beta)}{\partial y^T}=-\left[\frac{\beta}{1+\beta y_{(j)}}\right] \quad , \quad j=1,2\ \ldots.,r-1$$

$$=-\left[\frac{\beta}{1+\beta y_{(r)}}+\frac{(n-r)\beta}{1+\beta y_{(r)}}\right] \quad ,j=r$$

From (10)

$$\frac{\partial F(y_i;\hat{\Theta},\hat{\beta})}{\partial\hat{\Theta}}=\frac{(1+\hat{\beta}\ y_{(j)})^{\hat{\Theta}}\,log(1+\hat{\beta}\,y_{(j)})}{(1+\hat{\beta}\,y_{(j)})^{2\hat{\Theta}}}$$

$$,j=1,2\ \ldots.,r$$

$$f_j(y;\hat{\theta},\hat{\beta})=\frac{\hat{\beta}\hat{\theta}}{(1+\hat{\beta}y_j)^{\hat{\theta}+1}}$$

Then,

$$\hat{V}_\theta(j)=-\frac{(1+\hat{\beta}\ y_{(j)})^{\hat{\Theta}}\,log(1+\hat{\beta}\,y_{(j)})}{(1+\hat{\beta}\,y_{(j)})^{2\hat{\Theta}}}\cdot\frac{(1+\hat{\beta}\,y_{(j)})^{\hat{\theta}+1}}{\hat{\beta}\,\hat{\theta}}$$

$$=\frac{(1+\hat{\beta}\ y_{(j)})^{2\hat{\theta}+1}\,log(1+\hat{\beta}\,y_{(j)})}{(1+\hat{\beta}\,y_{(j)})^{2\hat{\Theta}}\hat{\beta}\hat{\theta}}$$

$$=-\frac{(1+\hat{\beta}\,y_{(j)})\,log(1+\hat{\beta}\ y_{(j)})}{\hat{\beta}\hat{\Theta}} \quad , \ j=1,2,\ldots.r-1$$

$$=-\frac{(1+\hat{\beta}\,y_{(r)})\,log(1+\hat{\beta}\ y_{(r)})}{\hat{\beta}\hat{\Theta}} \quad , \ j=r$$

Hence from (8)

$$\tilde{l}_{BN}(\beta)$$
$$=r\,log\,\beta+r\,log(\hat{\Theta}_\beta)-\sum_{j=1}^{r}log(1+\beta y_{(j)})-(\hat{\Theta}_\beta)\cdot\left[\sum_{j=1}^{r}log(1+\beta y_{(j)})+(n-\ r)\,log(1+\beta y_{(r)})\right]$$
$$+\frac{1}{2}log\left[\frac{\left(\sum_{j=1}^{r}log(1+\beta y_{(j)})+(n-r)\,log(1+\beta y_{(r)})\right)^2}{r}\right]$$
$$-\,log\left[\left(\sum_{j=1}^{r-1}\left(\left(\frac{\beta}{1+\beta y_{(j)}}\right)\left(\frac{(1+\hat{\beta}y_{(j)})log(1+\hat{\beta}y_{(j)})}{\hat{\beta}\hat{\Theta}}\right)\right)\right)+\left(\left(\frac{\beta}{1+\beta y_{(r)}}+\frac{(n-r)\beta}{1+\beta y_{(r)}}\right)\cdot\frac{(1+\hat{\beta}y_{(r)})\,log(1+\hat{\beta}y_{(r)})}{\hat{\beta}\hat{\Theta}}\right)\right]$$

where,

$$\hat{\Theta}_\beta=\frac{r}{\sum_{j=1}^{r}\log(1+\beta y_{(j)})+(n-r)\log(1+\beta y_{(r)})}$$

The corresponding estimator is $\hat{\hat{\beta}}_{BN}$, which will be computed numerically.

5. Simulation Study

In order to compare the performance of estimators which are obtained from the profile likelihood function and modified profile likelihood functions, a simulation study on point estimation for the Lomax scale parameter (parameter of interest) was carried out for different sample sizes, and different true parameters values of θ & β. Bias and mean square error (MSEs) are presented for all the following point estimators: $\hat{\beta}_p$, $\hat{\bar{\beta}}_{BN}$ and $\hat{\hat{\beta}}_{BN}$. $\hat{\beta}_p$ under both no and type II censoring. $\hat{\beta}_p$ denotes the profile likelihood estimator. $\hat{\bar{\beta}}_{BN}$ and $\hat{\hat{\beta}}_{BN}$ are the modified profile likelihood estimators derived from Barndorff-Nielsen's modified profile likelihood function based on an empirical covariance and an ancillary statistic approximation, respectively.

The following steps were followed:

(1) For a given value of $n,\beta,\theta,$ and r based on censored proportions 20% (failure rates of 80%) we generate a sample from the Lomax distribution.

(2) For a starting value of β (initial guess), we use the “optim” function in R to find the maximum profile likelihood estimator (MPLE) for the scale parameter (β).

(3) Using the estimate of β (found in step 2), we calculate the estimate of the shape parameter (θ).

(4) Using the estimated values of β (MLPE) and θ as starting values, we calculate the maximum modified profile likelihood estimators (adjusted MPLE).

(5) The previous steps are repeated 5000 times. The biases and the mean square errors of the estimators are computed. The results are shown in Tables 1-4.

Table 1. Bias of β for different sample sizes and true parameters value,$(\theta,\beta)=(1.0,1.0)$

Sample sizes	$\hat{\beta}_p$	$\hat{\bar{\beta}}_{BN}$	$\hat{\hat{\beta}}_{BN}$
n=50, r=40	0.0041898	−0.0273814	0.1378329
n=75, r=60	0.0180403	−0.0036065	0.1057426
n=100, r=80	0.0113555	−0.00481515	0.0755752

Table 2. Bias of β for different sample sizes and true parameters value,$(\theta,\beta)=(1.2,1.0)$

Sample sizes	$\hat{\beta}_p$	$\hat{\bar{\beta}}_{BN}$	$\hat{\hat{\beta}}_{BN}$
n=50, r=40	−0.0029896	−0.03752231	0.1447843
n=75, r=60	0.01656271	−0.00721947	0.1143707
n=100, r=80	0.01068609	−0.00722607	0.08313256

Based on the two tables above (Tabel 1 and 2) that contain bias for the following point estimators: $\hat{\beta}_p$, $\hat{\bar{\beta}}_{BN}$ and $\hat{\tilde{\beta}}_{BN}$ under type II censoring, the results show that when the proportion of censored observations is 20 % (Failure Rates of 80%), the modified profile maximum likelihood estimator based on an empirical covariance approximation ($\hat{\bar{\beta}}_{BN}$) has the smallest bias for sample sizes n=75 and n=100 and across all the true parameters values considered $(\theta, \beta) = (1.0,1.0), (1.2,1.0)$. It is also worth mentioning that as the sample size increases, the bias of the modified empirical covariance estimator $\hat{\bar{\beta}}_{BN}$ decreases.

Table 3. MSEs of β for different sample sizes and true parameters value,$(\theta, \beta) = (1.0,1.0)$

Sample sizes	$\hat{\beta}_p$	$\hat{\bar{\beta}}_{BN}$	$\hat{\tilde{\beta}}_{BN}$
n=50	1.430854	1.301745	1.969685
n=75	0.9000115	0.8466787	1.118009
n=100	0.6702818	0.6407744	0.7857691

Table 4. MSEs of β for different sample sizes and true parameters value,$(\theta, \beta) = (1.2,1.0)$

Sample sizes	$\hat{\beta}_p$	$\hat{\bar{\beta}}_{BN}$	$\hat{\tilde{\beta}}_{BN}$
n=50, r=40	1.637519	1.490294	2.27276
n=75, r=40	1.095941	1.0286	1.374207
n=100, r=80	0.7987492	0.7620716	0.9470012

From Table 3 and 4, we notice that the modified profile maximum likelihood estimator based on empirical covariance approximation ($\hat{\bar{\beta}}_{BN}$) has the smallest mean squared errors for all sample sizes and true parameters values considered. Therefore, we can conclude that based on mean squared errors, the best performing estimator under type II censoring is $\hat{\bar{\beta}}_{BN}$. It is worth noting that, under no censoring $(r = n)$, the standard profile likelihood estimator ($\hat{\beta}_p$) has the smallest bias and mean squared errors (MSEs) under the same sample sizes and true parameters values that we considered in type II censoring data.

6. Numerical Example with Real Data

In this section, we consider real-world data sets to demonstrate the proposed method and validate how our estimators perform in practice. We assume that the data is a random sample from the Lomax distribution.

Measurements of total rain volume in South Florida from cloud base following aircraft seeding penetration are included in the data. The research was based on radar-evaluated rainfall from 52 cumulus clouds in south Florida, 26 seeded clouds, and 26 control clouds. This data set was obtained from Simpson's [22] meteorological study and was further analyzed by Giles et al. [23], Helu et al. [24], and A. Baklizi et al. [9]. We obtained the profile likelihood estimator and modified profile likelihood estimators for the Lomax scale parameter using only measurements from the control group. For type II censoring, we consider subset of these measurements and impose a failure rate of 80%. The values of these estimators are shown in Table 5.

Table 5. Point estimates of the Lomax scale parameter (β)based on the real data set

Estimator	Point estimates
$\hat{\beta}_p$	0.01909989
$\hat{\bar{\beta}}_{BN}$	0.018055
$\hat{\tilde{\beta}}_{BN}$	0.02299024

It has been noticed that the point estimate obtained from the modified profile likelihood function based on an empirical covariance approximation ($\hat{\bar{\beta}}_{BN}$) is numerically smaller than the standard profile likelihood estimator ($\hat{\beta}_p$),and also, to the modified profile maximum likelihood estimator based on ancillary statistics approximation ($\hat{\tilde{\beta}}_{BN}$). This is to be expected, given that the results of the previous simulation study show the mean squared errors of this estimator ($\hat{\bar{\beta}}_{BN}$) is the least.

7. Conclusion

The Barndorff-Nielsen modified profile likelihood function, which is based on empirical covariance and an ancillary statistic approximation, is used to modify the standard maximum profile likelihood estimator for the Lomax scale parameter (parameter of interest). The numerical results of the simulation study under type II censoring show that the modified profile maximum likelihood estimator based on empirical covariance approximation outperforms not only the standard maximum profile likelihood estimator, but also the modified profile maximum likelihood estimator based on ancillary statistics approximation. It has almost smallest bias and always has the lowest mean squared errors for all sample sizes considered (n=50,75, and 100) with 20% censored observations and across all true values of the parameters considered. When there is no censoring in the data, the standard profile maximum likelihood estimator performs the best because it has the smallest bias and mean squared errors for the same sample sizes and true values of parameters that we considered when there is type II censoring.

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