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Research Article

A Simple Conservative Chaotic Oscillator with Line of Equilibria: Bifurcation Plot, Basin Analysis, and Multistability

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Here, a novel conservative chaotic oscillator is presented. Various dynamics of the oscillator are examined. Studying the dynamical properties of the oscillator reveals its unique behaviors. The oscillator is multistable with symmetric dynamics. Equilibrium points of the oscillator are investigated. Bifurcations, Lyapunov exponents (LEs), and the Poincare section of the oscillator's dynamics are analyzed. Also, the oscillator is investigated from the viewpoint of initial conditions. The study results show that the oscillator is conservative and has no dissipation. It also has various dynamics, such as equilibrium point and chaos. The stability analysis of equilibrium points shows there are both stable and unstable fixed points.

1. Introduction

Chaotic dynamics have been an interesting topic for many years. The chaotic Lorenz oscillator was proposed to model the atmosphere in 1963 [1]. There was a hypothesis that the chaotic attractors are associated with saddle equilibria. So, many well-known chaotic oscillators contain a saddle equilibrium [2, 3]. In 2011, Wei proposed a chaotic oscillator without an equilibrium point [4]. Also, in 2012, Wang and Chen presented a novel chaotic oscillator with one stable equilibrium [5]. These works have shown that a flow can have chaotic dynamics with any equilibria or without it [6, 7]. Many studies have discussed chaotic oscillators with lines of equilibria [8, 9]. Lyapunov exponent is a valuable measure in the study of chaos, and its calculation has been a hot topic [10].

Many research studies have been focused on proposing chaotic flows with various features [11–13]. Multiscroll dynamics were discussed in [14]. Hyperchaotic oscillators are attractive because of their complex dynamics [15]. The hyperchaotic dynamics are efficient in secure communications so that it is not possible to retrieve hidden messages [16]. These oscillators also have two positive LEs and high sensitivity to initial values [17]. In [18], a hyperchaotic oscillator with no equilibria was studied. Oscillators with self-excited and hidden attractors were discussed in [19, 20]. In [21], hidden dynamics in a piecewise linear oscillator were studied. Synchronization and control of chaotic flows have attracted much attention [22]. Multistability is a significant feature of dynamical systems [23]. The final circumstance of a multistable oscillator is determined by initial conditions in a constant set of parameters [24]. In case of undesirable multistability, the final state can be controlled by selecting the proper parameters to transform to monostability [25]. The multistability of a piecewise linear oscillator with various types of attractors was investigated in [26]. A multistable chaotic oscillator with different dynamics was proposed in [27]. In [28], a megastable system with a particular term was studied. A megastable system and its dynamics were discussed in [29]. The application of chaotic dynamics in encryption [30], secure communication [31], and robotics [32] has been a hot topic.

Chaotic dynamics can be classified as dissipative or conservative ones. In the dissipative dynamics, the phase space volume approaches zero when time goes to infinity; however, in conservative dynamics, volume is constant by changing time [33]. Conservative oscillators can conserve energy and display chaos without damping (although energy is not conserved in real mechanics) [33]. Studying conservative chaotic oscillators has been a hot topic [34]. Conservative dynamics of a chaotic oscillator based on Sprott A were discussed in [35, 36]. In [37], various groups of conservative oscillators were studied. The conservative flows were categorized into four groups based on their dissipation [37].

Here, a novel conservative oscillator with a line of equilibria is presented. The oscillator is a simple one with five terms. The oscillator is symmetric, which shows two symmetric coexisting dynamics. From the mathematical and computational point of view, there has been a noticeable interest in chaotic systems with special features [1–3]. The proposed chaotic system has some important features:

(a) It is conservative, and 3D conservative chaotic systems are rare [4–6]

(b) It has a line of equilibria, and 3D chaotic systems with a line of equilibria are rare [7–9]

(c) It is multistable, making the system more interesting in some applications [10]

The proposed system has all the features above, and we are aware of no other system with those properties combined. As far as we know, there is only one 4D conservative chaotic system with a line of equilibria in [11]. Therefore, the system itself is interesting and deserves attention. In Section 2, the oscillator is proposed, and its chaotic sea is discussed. Also, the symmetry of the oscillator and its line of equilibria are studied. In Section 3, various dynamics of the oscillator are discussed. The Poincare section, bifurcation diagram, Lyapunov exponents (LEs), and initial conditions are studied. In Section 4, the results are concluded.

2. The Conservative Oscillator

Here, a conservative oscillator is proposed as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ax + yz, \\ \dot{z} &= x^2 - by^2. \end{aligned} \tag{1}$$

The system is a three-dimensional quadratic oscillator containing five terms; thus, it can be considered a simple chaotic oscillator. The oscillator shows a conservative chaotic dynamic in $a = 1, b = 0.68$, and $(x_0, y_0, z_0) = (-0.57, -0.99, -0.71)$. The time series and state space of the chaotic sea are plotted in Figure 1. The LEs of the oscillator in the mentioned parameters are calculated with a run time of 120000. The LEs are obtained as $(0.0055, 0, -0.0055)$; therefore, the Kaplan–Yorke dimension (D_{KY}) is 3. The chaotic dynamic is conservative since D_{KY} is 3 and the sum of LEs is zero. As a result, the oscillator has no dissipation; i.e., energy is conserved when time goes to infinity.

Considering the equations of the oscillator, the system has symmetry by changing (x, y, z) to $(-x, -y, z)$. The two coexisting conservative dynamics are plotted in Figure 2 with two initial conditions $(-0.57, -0.99, -0.71)$ and $(0.57, 0.99, -0.71)$. The ranges of the two chaotic seas show that the two chaotic seas are entangled with each other. In other words, the variable intervals of the two multistable chaotic dynamics are approximately the same.

Calculating the equilibria of the oscillator reveals that it has a line of equilibria at $x = 0, y = 0$. To analyze the equilibria's stability, the Jacobian matrix and eigenvalues of the oscillator are calculated as equations (2)–(4) by considering parameters $a = 1$ and $b = 0.68$:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & \mathrm{z} & y \\ 2x & -1.36\mathrm{y} & 0 \end{bmatrix}, \tag{2}$$

$$|\lambda I - J| = \begin{vmatrix} \lambda & -1 & 0 \\ 1 & \lambda - z & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0, \tag{3}$$

$$\lambda\left(\lambda^2 - z\lambda + 1\right) = 0. \tag{4}$$

Eigenvalues are obtained as $\lambda_1 = 0$ and $\lambda_{2,3} = z \pm \sqrt{z^2 - 4}/2$. So, the stability of equilibrium points is dependent on their location of z. The real and imaginary parts of eigenvalues by changing z are shown in Figure 3. In positive z, the real parts of λ_2 and λ_3 are positive. So the equilibrium points with positive z are unstable. In negative z, there is not any positive eigenvalue. However, the type of equilibria cannot be determined in that interval using the eigenvalues. Numerical analysis shows that the equilibrium points are stable in negative z. The imaginary part of the eigenvalues reveals that the equilibria are spirals in the interval $z \in [-2, 2]$.

3. Dynamical Behavior

Various methods can be used to show a three-dimensional chaotic flow in two dimensions. One of the useful methods is the Poincare section [33]. The Poincare section of the oscillator with $a = 1, b = 0.68$, and $(x_0, y_0, z_0) = (-0.57 - 0.99 - 0.71)$ by selecting a section $= \{(x, y) \in R^2 | z = 0\}$ is presented in Figure 4. The values of the Poincare section are the values of trajectories crossing the plane $z = 0$. The symmetry of the oscillator's dynamics can be seen in this figure.

Bifurcation diagram is one important method to investigate various dynamics of chaotic oscillators. The bifurcation diagram of the oscillator by varying parameter a in the interval $[1, 1.28]$ is plotted in Figure 5(a). The bifurcation values are obtained using the maximum value of the x

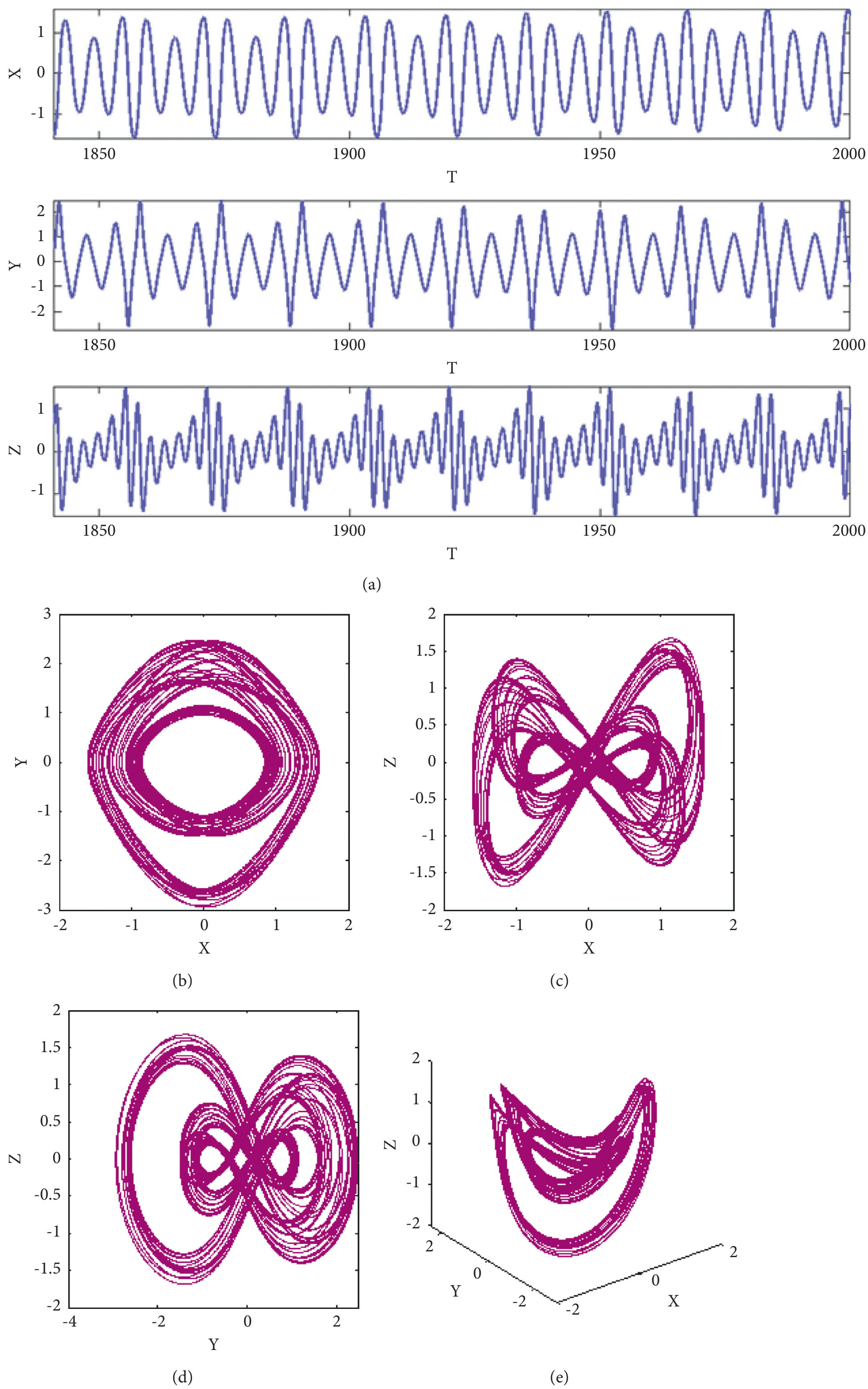


Figure 1: Time series (a) and chaotic sea with initial condition $(x0, y0, z0) = (-0.57 - 0.99 - 0.71)$ and parameters $(a, b) = (1, 0.68)$ in (b)X_Y, (c)$X - Z$, (d)$Y - Z$, and (e)X_Y_Z.

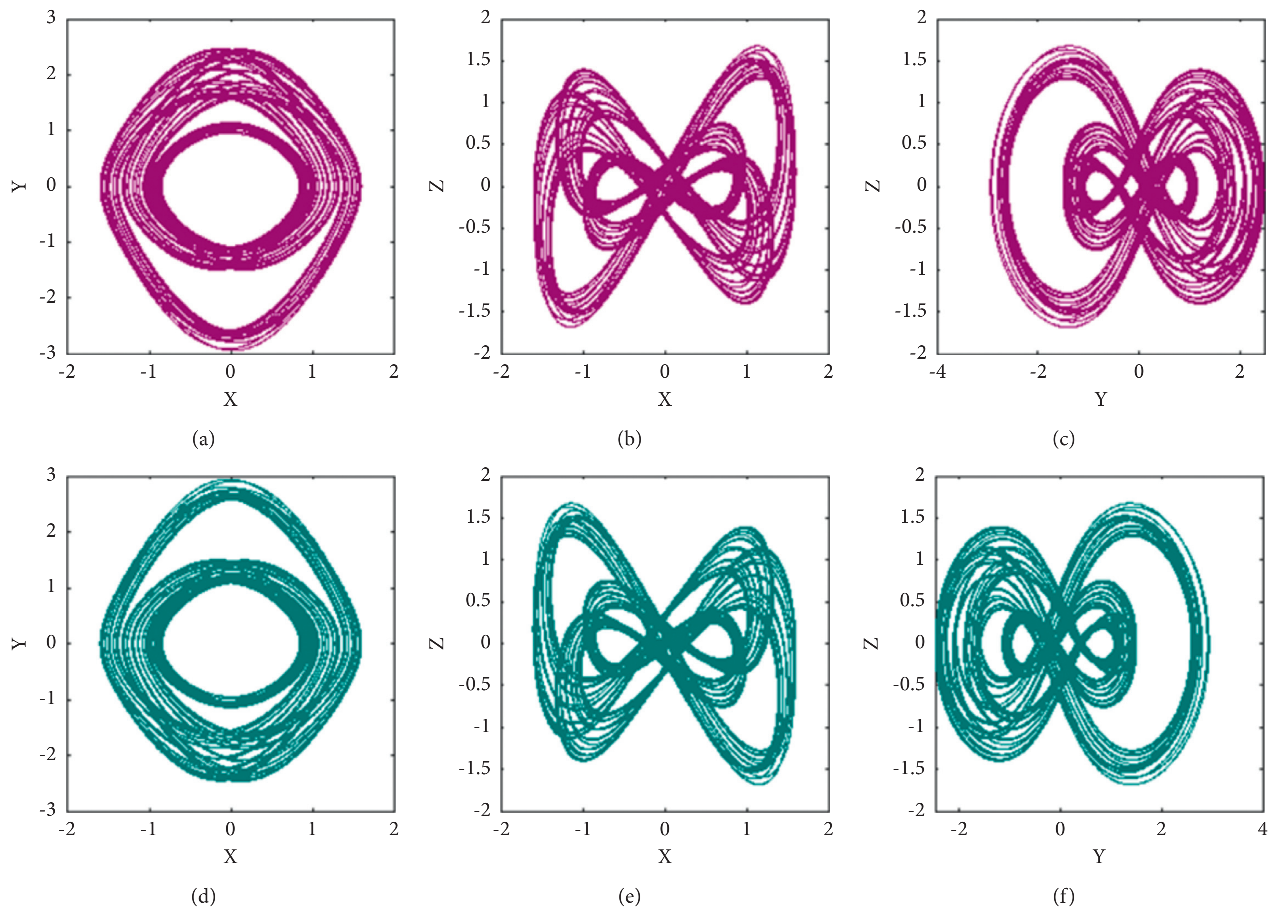


Figure 2: Chaotic sea of the oscillator with initial conditions $(-0.57, -0.99, -0.71)$, (a) in $X - Y$, (b) in $X - Z$, and (c) in $Y - Z$, and with initial conditions $(0.57, 0.99, -0.71)$, (d) in $X - Y$, (e) in $X - Z$, and (f) in $Y - Z$. The two dynamics are symmetric.

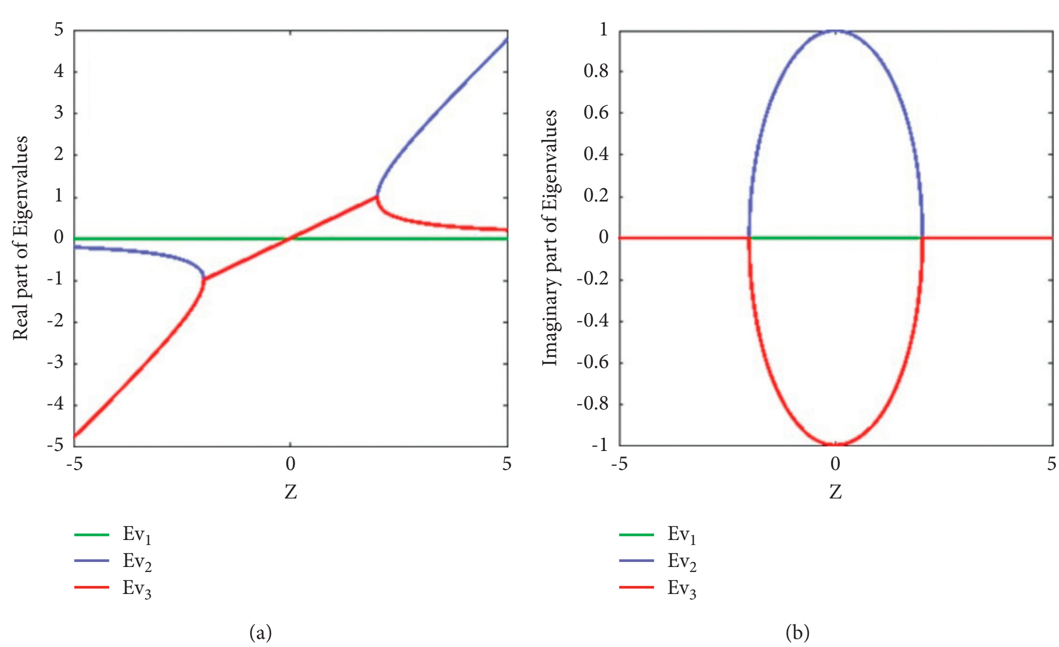


Figure 3: (a) Real and (b) imaginary part of eigenvalues for oscillator (1) with parameters $(a, b) = (1, 0.68)$.

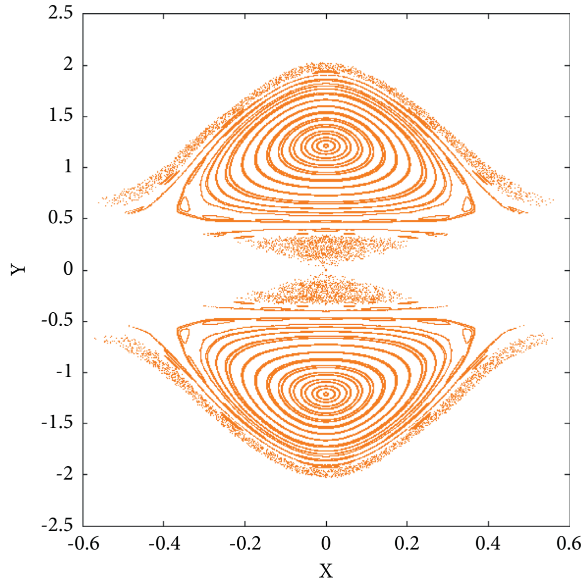
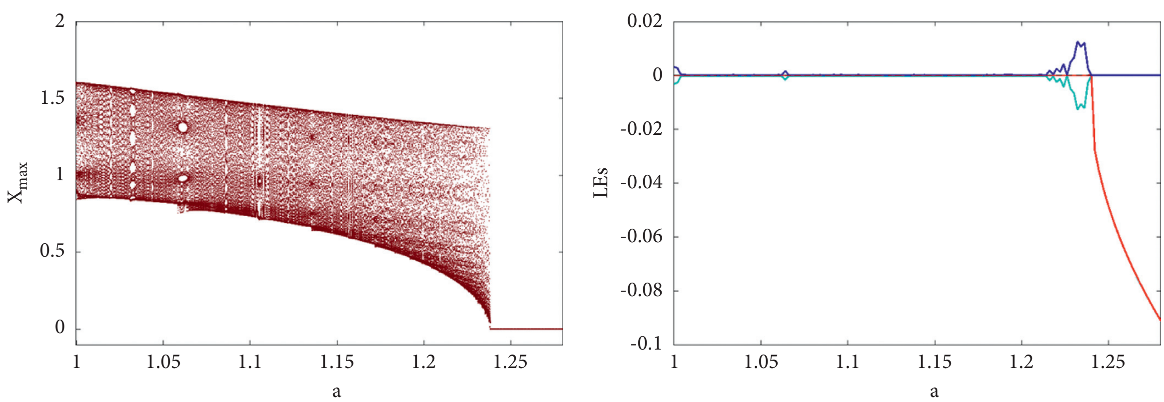
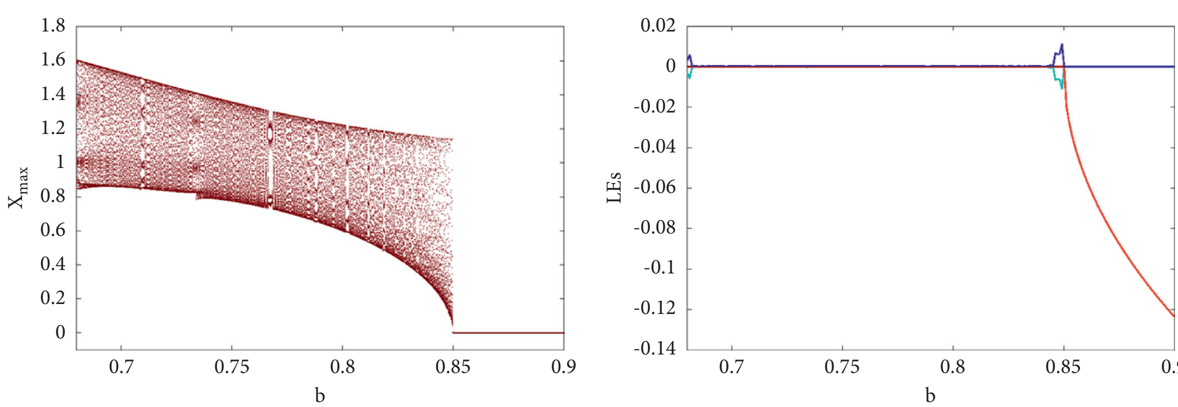


FIGURE 4: The Poincare section with crossing section.



(a) (b)

FIGURE 5: (a) Bifurcation diagram and (b) LEs of oscillator (1) by changing parameter a, with constant initial conditions $(x_0, y_0, z_0) = (-0.57, -0.99, -0.71)$.



(a) (b)

FIGURE 6: (a) Bifurcation diagram and (b) LEs of oscillator (1) by changing parameter b with constant initial conditions $(x_0, y_0, z_0) = (-0.57, -0.99, -0.71)$.

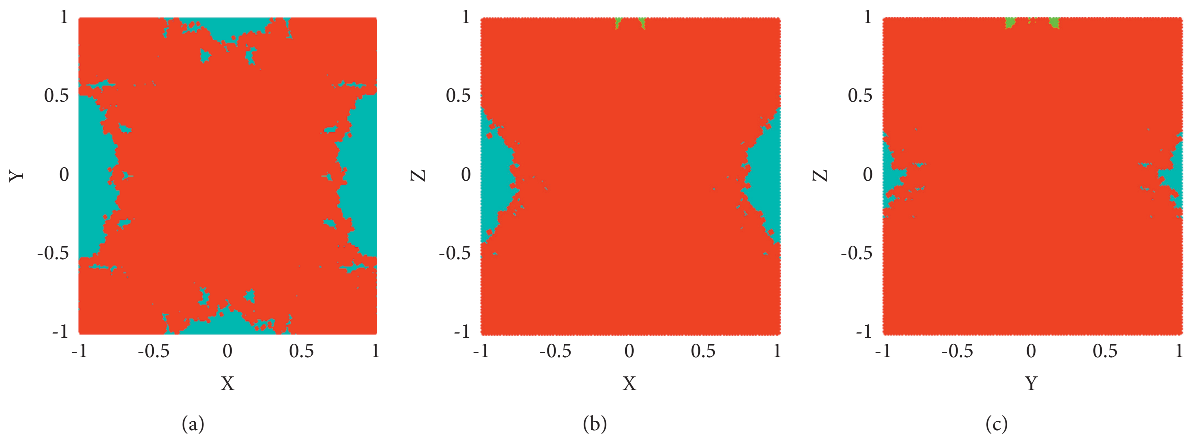


FIGURE 7: Initial values for various dynamics of the proposed oscillator in (a) X-Y plane, (b) X-Z plane, and (c) Y-Z plane. The oscillatory region is cyan, the equilibrium points are red, and the unbounded points are green.

variable in (1) with $b = 0.68$. The other variables show the same dynamics, so only $x_{\max}$ is presented. To analyze another parameter, $x_{\max}$ is plotted by changing b in the interval $[0.68, 0.9]$ with parameter $a = 1$ in Figure 6(a). The initial conditions for both diagrams are considered constant values as $(x_0, y_0, z_0) = (-0.57, -0.99, -0.71)$. The corresponding LEs by changing the parameters a and b are shown in Figures 5(b) and 6(b), respectively. The LEs are calculated with run time 50000 using the wolf method [10]. According to the bifurcation diagram and LEs, various dynamics can be seen by changing parameters. In $a = 1.239$ and $b = 0.85$, there is a sudden change in the dynamics of the oscillator in which the chaotic dynamics collapse.

Basin of attraction reveals the initial conditions that are attracted to various attractors of an oscillator. Figure 7 shows the initial values in the region of different dynamics. In Figure 7(a), the initial values of x and y variables are changing in the interval $x \in [-1, 1]$, $y \in [-1, 1]$. The initial value of the variable z is constant and is equal to zero. In addition, initial values in X-Z and Y-Z planes are investigated where $y_0$andx_0 are equal to zero, respectively. Cyan color demonstrates the oscillatory region while red color shows equilibrium points. Moreover, there are unbounded points that are shown with green color. The system has symmetry by changing (x, y, z) to $(-x, -y, z)$. The symmetric regions are consistent with the oscillator's symmetry, which was also seen in the Poincare section.

4. Conclusion

Here, an oscillator with conservative chaotic dynamics was proposed. By calculating LEs and D_{KY}, the dissipation of the oscillator was investigated. The summation of LEs was equal to zero, and D_{KY} was 3, which presented the conservative dynamics of the oscillator. The oscillator had a line of equilibria. Half of the line was unstable, while the other half was stable. The oscillator was symmetric. This symmetry reveals that the oscillator had a pair of the conservative chaotic sea. The symmetry of the oscillator's dynamics also was seen in the Poincare section. The bifurcation diagram of the oscillator was studied by changing parameters. The diagrams have shown the collapse of the chaotic dynamics by changing the bifurcation parameter. The dynamics of the bifurcation diagram were proved by LEs. The initial conditions that have resulted in various dynamics of the oscillator were also discussed. This study has revealed the uniqueness of the proposed conservative oscillator.

Data Availability

All the data used for the numerical analysis are mentioned within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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