Hidden and Self-Excited Collective Dynamics of a New Multistable Hyper-Jerk System with Unique Equilibrium

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Nonlinear dynamical systems with hidden attractors belong to a recent and hot area of research. Such systems can exist in different forms, such as without equilibrium or with a stable equilibrium point. This paper focuses on the dynamics of a new 4D chaotic hyper-jerk system with a unique equilibrium point. It is shown that the new hyper-jerk system effectively exhibits different hidden behaviors, which are hidden point attractor, hidden periodic attractor, and hidden chaotic state. Collective behaviors of the system are studied in terms of the equilibrium point, bifurcation diagrams, phase portraits, frequency spectra, and two-parameter Lyapunov exponents. Some remarkable and exciting properties are found in the new snap system, such as period-doubling transition, asymmetric bubbles, and coexisting bifurcations. Also, we demonstrate that it is possible to generate different varieties of two, three, four, or five coexisting hidden and self-excited attractors in the introduced model. In addition, the amplitude and offset of the hidden

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chaotic attractors are perfectly controlled for possible application in engineering. Furthermore, a circuit design has been implemented to support the physical feasibility of the proposed model.

**Keywords**: Hyper-jerk system; stable equilibrium; hidden attractor; multistability; control.

### 1. Introduction

Nonlinear problems are of interest to mathematicians, physicists, and other scientists because most physical systems are nonlinear. Complex phenomena, including chaos, can be observed when modeling some natural physical systems, such as a Hybrid Electric Vehicle (HEV) [Meli et al. 2013], Tamang Meli et al. (2021) or electrical hair dryers [Hu et al. 2018a; Hu et al. 2018b]. Nana et al. (2018) just to name a few. In artificial neural networks, for example, nonlinearities can be deliberately introduced to allow certain calculations. Under certain conditions, nonlinear systems exhibit sensitivity to initial values (IC), intermittency, and chaos [Strogatz 1994]. In a nonlinear dynamical system, chaotic behavior is the fingerprint of at least one positive Lyapunov exponent. Such behavior has been found in several categories of systems, including physical, biological, ecological systems, electronic circuits, mechanical systems, and so on [Strogatz 1994; Bao et al. 2020a]. Notice that although most chaotic systems are of hyperbolic type, there are still many that are not. In other words, it will be surprising to show that systems with only one stable equilibrium exhibit chaos. For this particular class of system, one would almost certainly expects asymptotically convergent behaviors. In this regard, Wang and Chen (2012) presented a surprising possibility of finding autonomous quadratic chaotic systems with only one stable equilibrium. Since this discovery, other interesting chaotic systems with one stable equilibrium have been reported and investigated [Pham et al., 2017]. Such systems from the viewpoint of computation have been classified as systems with hidden attractors. From a computational perspective, systems with stable equilibrium have been classified as systems with hidden attractors [Leonov & Kuznetsov, 2014]. Remember that if the basin of attraction for the attractor does not intersect with small neighborhoods of any equilibrium points, then that strange attractor is hidden. Otherwise, it is self-excited [Leonov et al., 2015]. This topic of chaotic systems with hidden strange attractors has been the subject of various discussions and research in the scientific community. Other chaotic systems in which hidden strange attractors occur are systems with infinite equilibria, stable or no equilibria, as discussed in [Yu et al., 2019; Zhang et al., 2019; Chen et al., 2021]. Also, systems with multistability [Bao et al., 2020b; megastability (Giakoumis et al., 2020; Safari et al., 2019; He et al., 2019; and extreme multistability [Chen et al., 2018] are of noticeable interest.

In a mechanical system, the derivative of the position gives velocity, the second derivative of the position gives acceleration, and the derivative of acceleration gives jerk. This is why jerk systems [Malasoma 2004] are expressed as $\dddot{\textbf{r}} = J(\dot{\textbf{r}}, \ddot{\textbf{r}}, \textbf{r})$, where $J = (\cdot)$ is called the “jerk function”. For such jerk systems, several have already been reported in the literature, and the results have indicated some very rich and striking phenomena, including symmetric properties, Hopf bifurcation, symmetry-breaking, symmetry bubbles, period-doubling scenario, the coexistence of multiple bifurcations mode, and existence of multiple attractors [Kengne et al., 2020]. Thus in jerk systems, an extensive repertoire of nonlinear behavior is recorded. In the literature [Leutcho et al., 2018; Munmuangsaen & Srisuchinwong, 2011], the fourth time derivative $d^4x/dt^4$ is called a “snap”. Hyper-jerk systems are classified as high-order dynamical systems but have the mathematical distinction of having a “simple” and “elegant” structure. For these reasons, new hyper-jerk systems/circuits capable of generating both series and parallel bubbles of bifurcation are introduced and analyzed in recent years [Leutcho et al., 2020; Leutcho et al., 2020]. The authors generated such complicated behaviors (i.e. coexisting bubbles, bursting oscillations, and so on) by introducing exponential nonlinear terms into the mathematical model. In 2018, Ren et al. (2018) introduced the first hyper-jerk chaotic system with nonlinear quadratic terms and having hidden dynamics. They exploited well-known dynamical tools to investigate its dynamics in its fractional order. Very recently,
2. The 4D Hyper-Jerk System

2.1. Model description

In this paper, the mathematical model of a new 4D autonomous hyper-jerk system is introduced. The model is expressed by the differential Eq. (1)

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= w, \\
\dot{w} &= -ax - by - cw - dy^2 + exz - f,
\end{align*}
\]  

where \(x, y, z, \) and \(w\) are the state variables; \(a, b, c, \) \(d, \) \(e, \) and \(f\) are six positive real constants with one bias term, which is \(f\). It is clear that the new model possesses only two quadratic nonlinear terms, which can be implemented easily with analog multiplier chips. Furthermore, no apparent symmetry can be observed from Eq. (1) despite the presence of quadratic nonlinearities since the model is variable under the substitution \((x, y, z, w) \rightarrow (-x, -y, -z, -w)\). More importantly, the new hyper-jerk oscillator Eq. (1) can be considered in the general hyper-jerk form as follows:

\[
\frac{d^4x}{dt^4} = -ax - bx + c\left(\frac{dx}{dt}\right)^2 + exz - f.  
\]  

(2)

2.2. Dissipation

Dissipation (i.e., volume contraction) is a common property in nonlinear dynamical systems. This dissipation is used to show if the phase space is conserved or not. An attractor can be characterized in nonlinear systems by calculating its dissipation. That is, if and only if this dissipation is negative. In this section, the dissipation is calculated using the definition as

\[
\Lambda = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial w^2} = -c.  
\]  

(3)

We clearly remark that the volume contraction is negative since coefficient \(c\) is positive. Thus, the new hyper-jerk system with its unique fixed point is dissipative and presents attractors.

2.3. Equilibrium points and stability

Let \(\dot{x} = \dot{y} = \dot{z} = \dot{w} = 0\) then, the equilibrium points of Eq. (1) is calculated as:

\[
E(\{x^*, y^*, z^*, w^*\}) | x^* = \frac{-f}{a}; y^* = z^* = w^* = 0. \]  

Thus the hyper-jerk
where the hyper-jerk system under consideration. We have provided Table 2 showing various behaviors with a stable equilibrium point. Based on Eq. (5), the system belongs to the category of hidden attractors if and only if the following conditions are satisfied according to the Routh–Hurwitz stability method:

\[ \lambda^4 + c\lambda^3 + \frac{f}{a} \lambda^2 + b\lambda + a = 0. \]  

Consider the following polynomial:

\[ p^4 + m_3p^3 + m_2p^2 + m_1p + m_0 = 0. \]  

All the roots of the real parts of Eq. (6) are negative coefficients condition of stability of the characteristic eigenvalues, which can be calculated as

\[ \lambda_i \] 

The table gives the values of the 2nd and 4th roots of the characteristic equation.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \lambda_{1,2} )</th>
<th>( \lambda_{1,4} )</th>
<th>Nature</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>(-0.2643 \pm 0.8560)</td>
<td>(-0.0057 \pm 0.0045)</td>
<td>Stable saddle-focus</td>
</tr>
<tr>
<td>0.52</td>
<td>(-0.2415 \pm 0.2451)</td>
<td>(-0.0285 \pm 0.0239)</td>
<td>Stable saddle-focus</td>
</tr>
<tr>
<td>1.0</td>
<td>(-0.2204 \pm 0.0954)</td>
<td>(-0.0496 \pm 0.0472)</td>
<td>Stable saddle-focus</td>
</tr>
<tr>
<td>1.45</td>
<td>(-0.2193 \pm 0.1643)</td>
<td>(-0.0561 \pm 0.0734)</td>
<td>Stable saddle-focus</td>
</tr>
<tr>
<td>1.6</td>
<td>(-0.2357 \pm 0.1422)</td>
<td>(-0.0495 \pm 0.0897)</td>
<td>Stable saddle-focus</td>
</tr>
<tr>
<td>1.8</td>
<td>(-0.3469 \pm 0.2128)</td>
<td>0.0769 \pm 0.0908\</td>
<td>Unstable saddle-focus</td>
</tr>
<tr>
<td>3.5</td>
<td>(-0.9226 \pm 0.8844)</td>
<td>0.6526 \pm 1.1405\</td>
<td>Unstable saddle-focus</td>
</tr>
</tbody>
</table>

Table 2. Eigenvalues \( \lambda_j \) \( j = 1, 2, 3, 4 \) of equilibrium point \( E(-f/a, 0, 0, 0) \) for \( b = 0.54 \), \( f = 2.39 \), \( e = 0.54 \), \( d = 1.07 \), \( e = 1.97 \).

3. Nonlinear Dynamics of the Oscillator

3.1. Bifurcation analysis and route to chaos

The dynamics of the new hyper-jerk system are numerically analyzed in this section using a fourth-order Runge–Kutta integrator with a fixed time grid \( \Delta t = 3 \times 10^{-3} \). The diagrams of bifurcation and graphs of Lyapunov exponents are two main tools that are presented to gain more information about different behavior to chaos in the new 4D chaotic hyper-jerk system when parameters are changed. The bifurcation diagram of the system represents the plot of the local maxima of the variable \( x \) or \( y \) in terms of the control parameter \( a \) and \( f \). At the same time, the graphs of Lyapunov exponents are obtained using the algorithm proposed by Wolf and co-workers [Wolf et al. 1985]. Sample results are provided in Figs. 1 and 2. In particular, Fig. 1 shows different paths related to the system’s dynamics, considering the variation of the parameter \( a \) in the range 1 \( \leq a \leq 2 \). Upward direction (blue) and downward direction (red) with the IC (0, 0, 0, 0, 0) for \( b = 0.54 \), \( f = 2.39 \), \( e = 0.54 \), \( d = 1.07 \), \( e = 1.97 \).

![Fig. 1. Bifurcation diagrams of w versus a of system (blue) plotted in the region 1 \( \leq a \leq 2 \). Upward direction (blue) and downward direction (red) with the IC (0, 0, 0, 0, 0) for b = 0.54, f = 2.39, e = 0.54, d = 1.07, e = 1.97.](image-url)
interval $1 \leq a \leq 2$. The graph in blue and red correspond to increasing and decreasing $\alpha$, respectively. In Fig. 2 the bifurcation diagrams of the variable $w$ versus $f$ are obtained using three methods. The blue and black diagrams correspond to the upward and downward directions, respectively, while the graph in red is obtained with fixed initial points.

Similarly, Fig. 3(a) shows a bifurcation graph of the state variable $x$ when varying $\alpha$ in the range $1.5 \leq \alpha \leq 1.565$ and the corresponding graph of Lyapunov exponents [see Fig. 3(b)]. From Fig. 3 it is observed that the new hyper-jerk system develops a reverse period-doubling transition to chaos when parameter $\alpha$ slowly increased or decreased in the range $1.5 \leq \alpha \leq 1.565$. Figure 4 demonstrates very well the transitions of the diagram provided in Fig. 3. That is, (a) period-1 limit cycle is plotted at $\alpha = 1.5$, (b) period-2 limit cycle at $\alpha = 1.53$, (c) period-4 limit cycle is shown for $\alpha = 1.537$, (d) a strange hidden chaotic attractor is observed for $\alpha = 1.55$, (e) period-4 limit cycle is obtained again for $\alpha = 1.561$ and (f) period-2 attractor at $\alpha = 1.565$. This sequence of behaviors found in the unique hyper-jerk system with a stable equilibrium also demonstrates the interesting path to chaos observed in many chaotic systems reported in the literature.

### 3.2. Two-parameter Lyapunov exponents (LE)

This section aims to investigate the general dynamics of the new hyper-jerk system \textcolor{blue}{(4)}. To gain more perception of the dynamics features and the performance of Eq. (4), two-parameter Lyapunov Exponents (LE) is provided. It is possible that the performance of the nonlinear systems also depends on the two-parameter LE diagram. The general dynamics behavior of the system is important for further application in engineering. The two-parameter graph also gives a possibility to use the oscillator in specific regions when parameters are changed simultaneously. In Rech \textit{et al.} [2017], Stegemann \textit{et al.} [2011], two-parameter LE is presented for chaotic or hyperchaotic systems to summarize their complex dynamical behavior. Two-parameter Lyapunov exponent is used to define a region of the possible existence of chaos or hyperchaos in the parameter space when two control bifurcations are changed simultaneously. In some situations, a two-parameter LE diagram
Fig. 4. Phase portrait of system (1) showing PD route to chaos by varying $\alpha$:
(a) Period-1 limit cycle for $\alpha = 1.5$, (b) period-2 limit cycle for $\alpha = 1.53$, (c) period-4 limit cycle for $\alpha = 1.537$, (d) strange hidden chaotic attractor for $\alpha = 1.55$, (e) period-4 limit attractor for $\alpha = 1.561$, and (f) period-2 limit cycle for $\alpha = 1.565$. 
Fig. 5. Two-parameter LE diagrams showing different oscillation modes (chaotic or periodic modes) of the hyper-jerk system (1) in (a–b) plane when \( f = 2.39, c = 0.54, d = 1.07, e = 1.97 \) in (a–d) plane when \( b = 0.54, f = 2.39, c = 0.54, e = 1.97 \) in (a–e) plane with \( b = 0.54, f = 2.39, c = 0.54, d = 1.07 \) in (f–a) plane for \( b = 0.54, c = 0.54, d = 1.07, e = 1.97 \) in (f–b) plane for \( a = 1.55, c = 0.54, d = 1.07, e = 1.97 \) and in (f–e) plane for \( b = 0.54, a = 1.55, c = 0.54, d = 1.07 \). Initial point is \((0.1, 0, 0, 0)\).

is exploited to demonstrate different regions of coexisting attractors and coexisting bifurcations in the parameter space, as reported in Negou & Kengne, 2019. The two-parameter Lyapunov exponent (LE) defined in this paper present the general oscillation modes of the new hyper-jerk system when adjusting two control parameters. Indeed, two-parameter Lyapunov exponents (LE) are drawn by simultaneously adjusting two control parameters of the 4D system through the establishment of appropriate colorful diagrams. The colorful graphs are obtained by numerically calculating the Lyapunov exponent spectrum using the algorithm proposed by the Wolf method Wolf et al. 1985, on a grid.
time of 400 $\times$ 400 values of the chosen parameters. However, the two-parameter bifurcation diagram is directly associated with the two-parameter LE since it allows to distinguish the different oscillation zones such as periodic oscillations (i.e. negative LE) and chaotic oscillations (i.e. positive LE). In this paper, we present two-parameter Lyapunov exponent diagrams that help us find all these dynamic behaviors in the introduced hyper-jerk system. The dynamics of the introduced hidden oscillator are detailed as shown in Fig. 3 in different parameters space of Eq. (1). It can be seen from these graphs that when two control parameters are varied, chaotic and, regular oscillation modes can be distinguished in terms of the calculated LE value. For example, blue or Cyan shows negative or zero LE; a continuously changing green-yellow scale represents positive LE while the most positive LE is shown by red color.

### 3.3. Coexisting bifurcation diagrams and multistability

This part’s main objective is to show the presence of a multitude of coexisting bifurcations and thus coexisting attractors in the new hyper-jerk system. The coexistence of many attractors strongly depends on the choice of initial conditions and parameter space. In addition, the coexistence of several solutions and thus multistability is carried out using bifurcation diagrams defined by sweeping the control parameters (i.e. upward and backward continuation). Clearly, exploiting the two-parameter diagrams presented in Fig. 4 for a discrete value of $b = 0.54$, $f = 2.39$, $c = 0.54$, $d = 1.07$, $e = 1.97$, and varying $a$ in the appropriate range, the coexisting bifurcation diagrams are computed. These diagrams are obtained using the well-known continuation technique [Meli et al., 2021; Tametang Meli et al., 2021], where the final state of each iteration is used as the initial condition for the next iteration. Sample results show coexisting bifurcation of the new hyper-jerk oscillator as depicted in Figs. 6(a) and 6(b). This figure represents a zoom of the bifurcation diagram of Fig. 1 in the range $1.523 \leq a \leq 1.5325$. Each graph corresponds to increasing the control parameter $a$ (see Fig. 6). These superimposed branches in Figs. 6(a) and 6(b) justify the coexistence of four different attractors in phase space, as shown in Figs. 7(a) and 7(b). That is a period-1 limit cycle attractor; a period-2 limit cycle and a strange chaotic attractor coexist with point attractor, as shown in Fig. 8(a).

![Fig. 6. Coexisting bifurcation diagrams obtained in the same regions of parameter $a$ ($1.523 \leq a \leq 1.5325$) plotted in the upward direction with different ICs for $b = 0.54$, $f = 2.39$, $c = 0.54$, $d = 1.07$, $e = 1.97$.](image)

In Fig. 6(b), two asymmetric period-2 limit cycles and an asymmetric strange attractor coexist with a point attractor.

Other zooms (see Figs. 8 and 10) of the bifurcation diagram of Fig. 6 are shown in Figs. 8 and 10, highlighting different dynamics of the hyper-jerk system. Up to four bifurcation branches coexist and can be used to explain different groups of coexisting attractors in the new oscillator. From the graphs in Fig. 8, four data correspond respectively to increase parameter $a$ (see Table 3 for more details). Thus, the coexisting branches of Fig. 8 help to demonstrate the coexistence of many asymmetric attractors (up to five asymmetric attractors) in the new oscillator (see Fig. 8). It can be seen in Fig. 8(a) that two strange chaotic attractors, a period-1 limit cycle and a period-2 limit cycle, coexist with a point
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Fig. 7. Coexistence of different attractors in system (1): (a) Period-1 attractor, period-2 attractor, chaotic attractor and point attractor for \( a = 1.523 \). IC1 \((-1.83, 0.2, 0.2, 0.5)\), IC2 \((-0.66, 0.2, 0.2, 0.5)\), IC3 \((-0.75, 0.2, 0.2, 0.5)\) and IC4 \((-2.5, 0.2, 0.2, 0.5)\) and (b) two period-2 attractors, a chaotic attractor and a point attractor for \( a = 1.529 \) with IC1 \((-0.75, 0.2, 0.2, 0.5)\), IC2 \((-0.66, 0.2, 0.2, 0.5)\), IC3 \((-1.53, 0.2, 0.2, 0.5)\), and IC4 \((-2.5, 0.2, 0.2, 0.5)\) respectively.

Similarly, Fig. 8 shows that coexisting branches can be exploited to justify the coexistence of four different asymmetric attractors without fixed-point attractors in the model. The method used to gain each diagram is summarized in Table 3 for more exploitation. These coexisting attractors can be periodic or chaotic. For example, by selecting four random initial values, two different asymmetric period-1 limit cycles coexist with two distinct strange attractors, as shown in Fig. 11(a). The situation where three different hidden chaotic attractors coexist with the period-1 limit cycle is highlighted in Fig. 11(b).

By using the same strategies, it is possible to track other coexisting branches and coexisting solutions. A strict analysis of the proposed hyper-jerk system, based on Table 3, reveals other regions where parallel bifurcations coexist. The results of this analysis are shown in Fig. 12–14. In particular, the coexistence of several parallel bifurcation branches with hysteresis can be visualized in Fig. 12 (see Table 3 for more details). All these
Fig. 9. Coexistence of five different attractors with different shape including: (a) Period-1 attractor, period-2 attractor, two chaotic attractors and point attractor for $a = 1.538$ with ICs $(-0.75, 0.2, 0.2, 0.5), (-1.8, 0.2, 0.2, 0.5), (-0.9, 0.2, 0.2, 0.5), (-1.56, 0.2, 0.2, 0.5)$, and $(-2.5, 0.2, 0.2, 0.5)$ and (b) period-2 attractor, period-4 attractors, two asymmetric chaotic attractors and point attractor for $a = 1.543$ using ICs $(-1.68, 0.2, 0.2, 0.5), (-1.98, 0.2, 0.2, 0.5), (-0.48, 0.2, 0.2, 0.5)$, $(-1.29, 0.2, 0.2, 0.5)$, and $(-2.5, 0.2, 0.2, 0.5)$, respectively.

Fig. 10. Zoom diagram of Fig. 9 in the range $1.577 \leq a \leq 1.584$ plotted in the downward direction with different ICs. This figure shows the other coexisting solution region in the system (1).
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Table 3. Procedures used to plot coexisting branches of Fig. [12] Other parameters are $b = 0.65, \ a = 1.55, c = 0.54, \ d = 1.07, \ \epsilon = 1.97$.

<table>
<thead>
<tr>
<th>Fig. No.</th>
<th>Color Graph</th>
<th>Parameter Range</th>
<th>Sweeping Direction</th>
<th>Initial Values ($x(0), y(0), z(0), w(0)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Green</td>
<td>$2 \leq f \leq 2.364$</td>
<td>Decreasing</td>
<td>(0, 0, 0.1, 0.6)</td>
</tr>
<tr>
<td>Black</td>
<td>$2 \leq f \leq 2.25$</td>
<td>Decreasing</td>
<td>(0, 0, 0.1, 0.24)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2.25 \leq f \leq 2.364$</td>
<td>Increasing</td>
<td>(0, 0, 0.1, 0.24)</td>
<td></td>
</tr>
<tr>
<td>Red</td>
<td>$2 \leq f \leq 2.25$</td>
<td>Decreasing</td>
<td>(0, 0, 0.1, 0.36)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2.25 \leq f \leq 2.364$</td>
<td>Increasing</td>
<td>(0, 0, 0.1, 0.36)</td>
<td></td>
</tr>
</tbody>
</table>

investigations demonstrate that the introduced hyper-jerk oscillator is capable of a plethora of coexisting behaviors. The demarcation region of initial values is defined to show different domains of some coexisting solutions. This demarcation region represents cross-sections of the basin of attraction in the different planes with other initial points fixed to zero, as shown in Fig. [12].

3.4. Control

3.4.1. Offset boosting control

In general, offset boosting is used in nonlinear systems to shift chaotic signals or the basin of attraction in phase spaces. This interesting feature can be exploited to induce multistability (i.e. coexisting attractors) in dynamical systems. Also, one can break the symmetry of a dynamical system by introducing a boosting controller [Li et al., 2020a; Lu et al., 2019; Lu et al., 2020]. A simple chaos generator of conditional symmetry induced by offset boosting is investigated using an efficient methodology of dynamics editing in [Li et al., 2020a]. Using the transformation of the variable $x \rightarrow q + x$ and substituting it into the new hyper-jerk system given

Fig. 11. Coexistence of four different asymmetric attractors without any point attractor: (a) Period-1 attractor, and three asymmetric chaotic attractor for $a = 1.584$ with ICs ($-2.5, 0.2, 0.2, 0.5$), ($-1.77, 0.2, 0.2, 0.5$), ($-0.84, 0.2, 0.2, 0.5$), and ($-0.93, 0.2, 0.2, 0.5$) and (b) a pair of period-1 attractors, and a pair of chaotic attractors for $a = 1.579$ and ICs ($-0.45, 0.2, 0.2, 0.5$), ($-1.8, 0.2, 0.2, 0.5$), and ($-1.47, 0.2, 0.2, 0.5$), respectively.

Fig. 12. Zoom diagram of Fig. [12] for the range of parameters $2 \leq f \leq 2.364$ plotted in the upward and downward directions with different ICs as summarised in Table [3].
Fig. 13. Coexistence of three different attractors with different shapes, including (a) a pair of chaotic attractors, (b) period-1 attractor and point attractor illustrated by time series in (c) for $f = 2.25$. Initial conditions $(x(0), y(0), z(0), w(0))$ are $(0, 0, 0.1, 0.24), (0, 0, 0.1, 0.28)$, and $(0, 0, 0.1, 0.36)$ respectively. The frequency spectra of the chaotic and periodic attractors are represented in (d).

Fig. 14. Solutions coexist in the system for $f = 2.323$ including (a) a chaotic attractor, (b) period-3 attractor and (c) period-2 attractor with IC1 $(0, 0, 0.1, 0.28)$, IC2 $(0, 0, 0.1, 0.39)$, and IC3 $(0, 0, 0.1, 0.18)$, respectively, and their corresponding frequency spectra in (d).
in Eq. (1), yields
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= w, \\
\dot{w} &= -a(q + x) - by - cw - d\dot{y}^2 + e(q + x)z - f.
\end{align*}
\tag{7}
\]

In Fig. 14, offset boosting is provided in \((x, z)\) and \((x, w)\) planes when adjusting the constant \(q\). We observe that when \(q > 0\), the chaotic signal \(x\) is shifted in the negative direction, and when \(q < 0\), this chaotic signal shifts in the positive direction.

3.4.2. Total amplitude control

Consider the following transformation necessary to control the amplitude in the system (1):
\[
x = \frac{1}{d} \hat{x}, \quad y = \frac{1}{d} \hat{y}, \quad z = \frac{1}{d} \hat{z}, \quad w = \frac{1}{d} \hat{w}.
\tag{8}
\]

When the transformation, given in Eq. (8), is applied to the nonlinear Eq. (1), we obtained the new 4D hyper-jerk system as follows:
\[
\begin{align*}
\dot{\hat{x}} &= \hat{y}, \\
\dot{\hat{y}} &= \hat{z}, \\
\dot{\hat{z}} &= \hat{w}, \\
\dot{\hat{w}} &= -a\hat{x} - b\hat{y} - c\hat{w} - \frac{\hat{y}^2}{d} + \frac{e\hat{x}\hat{z}}{d} - \frac{f}{d}.
\end{align*}
\tag{9}
\]

One can remark that Eq. (9) is identical to Eq. (1) for \(d = 1\). Thus, the constant term \(d\) in the 4D hyper-jerk oscillator can be exploited to control the amplitude of all variables proportionally and simultaneously according to \(1/d\). This control technique is called total amplitude control (TAC) \cite{Gu et al., 2021; Leutcho et al., 2021; Li & Sprott., 2014; Zang et al., 2021} because all the states in the chaotic hyper-jerk system can be controlled when changing the coefficient \(d\). The amplitude control of the strange chaotic attractor of the system (1) in different planes when changing \(d\) is shown in Fig. 15. In electronic circuit design, this amplitude coefficient can be realized using only a potentiometer.
Fig. 16. Offset boosting of the chaotic signal in the new hyper-jerk system for varying boostable states $q$ as follows: Green attractor for $q = -2.5$, red attractor for $q = -0.86$, and blue attractor for $q = 0.3$. It is observed that chaotic signals can be moved through the $x$-axis. The initial point is $(0, 0, 0.1, 0.18)$, and the rest of the parameters are those in Fig. 1.

Fig. 17. (a)–(d) Amplitude control of the chaotic signal of system (1) plotted for $d = 1.065$ (blue), $d = 1.1$ (red) and $d = 1.22$ (yellow). IC $(0, 0, 0.1, 0.18)$ and the rest of the parameters are those in Fig. 1.
3.5. Antimonotonicity

Let us emphasize that the birth of periodic orbits followed by their annihilation by an inverse PD transition when a control parameter is slowly modified is called an antimonotonicity [Kyprianidis et al., 2000]. This interesting property has been investigated in many nonlinear systems, including laser systems [Pailitz & Lauterborn, 1985], Chua circuit [Dawson et al., 1992], Duffing oscillator [Kengne et al., 2018], and autonomous MLC oscillator [Kocarev et al., 1993]. In 2018, the first exploration of antimonotonicity in 4D hyper-jerk circuits with hyperbolic sine nonlinearity was conducted [Leutcho et al., 2018]. In [Leutcho et al., 2020b], the authors proposed different bifurcations in which bubbles coexist in series or in parallel. Also, a chaotic bubble can coexist with a periodic bubble when the symmetric of a chaotic system is broken.

Fig. 18. Asymmetric bubbles of bifurcation computed for the same parameter setting in Fig. 1 in the range $1.51 \leq a \leq 1.57$ with initial conditions $(-1.82, 0.2, 0.2, 0.5)$ when scanning the parameter upward (continuation method).
as presented in [Kengne et al., 2020a]. Recall beforehand that for a nonlinear system to change period-doubling bifurcations forward and backward, the presence of periodic islands in the parameter space is required [Negou & Kengne, 2019]. In Fig. 18, a sample result of antimonotonicity (i.e. asymmetric bubbles) in the new 4D hyper-jerk system for a specific parameter \( b \) is presented. For \( b = 0.546 \), period-2 bubble is obtained, at \( b = 0.54 \), period-4 bubble is formed, at \( b = 0.534 \), the bifurcation illustrates period-8 bubble. The first chaotic bubble is formed at \( b = 0.532 \). It is easy to see that the type of bubble changes when \( b \) is correctly chosen. As \( b \) is further decreased smoothly, more asymmetric bubbles are generated.

4. Design of the Circuit

PSIM (Power Simulation) is software used to design and simulate power electronics devices, motor, and order power devices. It can also be used to simulate simple electronic circuits based on conventional components. An interesting aspect of using such simulator software like PSIM based simulations (PSIM Professional Version 9.0.3.400 x32) is the possibility of easily introducing the initial state of the components like capacitor and inductor, thus in the wished case track the different coexisting attractor in the circuit. To achieve that goal, let us consider the electronic circuit depicted in Fig. 19. The circuit consists of components, such as operational amplifiers, capacitors, resistors, analogue devices AD633 multipliers.

Applying Kirchhoff’s laws, the circuit of Fig. 19 is described by the following equations:

\[
\begin{align*}
\frac{dV_1}{dt} & = \frac{1}{RC_1}V_2, \\
\frac{dV_2}{dt} & = \frac{1}{RC_2}V_3, \\
\frac{dV_3}{dt} & = \frac{1}{RC_3}V_4, \\
\frac{dV_4}{dt} & = -\frac{1}{Ra}C_4 V_1 - \frac{1}{Rb}C_4 V_2 - \frac{1}{Rc}C_4 V_4 - \frac{1}{Rd}C_4 V_2^2 + \frac{1}{Re}C_4 V_1 V_3 - \frac{1}{Rf}C_4 V_{cc}.
\end{align*}
\]

(10)

In order to avoid the saturation of the operational amplifiers, we reduce the scale of the amplitudes of the system by proceeding to the following change of variables:

\[
V_1 = \frac{X}{10 \, \text{V}}, \quad V_2 = \frac{Y}{10 \, \text{V}}, \quad V_3 = \frac{Z}{10 \, \text{V}}, \quad V_4 = \frac{W}{10 \, \text{V}}
\]

(11)
We obtained the following nonlinear fourth-order differential equation:

\[
\begin{align*}
\frac{dX}{dt} &= \frac{1}{RC_1} Y, \\
\frac{dY}{dt} &= \frac{1}{RC_2} Z, \\
\frac{dZ}{dt} &= \frac{1}{RC_3} W, \\
\frac{dW}{dt} &= -\frac{1}{R_a C_4} Y - \frac{1}{R_b C_4} Y - \frac{1}{R_C C_4} W \\
&\quad - \frac{10}{10R_a C_4} Y^2 + \frac{10}{10R_b C_4} XZ - \frac{10}{R_C C_4} V_{cc}.
\end{align*}
\]

(12)

Choosing time changes as \( t = \tau RC \) with \( C = C_1 = C_2 = C_3 = C_4 = 10 \text{nF} \) and \( R = 10 \text{K}\Omega \), the parameters of Eq. (1) are expressed in terms of the values of capacitors and resistors as follows:

\[
\begin{align*}
\frac{R}{R_a} &= a; \quad \frac{R}{R_b} = b; \\
\frac{R}{R_C} &= c; \quad \frac{R}{10R_a} = d; \\
\frac{R}{10R_b} &= e; \quad \frac{10R_{V_{cc}}}{10R_f} = f.
\end{align*}
\]

(13)

From Fig. 20, one can observe some sample number of attractors confirming the phenomenon of a large number of attractors observed during the numerical integration.

Fig. 20. PSIM-based simulation result showing some of the coexisting attractors depicted in Fig. 10(a). This figure shows the coexistence of periodic attractors and one chaotic attractor with initial conditions \((x(0), y(0), z(0), w(0)) :\) \((-0.0247, 0.02, 0.03, 0.05), (-0.2, 0.02, 0.03, 0.05), (-0.02, 0.02, 0.03, 0.05), (-0.2, 0.02, 0.03, 0.05). The other resistors are fixed as \( R_a = 6.337 \text{K}\Omega, R_b = 18.52 \text{K}\Omega, R_c = 18.52 \text{K}\Omega, R_d = 0.93 \text{K}\Omega, R_e = 0.51 \text{K}\Omega, R_f = 627.6 \text{K}\Omega.\)
5. Concluding Remarks

This contribution has focused on the dynamics of a new chaotic hyper-jerk system with a unique equilibrium. The nonlinear behavior observed in the newly introduced hyper-jerk system comes from two nonlinear quadratic terms. Based on the traditional nonlinear diagnostic tools such as bifurcation diagrams, phase portrait, frequency spectra, and two-parameter Lyapunov exponents, it has been found that the new hyper-jerk system effectively exhibits three types of hidden attractors, including hidden point attractor, hidden periodic attractor, and hidden chaotic attractor. The results indicate some interesting properties in the new hyper-jerk system, such as PD bifurcation, antimonotonicity, hysteretic dynamics, and coexisting strange attractors (up to five asymmetric hidden and self-excited attractors were found). The amplitude and offset of the hidden signals have been studied and perfectly controlled. Finally, the electronic analog of the model has been constructed and simulated in PSIM to confirm the occurrence of hidden behaviors.

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