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Error-feedback temperature regulation for a reverse flow reactor driven by a distributed parameter exosystem

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ABSTRACT

This work is devoted to solve the state and error-feedback control problems for a catalytic reverse flow reactor (CFRR), which is modeled by nonlinear partial differential equations (PDEs). These two regulation problems will be solved based on the linearized infinite-dimensional representation. The objective is to track a desired output reference under the presence of disturbances. Both the reference trajectory and the disturbance profiles are generated by a distributed parameter exosystem. First, a state feedback stabilizing regulator is designed which drives the process output to a reference trajectory. The second main aim is to develop a dynamical controller that uses the tracking error as an input. Furthermore, it has been demonstrated that the closed-loop plant is exponentially stable and the tracking error asymptotically goes to zero. The developed regulators are evaluated through numerical simulations for the case study of methane combustion.

1. Introduction

Catalytic flow reverse reactors (CFRR) are fixed-bed chemical reactors such that the flow direction is switched periodically. The dynamics of the CFRR process is best modeled by nonlinear partial differential equations (PDEs) obtained from material and energy balances (see [1,2]). The main advantage of this process is the heat trap effect permitting self-thermal operation. It is a captivating way of operation for exothermic chemical reactions and the temperature delivered from the process can be employed in the best way possible to keep the optimal operation of process. CFRRs are widely used in many chemical processes for example methane combustion, oxidation of sulfur dioxide and oxidation of VOCs (see [1,3]). Moreover, some recent studies showed that CFRRs can be also used for endothermic reactions, such as methane steam reforming (see [3]). The heat trap feature of this process may cause unlimited increase of temperature, therefore temperature regulation is a very important issue for the CFRR process and it can be done via gas removal technique or through a heat exchanger. Control of chemical processes via heat exchanger has been the subject of many research works (see e.g. [4–7]). Here, the focus is on temperature regulation through gas removal, which is more advantageous over heat exchanger cooling (see [8]).

Control of PDEs is an important and rich area of research. Usually, two main approaches are adopted. Discretization-based approach by which the PDEs model is converted (via finite-difference, finite-element, spectral decomposition, etc.) into an ODEs model and the latter is used for control design purpose (see e.g. [9,10]). The main advantage of this approach is the fact that control of ODEs is well established but a major drawback is that high number of discretization points in needed to preserve the distributed nature of the system. The second approach is based on infinite-dimensional representation. In this approach, the PDEs model is reformulated as a differential equation on an abstract space (see [11,12]). Many control techniques have been developed in the framework of systems that are governed by PDEs, such as optimal control [13,14], model predictive control [15,16] and PI control [17–21].

Feedback regulation is widely employed in modern controlled systems and it plays a major role to improve the performances of the control systems. It is a mechanism that utilizes information from the system measurements to manipulate a variable in order to achieve the desired outcome. The concept of feedback regulation is well developed for both finite-dimensional systems. Moreover, feedback control of infinite-dimensional systems has attracted lots of attention and efforts (see e.g [12,22,11]) and variety of techniques have been adopted for many types of PDEs systems, including hyperbolic and parabolic PDEs (see e.g. [23,14]). In [24], state-feedback linear–quadratic regulator has been designed for VOC combustion in a CFRR by manipulating internal electric heating and dilution as inputs to control the hot spot temperature. The problem has been investigated by implementing finite-difference discretization method of the linearized plant to regulate the temperature at a single position. In [25], state

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feedback temperature regulation problem for the CFRR model has been solved by using linear–quadratic optimal control method based on the infinite-dimensional model representation. The regulation objective is to drive the process trajectories towards a stationary state and this is achieved by gas removal through control of the fluid velocity. Moreover and due to the fact that access to the process state is not feasible, an observer has been designed to construct an output feedback regulator in [26]. The design utilizes the duality feature between control and estimation problems. The previous works ignore the presence of disturbances which is not realistic in chemical processes. Indeed, disturbances in a catalytic reactor create challenging problems due to the propagation of these disturbances through the reactor bed [27]. The focus in this work is to investigate the feedback temperature regulation problem in order to track a desired output trajectory and stabilizes the closed-loop system in the presence of disturbances.

The regulator problem (sometimes called servomechanism) is one of the most major approaches in the theory of system control. This problem consists of the design of a feedback controller to track a desired output profile and reject disturbances while preserving stability of the closed-loop process. Usually, the desired output trajectory and disturbances are produced by an exosystem. In general, the regulator problem has two main versions. One is the state feedback regulator problem in which the controller is constructed with full access to the states of the process and the exosystem. The second version is the more realistic error feedback regulator problem where only the tracking error is accessible for measurement. The regulator problem has been studied broadly for finite-dimensional systems and many results have been developed based on the internal model principle (see. e.g. [28–30]). Great efforts have been devoted to reach out similar outcomes for infinite-dimensional systems. The case of bounded control and observation was investigated in [31,22] and then extended to the unbounded case in [32]. These research works focused mainly on plant driven by finite-dimensional exosystems that has been enlarged later [33] to infinite-dimensional exosystems. Here, the regulator problem will be investigated for the CFRR process in order to regulate the temperature inside the reactor under the presence of disturbances. The investigation is based on the infinite-dimensional model of the CFRR and also by considering disturbances that are generated by a distributed parameter exosystem.

The paper is organized as follows. Section 2 is devoted to the mathematical PDEs model together with its linearized infinite-dimensional version. The state-feedback regulation problem is investigated in Section 3. Indeed, the state-feedback operator is expressed through the solution of a Riccati differential equation and the disturbance-feedback operator is expressed through the solution of a Sylvester differential equation. In Section 4, the error-feedback regulator is solved assuming that only the tracking error is available. Indeed, a stabilizing dynamical controller is designed to track a desired output trajectory and reject the disturbances. Numerical simulations are executed for the case study of combustion of methane to demonstrate the achievements of the developed regulators.

Notations. The following notations are needed throughout the paper:

- $\mathbb{R}$ is the set of real numbers.
- $L^2(0,1)$ is the space of square integrable functions on $[0,1]$.
- $\langle\cdot,\cdot\rangle$ is the usual inner product, i.e. for any $f_1, f_2 \in L^2(0,1)$,

$$\langle f_1, f_2\rangle = \int_0^1 f_1(\xi) f_2(\xi) d\xi.$$

- $L^\infty(0,1)$ is the space of bounded functions on $(0,1)$.
- $\mathcal{L}(H)$ is the space of bounded linear operators from H to H.
- $\mathcal{L}(H_1, H_2)$ is the space of bounded linear operators from H_1 to H_2.
- A^* is the adjoint operator of A.

Definition. A function f is absolutely continuous on the interval $[0,1]$ if f has a derivative f' almost everywhere, $f' \in L^1(0,1)$ and

$$f(x) = f(0) + \int_0^x f'(\xi) d\xi, \quad \forall x \in [0,1]$$

2. Mathematical model

Based on the material and energy balances, the dynamics of CFRR can be described by a set of PDEs. Here, it is assumed that the fluid and solid temperatures and concentrations are uniform along the reactor [2]. Moreover, it is assumed that convection is dominant over diffusion. If T is the temperature in the reactor and Y is the molar fraction of the chemical substance, then the mathematical PDEs model is given by:

$$\left\{\begin{array}{rcl} \epsilon\dfrac{\partial Y}{\partial t} + \phi v_{in}\dfrac{\partial Y}{\partial \xi} &=& -k_0 \exp\left(\dfrac{-E}{R_g T}\right) Y \\ \eta\dfrac{\partial T}{\partial t} + \phi v_{in}\rho\dfrac{\partial T}{\partial \xi} &=& (-\Delta H_r) k_0 \exp\left(\dfrac{-E}{R_g T}\right) Y \end{array}\right. \tag{1}$$

where ϕ is the input variable and it represents the fraction of the inlet gas in the reactor. t and ξ represent time and space variables, respectively and $v_{in}, \epsilon, \Delta H_r, R_g$ and E are inlet gas flow velocity, void fraction, heat of reaction, universal gas constant and the activation energy respectively. Moreover, the constants k_0, η and ρ are given by

$$k_0 = (1-\epsilon)\mu_{eff} k_\infty, \quad \eta = \rho_s(1-\epsilon)Cp_s \text{ and } \quad \rho = \rho_g Cp_g$$

where $k_\infty, \mu_{eff}, \rho_g, C_{P_g}, \rho_s$ and C_{P_s} are stoichiometric coefficient, effectiveness factor, density of gas phase, specific heat of gas phase, density of solid phase and specific heat of solid phase, respectively. The boundary and initial conditions are given by

$$\begin{array}{rcl} Y(0,t) &=& Y_{in} \text{ and } T(0,t) = T_{in} \\ Y(\xi,0) &=& Y_0(\xi) \text{ and } T(\xi,0) = T_0(\xi) \end{array} \tag{2}$$

Y_{in} and T_{in} are the inlet molar fraction and inlet temperature, respectively. On the other hand, Y_0 and T_0 are the initial molar fraction and the initial temperature, respectively. Now, let us implement the following transformation

$$\tilde{Y} = \frac{Y_{in} - Y}{Y_{in}} \quad \text{and} \quad \tilde{T} = \frac{T - T_{in}}{T_{in}} \tag{3}$$

which converts the PDEs model (1) into the following dimensionless model

$$\left\{\begin{array}{rcl} \dfrac{\partial \tilde{Y}}{\partial t} &=& \phi v_1 \dfrac{\partial \tilde{Y}}{\partial \xi} + k_1(1-\tilde{Y})\mathcal{E}(\tilde{T}) \\ \dfrac{\partial \tilde{T}}{\partial t} &=& \phi v_2 \dfrac{\partial \tilde{T}}{\partial \xi} + k_2(1-\tilde{Y})\mathcal{E}(\tilde{T}) \end{array}\right. \tag{4}$$

with the boundary conditions $\tilde{Y}(0,t) = \tilde{T}(0,t) = 0$. The function $\mathcal{E}$ is given by $\mathcal{E}(x) = e^{\frac{\mu}{1+x}}$ and v_1, v_2, μ, k_1 and k_2 are given in terms of model parameters via the following equations

$$v_1 = -\frac{v_{in}}{\epsilon}, \quad v_2 = -\frac{v_{in}\rho}{\eta}, \quad \mu = \frac{-E}{R_g T_{in}}, \quad k_1 = \frac{k_0}{\epsilon},$$

$$k_2 = \frac{(-\Delta H_r)k_1\epsilon}{\eta}\frac{Y_{in}}{T_{in}}$$

In a catalytic process, the heat capacity of solid catalyst is usually three times of magnitude higher than the fluid phase, which means that the system of equations (4) possesses two-time scale property and as a result the reactant dynamics are faster than the temperature dynamics. Therefore, one can assume that the mass balance in the fluid phase is at qausi-steady state and the accumulation term in the first equation of system (4) is excluded, which leads to the following differential equation

$$\frac{d\tilde{Y}}{d\xi} = -\frac{k_1}{\phi v_1}(1-\tilde{Y})\mathcal{E}(\tilde{T}), \quad \tilde{Y}(0) = 0 \tag{5}$$

The above equation can be solved easily by the method separation of variables

$$\int_0^{\xi} \frac{d\tilde{Y}}{1-\tilde{Y}} d\xi = \int_0^{\xi} -\frac{k_1}{\phi v_1}\mathcal{E}(\tilde{T}) d\xi$$

and therefore the explicit solution is given by

$$\tilde{Y}(\xi) = 1 - \exp\left(\int_0^{\xi} \frac{k_1}{\phi v_1}\mathcal{E}(\tilde{T}) d\tilde{\xi}\right) \tag{6}$$

On the other hand, it has been proved in [34] that the fast system generates an exponentially stable trajectory, which means that if a two-time scale decomposition of system (4) is performed, the dynamics of the reactant will not affect the stability of the entire process (more details can be found in [35]).

By substituting the solution $\tilde{Y}$ in the other equation of (4), one can get the following single PDE with the temperature $\tilde{T}$ as the state and ϕ as the input.

$$\frac{\partial \tilde{T}}{\partial t} = \phi v_2 \frac{\partial \tilde{T}}{\partial z} + k_2 \exp\left(\int_0^{\xi} \frac{k_1}{\phi v_1}\mathcal{E}(\tilde{T}) d\tilde{\xi}\right)\mathcal{E}(\tilde{T}), \qquad \tilde{T}(0,t) = 0 \tag{7}$$

The associated steady-state equation is given by the following first-order differential equation

$$\frac{d\tilde{T}_e}{d\xi} = -\frac{k_2}{\phi_e v_2} \exp\left(\int_0^{\xi} \frac{k_1}{\phi_e v_1}\mathcal{E}(\tilde{T}_e) d\tilde{\xi}\right)\mathcal{E}(\tilde{T}_e), \qquad \tilde{T}_e(0) = 0$$

For a given profile $\phi_e \in L^{\infty}(0,1)$ and according to the existence theorem for first-order differential equations, the above equation admits a solution $\tilde{T}_e \in L^{\infty}(0,1)$. It should be noted that this is associated with a standard unidirectional flow operation. However, in the case of reverse flow operation, the high temperatures near the reactor exit can used to preheat the reactor feed. In this case, a quasi-steady state operation may be reached in which the reactor temperature has a maximum value near the center of the reactor. In the rest of the paper, this quasi-steady state will be called a stationary state and will be denoted $\tilde{T}_e$.

For the purpose of solving the state-feedback and error-feedback control problems, linearization of PDE (7) is needed. Let us consider $L^2(0,1)$ as the state space. Let $\tilde{T}_e \in L^{\infty}(0,1)$ and $\phi_e \in L^{\infty}(0,1)$ be the dimensionless stationary profiles of the model (7). Now, let us define the deviated state and input trajectories, respectively:

$$x(t) := x(\cdot,t) = \tilde{T}(\cdot,t) - \tilde{T}_e(\cdot) \in L^2(0,1) \text{ and } u(t) = \phi(t) - \phi_e \in \mathbb{R} \tag{8}$$

Hence the linearization of Eq. (7) around its stationary state conducts to the resulting abstract differential equation on the infinite-dimensional space $L^2(0,1)$:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \in L^2(0,1)\,. \end{cases} \tag{9}$$

such that A is the operator defined on its domain:

$$D(A) = \{x : x \text{ is absolutely continuous.}, \quad \frac{dx}{d\xi} \in L^2(0,1) \text{ and } x(0) = 0\} \tag{10}$$

by
$$Ax = \alpha\frac{dx}{d\xi} + \beta(\xi)x \qquad \forall x \in D(A) \tag{11}$$

where $\alpha = \phi_e v_2 < 0$ and $\beta \in L^{\infty}(0,1)$ is the Jacobian of the nonlinear part $k_2 \exp\left(\int_0^{\xi} \frac{k_1}{\phi v_1}\mathcal{E}(\tilde{T})d\tilde{\xi}\right)\mathcal{E}(\tilde{T})$ in Eq. (7) with respect to x and by simple calculations it is given by

$$\beta(\xi) = k_2\mathcal{E}(\tilde{T}_e)\exp\left(\int_0^{\xi} \frac{k_1\mathcal{E}(\tilde{T}_e)}{\phi_e v_1} d\tilde{\xi}\right) \times \left[\int_0^{\xi} \frac{-k_1\mu\mathcal{E}(\tilde{T}_e)}{\phi_e v_1(1+\tilde{T}_e)^2} d\tilde{\xi} - \frac{\mu}{(1+\tilde{T}_e)^2}\right]$$

Also, $B \in \mathcal{L}(\mathbb{R}, L^2(0,1))$ is the linear bounded operator given by

$$B = \gamma(\xi)I, \tag{12}$$

such that I represents the identity operator and $\gamma \in L^{\infty}(0,1)$ is the Jacobian of the right side of Eq. (7) with respect to ϕ and is given by

$$\gamma(\xi) = v_1\frac{d\tilde{T}_e}{d\xi} + k_2\mathcal{E}(\tilde{T}_e)\exp\left(\int_0^{\xi} \frac{k_1\mathcal{E}(\tilde{T}_e)}{\phi_e^2 v_1} d\tilde{\xi}\right) \times \exp\left(\int_0^{\xi} \frac{k_1\mathcal{E}(\tilde{T}_e)}{\phi_e v_1} d\tilde{\xi}\right)$$

It can be noted that exponential stability of the uni-flow direction plant is a trivial result of [13, Theorem 2], i.e there are $\nu > 0$ and $\Lambda > 0$ such that

$$\|e^{At}\| \le \Lambda e^{-\nu t}$$

Disturbances in a catalytic reactor create challenging problems due to the propagation of these disturbances through the reactor bed [27]. The objective here is to solve the state-feedback regulation and error-feedback regulation problems associated with the linearized CRFF (9) with the presence of disturbances. In this case, the state–space system can be written as

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + w(t), \quad t > 0, \ x(0) \in L^2(0,1) \\ y(t) &= Cx(t)\,. \end{cases} \tag{13}$$

where $w(t) \in L^2(0,1)$ represents disturbance and the output function is defined as follows

$$y(t) = Cx(t) = \int_0^1 c(\xi)x(t,\xi)d\xi = \langle c, x\rangle \tag{14}$$

The output function acts for the (weighted) average temperature inside the reactor, such that the function c is continuous on $[0,1]$ and it plays the role of a weight function. The disturbance $w(t)$ and the desired output $y_r(t)$ to be tracked by $y(t)$ are governed by the following distributed parameter exosystem

$$\begin{cases} w_t(t,\xi) &= \alpha w_{\xi}(t,\xi) + \zeta(\xi)w(t,\xi) = Sw \quad t > 0, \\ & \quad w(t,1) = 0 \\ y_r(t) &= Qw(t) = \int_0^1 q(\xi)w(t,\xi)d\xi \\ &= \langle q, w\rangle \quad t \ge 0, \quad t \ge 0 \end{cases} \tag{15}$$

where q is a space-varying continuous function on the interval $[0,1]$. Assume that if there is a subinterval $I \subset [0,1]$ in which the function $c = 0$ then the function q is equal to zero on I. This assumption is not restrictive since both functions represent the weight functions for the output and the desired output. Denote by $e(t)$ the error between the output and the desired output

$$e(t) = y(t) - y_r(t) = Cx(t) - Qw(t)$$

3. State-feedback regulator problem

This section is devoted to solve the state-feedback control problem. Indeed, the main objective here is to find a controller under the form

$$u(t) = Kx(t) + Lw(t) \tag{16}$$

where K and L are bounded operators, i.e $K, L \in \mathcal{L}(L^2(0,1), \mathbb{R})$ and satisfies:
(i) The operator $A + BK$ generates an exponentially stable C_0-semigroup. (ii) The error $e(t) = Cx(t) - Qw(t)$ converges asymptotically to zero, where x and w are generated by the system

$$\begin{cases} \dot{x}(t) &= (A+BK)x(t) + (BL+I)w(t) \\ \dot{w}(t) &= Sw(t) \end{cases} \tag{17}$$

First, let us focus on the design of the stabilizing feedback operator K. By using the fact that exponential stability is strongly related to the existence of solution of a certain Lyapunov equation (see e.g. [11, Theorem 5.1.3]), the following theorem shows that a stabilizing feedback can be found through the solution of a differential Riccati equation.

Theorem 1. *Let us consider the linearized CFRR process given by (9). If the following differential Riccati-type equation*

$$\alpha \frac{d\theta}{d\xi} x = 2\beta\theta x + nx - 2\gamma\theta\langle\gamma\theta, x\rangle = 0, \quad \theta(1) = 0 \tag{18}$$

admits a unique positive solution θ, then the stabilizing feedback operator is given by

$$Kx = -\langle\gamma\theta, x\rangle \tag{19}$$

The function n is a positive design function, which means it can be adjusted to design a feedback to stabilize the closed-loop system at a desired rate.

Proof. It can be easily shown that if Q_0 is the nonnegative self-adjoint solution of the following Riccati equation for all $x \in D(A)$

$$\begin{aligned}&\left(A^*Q_0 + Q_0A + N - 2Q_0BB^*Q_0\right)x \\ &= 0, \ \forall x \in D(A), Q_0D(A) \subset D(A^*)\end{aligned} \tag{20}$$

when N is positive definite design operator (to be chosen), then $K = -B^*Q_0$ is a stabilizing feedback operator. Indeed, it is enough to see that the Riccati equation (20) can be rewritten as follows

$$(A - BB^*Q_0)^*Q_0 + Q_0(A - BB^*Q_0) + N = 0$$

Now let us investigate the operator Riccati equation (20) in the case of the linearized CFRR system (9). Let us choose $N = n(\xi)I$ and assume that Eq. (20) admits a solution of the form $Q_0 = \theta(\xi)I$. This can be considered as a valid assumption unless a contradiction arises in the development latter. The motivation behind this assumption is to convert the operator Riccati equation (20) into a scalar differential Riccati equation. Indeed, under this assumption, Eq. (20) can be written for any $x \in D(A)$

$$-\alpha\frac{d(\theta x)}{d\xi} + \beta\theta x + \theta\left(\alpha\frac{dx}{d\xi} + \beta x\right) + nx - 2\theta\gamma\langle\gamma, \theta x\rangle = 0$$

The last term is a result of the fact that $B^*w = \langle\gamma, w\rangle, \forall w \in L^2(0,1)$. The previous equation leads to the following

$$-\alpha\frac{d(\theta x)}{d\xi} + \beta\theta x + \theta\alpha\frac{dx}{d\xi} + \theta\beta x + nx - 2\theta\gamma\langle\gamma, \theta x\rangle = 0$$

which implies

$$-\alpha\frac{d\theta}{d\xi}x - \alpha\theta\frac{dx}{d\xi} + \beta\theta x + \theta\alpha\frac{dx}{d\xi} + \phi\beta x + nx - 2\theta\gamma\langle\gamma, \theta x\rangle = 0$$

Consequently, θ is the positive solution of Eq. (18). The condition $\theta(1) = 0$ is a consequence of the condition $Q_0D(A) \subset D(A^*)$. Moreover, the stabilizing feedback expression (19) is a result of the fact that $K = -B^*Q_o$. □

Remark 1. Note that Eq. (18) is not a standard Riccati differential equation due to the last term, which is a consequence of the form of B^*. Indeed, the operator B^* acts on functions $x \in L^2(0,1)$ to produce the weighted average value with respect to γ as a weight function. If the average value is substituted by the original space-varying function, then this will lead to the following standard Riccati differential equation

$$\alpha\frac{d\theta}{d\xi} = 2\beta\theta + n - 2\gamma^2\theta^2 = 0, \quad \theta(1) = 0 \tag{21}$$

which admits a unique positive solution θ. (see [36, Theorem 4.2.1])

Now let us investigate the state-feedback control problem, more precisely let us design the operator L. In order to express the operator L, the following lemma is required.

Lemma 1. *If there exist $\Pi \in \mathcal{L}(L^2(0,1))$ and $\Gamma \in \mathcal{L}(L^2(0,1), \mathbb{R})$ such that the following Sylvester equations*

$$A\Pi - \Pi S + B\Gamma + I = 0 \tag{22}$$

$$C\Pi - Q = 0 \tag{23}$$

hold, then the tracking error $e(t)$ converges asymptotically to zero.

Proof. Assume that Π and Γ are solutions of (22)–(23). Let us choose $L = \Gamma - K\Pi$ and substitute in Eq. (17) to get the following

$$\dot{x}(t) = (A+BK)x(t) + (B(\Gamma - K\Pi) + I)w(t)$$

By using Eq. (22), one gets

$$\begin{aligned}\dot{x}(t) &= (A+BK)x(t) + ((\Pi S - A\Pi) - BK\Pi)w(t) \\ &= (A+BK)x(t) + \Pi Sw(t) - (A+BK)\Pi w(t) \\ &= (A+BK)(x(t) - \Pi w(t)) + \Pi\dot{w}(t)\end{aligned}$$

By taking $\tilde{x}(t) = x(t) - \Pi w(t)$, the equation above can be written as

$$\dot{\tilde{x}}(t) = (A+BK)\tilde{x}(t)$$

Since $A + BK$ is the generator of an exponentially stable C_0-semigroup on $L^2(0,1)$, then

$$\tilde{x}(t) = e^{(A+BK)t}\tilde{x}(0)$$

which is equivalent to

$$x(t) = e^{(A+BK)t}(x(0) - \Pi w(0)) + \Pi w(t)$$

Therefore, the error $e(t)$ can written as

$$e(t) = Cx(t) - Qw(t) = Ce^{(A+BK)t}(x(0) - \Pi w(0)) + (C\Pi - Q)w(t)$$

Using Eq. (23) and the exponential stability of $e^{(A+BK)t}$ and the fact that C is bounded, it is trivial to conclude that $e(t) \to 0$ when $t \to \infty$. □

Now, let us investigate Eqs. (22) and (23) in an explicit way for the CFRR linearized system. The following theorem shows that the state-feedback operator L can be expressed through the solution of a Sylvester differential equation.

Theorem 2. *Let us consider the linearized CFRR process given by (13) with disturbances generated by the exosystem (15). Let θ be the unique solution of Eq. (18). If there exist two functions κ and π such that the following equations are satisfied for all $w \in D(S)$*

$$\alpha \frac{d\pi}{d\xi} w + (\beta - \zeta)\pi w + \gamma \langle \kappa, w \rangle + w = 0, \quad \pi(0) = 0 \tag{24}$$

$$\pi c = q \tag{25}$$

then the state-feedback control given by

$$u(t) = H \begin{pmatrix} x \\ w \end{pmatrix} = -\langle \gamma\theta, x \rangle + \langle \kappa + \gamma\theta\pi, w \rangle \tag{26}$$

drives the tracking error $e(t)$ to zero when t goes ∞.

Proof. First note that Eq. (22) is a Sylvester equation that admit a unique solution Π since the spectrums of A and S are both empty, which means that the two operators do not share any eigenvalue. Let us assume that $\Pi = \pi(\xi)I$, where the function π is absolutely continuous and $\frac{d\pi}{d\xi} \in L^2(0, 1)$. If one assumes that $\pi c = q$, then it is obvious that Eq. (23) is satisfied. On the other hand, let us define the operator Γ for any $w \in L^2(0, 1)$ as follows

$$\Gamma w = \langle \kappa, w \rangle$$

where κ is a continuous function. In this case, Eq. (22) can be written for any $w \in D(S)$ as follows

$$A\Pi w - \Pi S w + B\Gamma w + w = 0$$

By using the expressions of the operators A and S given by (11) and (15), respectively, one gets

$$\alpha \frac{d(\pi w)}{d\xi} + \beta\pi w - \pi\alpha \frac{dw}{d\xi} - \pi\zeta w + \gamma\langle \kappa, w \rangle + w = 0$$

By simply using product rule for derivative, one obtains the following equation

$$\alpha \frac{d\pi}{d\xi} w + \alpha\pi \frac{dw}{d\xi} + \beta\pi w - \pi\alpha \frac{dw}{d\xi} + \gamma\langle \kappa, w \rangle + w = 0$$

Consequently, the functions π and κ satisfy the Eqs. (24) and (25). □

Remark 2. It is clear that Eq. (22) admits a unique solution due to the fact that the spectrums of A and S are both empty. In order to solve this equation, it has been converted to the scalar differential Sylvester Eq. (24), which is easier to solve together with Eq. (25). Indeed, Eq. (25) is an algebraic equation that can be solved first to get the function π, then by plugging the solution π in Eq. (24), one can get the solution κ. This process is valid due to the assumption given after Eq. (15) since this will help to avoid the case when c equals to zero on a subinterval in which q is not zero.

4. Error-feedback control problem

In order to implement the state-feedback controller (16), full information on the state and disturbance should be available, which is not realistic from application point of view. This Motivates the need to investigate the more practical error-feedback control problem, where the objective is to find a controller of the form

$$\begin{cases} \begin{pmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{w}}(t) \end{pmatrix} = \mathcal{A} \begin{pmatrix} \hat{x}(t) \\ \hat{w}(t) \end{pmatrix} + Ge(t) \\ u(t) = K\hat{x}(t) + L\hat{w}(t) = H \begin{pmatrix} \hat{x}(t) \\ \hat{w}(t) \end{pmatrix} \end{cases} \tag{27}$$

such that the operator

$$\mathfrak{A} = \begin{pmatrix} A & BH \\ GC & \mathcal{A} \end{pmatrix}$$

generates an exponentially stable C_0-semigroup and the error $e(t) = Cx(t) - Qw(t)$ converges to zero as $t \to \infty$.

First, remember that the expression of the stabilizing feedback operator H is given in Theorem 2 in term of the functions π and κ provided that these two functions satisfy Eqs. (24)–(25). Moreover, this guarantees that the operators $\Pi = \pi(\xi)I$ and $L = \langle \kappa, \cdot \rangle$ satisfy Eqs. (22)–(23) (see the proof of Theorem 2). Now let us consider the operators A_0 and C_0 given by

$$A_0 = \begin{pmatrix} A & B_dF \\ 0 & S \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} = \alpha \frac{d}{d\xi} \begin{pmatrix} x \\ w \end{pmatrix} + \tilde{M} \begin{pmatrix} x \\ w \end{pmatrix}$$

$$C_0 \begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} C & -Q \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} = \langle c, x \rangle - \langle q, w \rangle$$

where the domain of A_0 is given by

$$D(A_0) = \{\tilde{x} \in (L^2(0, 1))^2 : \tilde{x} \text{ absolutely continuous}, \\ \frac{d\tilde{x}}{d\xi} \in (L^2(0, 1))^2, \tilde{x}(0) = 0\}$$

Following similar proof of [31, Theorem 4.2], it can be shown that if $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ is the output injection operator for the pair (C_0, A_0) and the operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} A + BK - G_1C & B_dF + B(\Gamma - K\Pi) + G_1Q \\ -G_2C & S + G_2Q \end{pmatrix}$$

then the tracking error converges to zero as t goes to infinity. In order to complete the design of the controller (27), the stabilizing output injection operator G should be found. The following theorem states that the output injection operator G can be found through the solution of a matrix differential equation

Theorem 3. *Let us consider the pair (C_0, A_0) and let M_0 a definite positive design matrix. If the matrix $\tilde{\Phi} = \begin{pmatrix} \tilde{\phi}_{11} & \tilde{\phi}_{12} \\ \tilde{\phi}_{21} & \tilde{\phi}_{22} \end{pmatrix}$ is the unique nonegative solution of the matrix differential Riccati equation*

$$-v \frac{d\tilde{\Phi}}{d\xi} = \tilde{M}\tilde{\Phi} + \tilde{\Phi}\tilde{M}^* + M_0 - \tilde{\Phi}C_0^*C_0\tilde{\Phi}, \quad \tilde{\Phi}(0) = 0 \tag{28}$$

then $G = -\tilde{\Phi}C_0^* = \begin{pmatrix} q\tilde{\phi}_{12} - c\tilde{\phi}_{11} \\ q\tilde{\phi}_{22} - c\tilde{\phi}_{21} \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ is the stabilizing output injection of the pair (C_0, A_0).

Proof. By duality principle, G is the stabilizing output injection operator of the pair (C_0, A_0) is equivalent that G^* is the stabilizing state-feedback operator of the pair (A_0^*, C_0^*). Hence, G^* can be found through the same idea of Theorem 1, which is based on the solution of the associated operator Riccati equation

$$\left[A_0\tilde{Q}_0 + \tilde{Q}_0A_0^* + M_0 - 2\tilde{Q}_0C_0^*C_0\tilde{Q}_0\right]\tilde{x} \\ = 0, \ \forall \tilde{x} \in D(A_0^*), \ \tilde{Q}_0D(A_0^*) \subset D(A_0)$$

Since the above equation admits a unique nonegative solution, let us assume that the solution is under the form $\tilde{Q}_0 = \tilde{\Phi}_0I$, therefore, by using the expression of the operators A_0 and its adjoint, one can write

$$v\frac{d(\tilde{\Phi}_0\tilde{x})}{d\xi} + M_0\tilde{\Phi}_0\tilde{x} - v\tilde{\Phi}_0\frac{d\tilde{x}}{d\xi} + \tilde{\Phi}_0\tilde{M}^*\tilde{x} + M_0\tilde{x} - \tilde{\Phi}_0C_0^*C_0\tilde{\Phi}_0\tilde{x} = 0$$

which means that $\tilde{\Phi}_0$ is the unique nonnegative solution of the matrix Riccati differential equation (28). Furthermore, the condition $\tilde{\Phi}(0) = 0$ is a result of the inclusion condition $\tilde{Q}_0D(A_0^*) \subset$

$D(A_0)$. Moreover, the fact that $\tilde{\Phi}_0$ is nonnegative implies that $\tilde{Q}_0$ is nonnegative. Indeed, it is a consequence of the fact that

$$\langle \tilde{Q}_0 x, x\rangle = \int_0^1 \tilde{x}^T(\xi)\tilde{\Phi}_0(\xi)x(\xi)d\xi \quad \square$$

Remark 3. Existence and uniqueness of the nonnegative solution of the matrix differential Riccati equation (28) is guaranteed by [36, Theorem 4.1.6] since M_0 and $C_0^*C_0$ are positive definite matrices.

Based on the form of the dynamical controller (27) and also by using the results of Theorems 2 and 3, the error feedback controller can be written explicitly as follows

Corollary 1. *Let us consider the linearized CFRR process with disturbance given by (13). Let n be a positive design function and M_0 a positive definite design matrix. If θ is the unique positive solution of Eqs. (18) and π and κ are the solutions of Eqs. (24)–(25) and the functions $\{\tilde{\phi}_{i,j}\}_{1\le i,j\le 2}$ are the entries of the nonnegative solution of the matrix differential equation (28). Then the error feedback controller given by*

$$\begin{cases} \dot{\hat{x}}(t) = \alpha\hat{x}_\xi + \beta\hat{x} - \gamma\langle\gamma\theta,\hat{x}\rangle + (c\tilde{\phi}_{11} - q\tilde{\phi}_{12})\langle c,\hat{x}\rangle + \hat{w} \\ \qquad\qquad + \gamma\langle\kappa\gamma\theta\pi,\hat{w}\rangle + (q\tilde{\phi}_{12} - c\tilde{\phi}_{11})\langle q,\hat{w}\rangle \\ \qquad\qquad + (q\tilde{\phi}_{12} - c\tilde{\phi}_{11})e(t) \\ \dot{\hat{w}}(t) = \alpha\hat{w}_\xi + \zeta\hat{w} + (q\tilde{\phi}_{22} - c\tilde{\phi}_{21})\left[\langle q,\hat{w}\rangle - \langle c,\hat{x}\rangle + e(t)\right] \\ u(t) = -\langle\gamma\theta,\hat{x}\rangle + \langle\kappa + \gamma\theta\pi,\hat{w}\rangle \end{cases} \tag{29}$$

drives the output of linearized CFRR to track the reference trajectory y_r. Furthermore, the closed-loop system under the error feedback input u is exponentially stable.

Robustness of the error-feedback controller (29) is an important issue. It is known that the regulator problem is based on the internal model principle and also any error-feedback controller which achieves closed-loop stability also achieves robust output regulation if and only if the controller incorporates a suitably reduplicated model of the dynamic structure of the exosystem (see [28]). In Corollary 1, it has been shown that, under some conditions, the error-feedback controller (29) drives the output to track the reference y_r and stabilizes the closed-loop system. Now, let us make a perturbation of the parameters α, β, γ and c of system (13) to $\alpha_p, \beta_p, \gamma_p$ and c_p, which leads to the perturbed system (A_p, B_p, C_p). The main question here is under what conditions the controller (29) solves the error-feedback regulation problem associated with the perturbed system. For this purpose, let us consider the following assumptions:
(A1) The operator A_p generates a C_0-semigroup and the operators B_p, C_p are bounded operators.
(A2) The exponential stability of the closed-loop system is preserved.

In [33], it has been shown that, under some conditions on the perturbed plant, the error-feedback controller (27) robustly regulates and stabilizes the closed-loop system. The following corollary is an immediate consequence of [33, Theorem 9].

Corollary 2. *Assume that the controller (29) is robustly regulating in the sense of [33, Definition 8]. If the conditions (A1) and (A2) are satisfied and there exist π and κ satisfying the perturbed version of Eqs. (24)–(25). Then the error-feedback controller (29) drives the output of the perturbed linear CFRR to track the reference trajectory y_r and exponentially stabilizes the closed-loop system.*

Table 1
Model parameters.

Parameter	Value	Unit
ϵ	0.51	s^{-1}
v_{in}	1	ms^{-1}
M	0.029	kg/mole
k_∞	1.35E5	s^{-1}
R_g	8.314	$J\ mol^{-1}\ K^{-1}$
E	54400	$J\ mol^{-1}$
ρ_g	1240	$kg\ l^{-1}$
ρ_s	1240	$kg\ l^{-1}$
ΔH_r	−802E3	$J\ mol^{-1}$
Cp_f	1066	$J\ kg^{-1}\ K^{-1}$
Cp_s	1020	$J\ kg^{-1}\ K^{-1}$
P	101,325	$J\ m^{-3}$
μ_{eff}	0.1	

5. Case study: Methane lean combustion

Methane is the second most dangerous greenhouse gas and is contributing 20 times more to global warming than carbon dioxide. In order to perform complete oxidation of methane, catalytic combustion is one of the most efficient and promising technologies. Here, the theoretical results developed earlier are to be illustrated through numerical simulations for the methane combustion case study.

$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$$

Remember that the feedback regulators are developed on the basis of the linearized model (9), which describes the plant with one flow direction. The reverse flow direction can be also described by the same model taking into consideration the velocity sign and the boundary condition. This means that the feedback regulators developed earlier can be slightly modified during the reverse flow direction. The main objective is to achieve tracking of the desired output and closed-loop stability under the designed regulators. The values of the plant parameters to be utilized in the numerical simulations are depicted in the Table 1.

Here we consider the exosystem (15) with $\zeta = 1$, which means that w is the solution of the PDE

$$w_t = \alpha w_\xi + w$$

which can be solved easily by the method of characteristics to get the disturbances as follows

$$w(t,\xi) = f(\xi + \alpha t)\exp\left(-\alpha^{-1}\xi\right)$$

where f is an arbitrary function. One disadvantage of the error-feedback controller is the fact that the desired trajectory and the disturbance signals are produced from the same exosystem. However, the function f gives us some degrees of freedom. Indeed, disturbances in fixed-bed reactors are propagated through the reactor-bed as wave-like signals. Therefore, the function f is chosen as a sine function to fit this type of disturbances and also to track a sinusoidal trajectory. If $f(\xi) = \sin(\xi)$, then the solution in this case is given by

$$w(t,\xi) = \sin(\xi + \alpha t)\exp\left(-\alpha^{-1}\xi\right)$$

Now if we choose $q(\xi) = \exp\left(\alpha^{-1}\xi\right)$, then the reference trajectory to be tracked is given by

$$y_r(t) = \langle q, w\rangle = \int_0^1 \sin(\xi + \alpha t)d\xi = \cos(\alpha t) - \cos(1 + \alpha t) \tag{30}$$

To perform numerical simulations for the reverse flow plant, a full cycle of 600 s is considered here. In order to control the temperature in the CFRR, the fluid flow velocity is manipulated at the gas removal location. The stationary state of the CFRR is

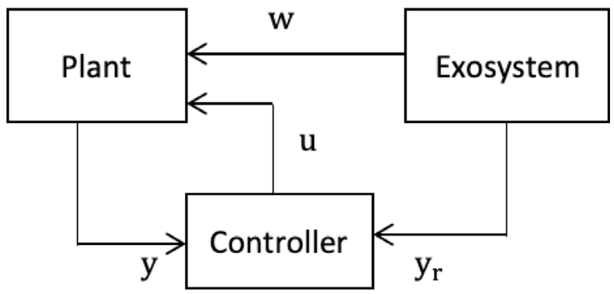

Fig. 1. State-feedback control block diagram.

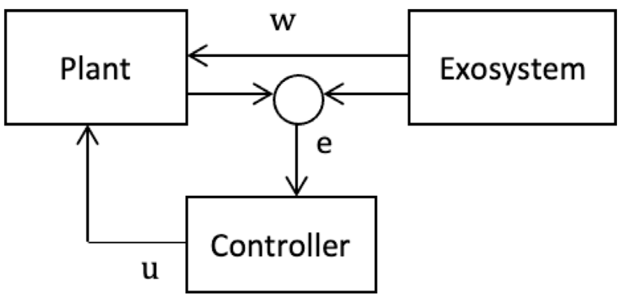

Fig. 2. Error-feedback control block diagram.

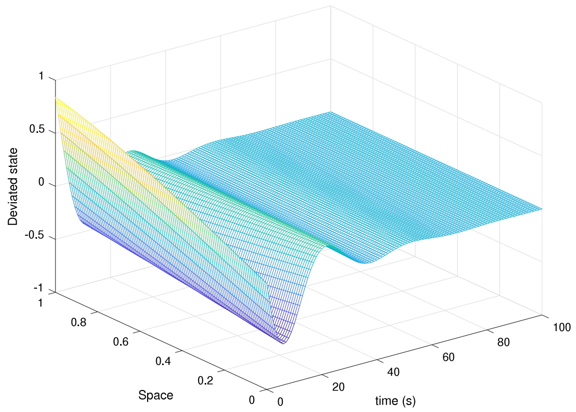

Fig. 3. Left: Closed-loop temperature deviation x under state-feedback regulator.

computed by solving Eq. (7) at stationary state using the inlet values $Y_{in} = 0.03$ and $T_{in} = 298$ K. First let us assess the performances of the state-feedback regulator. For this purpose, Eq. (18) is solved to get the function θ and also Eqs. (22)–(23) are solved to get the functions π and κ. Sampling of 100 points is utilized to discretize the closed-loop equations. The resulting deviated response of the nonlinear closed-loop system is given in Fig. 1. It is easy to observe that the state-feedback regulator stabilizes the system. Moreover, Fig. 2 shows that the output of the closed-loop system tracks the desired output given by (30). On the other hand, to assess the performances of the error-feedback regulator, we need to solve the matrix Riccati differential equation (28) to get the function $\tilde{\phi}$ and then solve the error-feedback controller equations (29). The response of the nonlinear closed-loop system is shown in Fig. 3 and the outputs are shown in Fig. 4. Here again it can observed that the designed error-feedback regulator stabilizes the system and guarantees the tracking of the desired output. (See Figs. 5 and 6.)

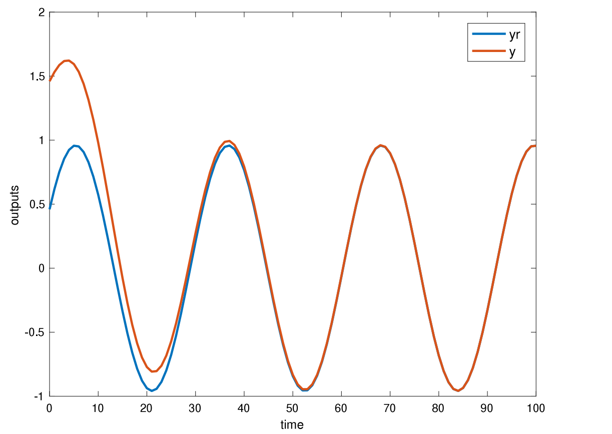

Fig. 4. The output y and the reference trajectory y_r under state-feedback regulator.

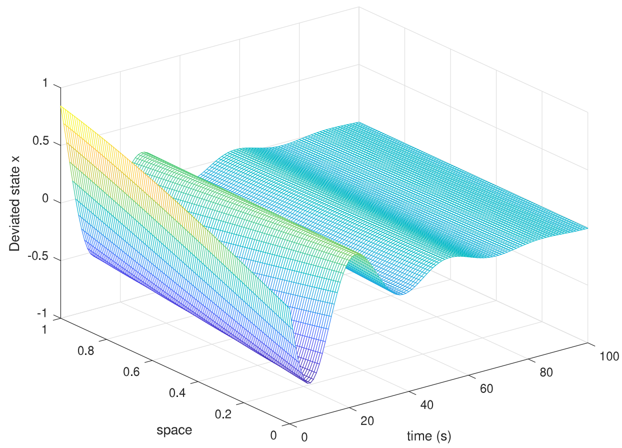

Fig. 5. Left: Closed-loop temperature deviation x under error-feedback regulator.

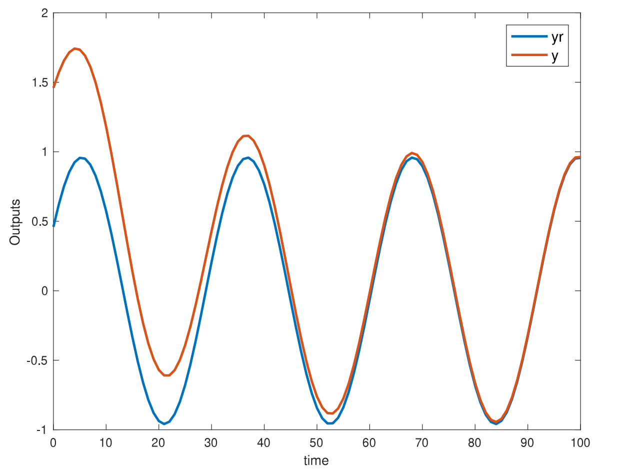

Fig. 6. The output y and the reference trajectory y_r under error-feedback regulator.

6. Conclusion

State feedback and error feedback regulation problems have been solved to control the temperature through gas removal strategy in the catalytic flow reversal reactor. The investigation of these problems is based on the infinite-dimensional linearized version of the process PDE model. Moreover, it is assumed that the disturbances and reference trajectory are generated by a distributed parameter system. The developed regulators has been tested through numerical simulations for the case study of methane combustion.

Extension of the state feedback and error feedback regulation problem for a general class of diffusion–convection–reaction processes is the subject of future investigation.

CRediT authorship contribution statement

Ilyasse Aksikas: Conceptualization, Methodology, Writing – original draft, Investigation, Numerical simulations, Writing – reviewing and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] Y.S. Matros, G.A. Bunimovich, Reverse-flow operation in fixed bed catalytic reactors, Catal. Rev. Sci. Eng. 38 (1) (1996) 1–68.
[2] G. Eigenburger, U. Nieken, Catalytic combustion with periodic flow reversal, Chem. Eng. Sci. 43 (8) (1988) 2109–2115.
[3] P. Marin, F.V. Diez, S. Ordonez, Reverse flow reactors as sustainable devices for performing exothermic reactions: Applications and engineering aspects, Chem. Eng. Process. 135 (2019) 175–189.
[4] L. Pekaf, R. Prokop, Algebraic robust control of a closed circuit heating-cooling system with a heat exchanger and internal loop delays, Appl. Therm. Eng. 113 (2017) 1464–1474.
[5] M. Fratczak, J. Czeczot, P. Nowak, M. Metzger, Practical validation of the effective control of liquid-liquid heat exchangers by distributed parameter balance-based adaptive controller, Appl. Therm. Eng. 129 (2018) 549–556.
[6] A. Maidi, J. Corriou, Chapter17-PDE control of heat exchangers by input-output linearization approach, in: Libor Pekaf (Ed.), Advanced Analytic and Control Techniques for Thermal Systems with Heat Exchangers, Academic Press, 2020, pp. 367–386.
[7] A. Maidi, M. Diaf, J. Corriou, Boundary geometric control of a counter-current heat exchanger, J. Process Control 19 (2) (2009) 297–313.
[8] F. Aubé, H. Sapoundjiev, Mathematical model and numerical simulations of catalytic flow reversal reactors for industrial applications, Comput. Chem. Eng. 24 (12) (2000) 2623–2632.
[9] D. French, J. King, Approximation of an elliptic control problem by the finite element method, Numer. Funct. Anal. Optim. 12 (1991) 299–314.
[10] R. McKnight, W. Bosarge, The Ritz-Galerkin procedure for parabolic control problems, SIAM J. Control 11 (3) (1973) 510–524.
[11] R.F. Curtain, H.J. Zwart, Introduction to Infinite-Dimensional Linear Systems Theory, Springer-Verlag, 1995.
[12] F.M. Callier, J. Winkin, LQ-optimal control of infinite-dimensional systems by spectral factorization, Automatica 28 (4) (1992) 757–770.
[13] I. Aksikas, A. Fuxman, J.F. Forbes, J.J. Winkin, LQ-control design of a class of hyperbolic PDE systems: Application to fixed-bed reactor, Automatica 45 (2009) 1542–1548.
[14] I. Aksikas, A.A. Moghadam, J.F Forbes, Optimal linear–quadratic control of coupled parabolic–hyperbolic PDEs, Internat. J. Control 90 (10) (2017) 2152–2164.
[15] S. Dubljevic, P. Mhaskar, N.H. El-Farra, P. Christofides, Predictive control of transport-reaction processes, Comput. Chem. Eng. 29 (2005) 2335–2345.
[16] H. Pakravesh, I. Aksikas, M. Votsmeier, S. Dubljevic, R.E. Hayes, J.F. Forbes, Characteristics-based model predictive control of selective catalytic reduction in diesel-powered vehicles, J. Process Control 47 (2016) 98–110.
[17] J.M. Coron, A. Hayat, PI controllers for 1-D nonlinear transport equation, IEEE Trans. Automat. Control 64 (11) (2019) 4570–4582.
[18] V. Dos Santos, J.M. Coron, B. d'Andrea Novel, Boundary control with integral action for hyperbolic systems of conservation laws: Stability and experiments, Automatica 44 (5) (2008) 1310–1318.
[19] H. Lhachemi, C. Prieur, E. Trelat, PI regulation of a reaction-diffusion equation with delayed boundary control, IEEE Trans. Automat. Control 66 (4) (2020) 1573–1587.
[20] H. Lhachemi, C. Prieur, Finite-dimensional observer-based PI regulation control of a reaction-diffusion equation, IEEE Trans. Automat. Control (2021) in press.
[21] M. Barreau, F. Gouaisbaut, A. Seuret, Practical stability analysis of a drilling pipe under friction with a PI-controller, IEEE Trans. Control Syst. Technol. 29 (2) (2019) 620–634.
[22] J.M. Schumacher, Finite-dimensional regulators for a class of infinite-dimensional systems, Syst. Control Lett. 3 (1) (1983) 7–12.
[23] X. Xu, S. Dubljevic, Output regulation problem for a class of regular hyperbolic systems, Internat. J. Control 89 (1) (2016) 113–127.
[24] D. Edouard, H. Hammouri, X. Zhou, Control of a reverse flow reactor for VOC combustion, Chem. Eng. Sci. 60 (2005) 1661–1672.
[25] A. Fuxman, I. Aksikas, J.F. Forbes, R.E. Hayes, LQ-feedback control of a reverse flow reactor, J. Process Control 18 (2008) 654–662.
[26] I. Aksikas, Optimal output-feedback temperature regulation of a catalytic reverse flow reactor PDEs model, J. Process Control 103 (2021) 1–7.
[27] B.E. Stangeland, A.S. Foss, Control of a fixed-bed chemical reactor, Ind. Eng. Chem. Fundam. 9 (1) (1970) 38–48.
[28] B.A. Francis, W.M. Wonham, The internal model principle of control theory, Automatica 12 (1976) 457–465.
[29] E.J. Davison, The robust control of a servomechanism problem for linear time-invariant multivariable systems, IEEE Trans. Automat. Control 21 (1) (1976) 25–34.
[30] B.A. Francis, The linear multivariable regulator problem, SIAM J. Control Optim. 15 (3) (1977) 486–505.
[31] C. Byrnes, I.G. Lauko, D.S. Gilliam, V.I. Shubov, Output regulation for linear distributed parameter systems, IEEE Trans. Automat. Control 45 (2000) 2236–2252.
[32] V. Natarajan, D.S. Gilliam, G. Weiss, The state feed- back regulator problem for regular linear systems, IEEE Trans. Automat. Control 59 (2014) 2708–2723.
[33] T. Hamalainen, S. Pohjolainen, Robust regulation of distributed parameter systems with infinite-dimensional exosystems, SIAM J. Control Optim. 48 (8) (2010) 4846–4873.
[34] P.D. Christofides, Nonlinear and Robust Control of PDE Systems: Methods and Applications to Transport-Reaction Processes, Birkhäuser, 2001.
[35] A. Fuxman, Model-Based Control of Catalytic Reactors with Reverse Flow Operation (Ph.D. thesis), University of Alberta, 2008.
[36] H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, Matrix Riccati Equations in Control and Systems Theory, Birkhäuser Verlag, Basel, 2003.