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A Note on Parabolic Maximal Operators along Surfaces of Revolution via Extrapolation

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Abstract: Under weak conditions on the kernels, we obtain sharp L^p bounds for rough parabolic maximal integral operators over surfaces of revolution. By virtue of these bounds along with Yano's extrapolation argument, we confirm the L^p boundedness of these maximal operators under weaker conditions on the kernels. Our obtained results represent substantial extensions and improvements of some known results on maximal operators with rough kernels on symmetric spaces.

Keywords: extrapolation; boundedness; mixed homogeneity; rough kernels; maximal integrals

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1. Introduction

Throughout this work, we assume that $\mathbf{S}^{n-1}$, $n \geq 2$, is the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma_n(\cdot)$. Furthermore, we assume that q' denotes the exponent conjugate to q defined by $1/q' + 1/q = 1$.

For $j \in \{1, 2, \cdots, n\}$, let α_j be fixed real numbers in the interval $[1, +\infty)$. Consider the function $\Psi : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}$ defined by $\Psi(\rho, z) = \sum_{j=1}^{n} \frac{z_j^2}{\rho^{2\alpha_j}}$ with $z = (z_1, z_2, \ldots, z_n) \in \mathbf{R}^n$. For a fixed $z \in \mathbf{R}^n$, we denote the unique solution to the equation $\Psi(\rho, z) = 1$ by $\rho \equiv \rho(z)$. The metric space $(\mathbf{R}^n, \rho)$ is called the mixed homogeneity space related to $\{\alpha_j\}_{j=1}^n$. Let D_ρ with $\rho > 0$ be the diagonal $n \times n$ matrix:

$$D_\rho = \begin{bmatrix} \rho^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \rho^{\alpha_n} \end{bmatrix}.$$

The change of variables regarding the space $(\mathbf{R}^n, \rho)$ is presented as follows:

$$\begin{aligned} z_1 &= \rho^{\alpha_1} \cos\vartheta_1 \cdots \cos\vartheta_{n-2} \cos\vartheta_{n-1}, \\ z_2 &= \rho^{\alpha_2} \cos\vartheta_1 \cdots \cos\vartheta_{n-2} \sin\vartheta_{n-1}, \\ &\vdots \\ z_{n-1} &= \rho^{\alpha_{n-1}} \cos\vartheta_1 \sin\vartheta_2, \\ z_n &= \rho^{\alpha_n} \sin\vartheta_1. \end{aligned}$$

This gives that $dz = \rho^{\alpha-1} J(z') d\rho d\sigma(z')$, where

$$\alpha = \sum_{j=1}^{n} \alpha_j, \quad J(z') = \sum_{j=1}^{n} \alpha_j (z_j')^2, \quad z' = D_{\rho^{-1}} z \in \mathbf{S}^{n-1},$$

and $\rho^{\alpha-1} J(z')$ is the Jacobian of our transformation.

It was proven in [1] that $J(z')$ is in $C^\infty(\mathbf{S}^{n-1})$ and

$$1 \leq J(z') \leq C \ \text{ for some } \ C \geq 1.$$

Let h be a measurable function on $\mathbf{R}^+$ and $\Theta \in L^1(\mathbf{S}^{n-1})$, which satisfies the conditions

$$\int_{\mathbf{S}^{n-1}} \Theta(z')J(z')d\sigma(z') = 0, \tag{1}$$

$$\Theta(D_\rho z) \;=\; \Theta(z), \;\; \forall \rho > 0. \tag{2}$$

For an appropriate function $\varphi : \mathbf{R}^+ \to \mathbf{R}$, we define the class of maximal operators $\mathfrak{M}_{P,\Theta,\varphi}$ initially for C_0^∞ functions on the symmetric space $\mathbf{R}^{n+1}$ by

$$\mathfrak{M}_{P,\Theta,\varphi}(f)(z, z_{n+1}) = \sup_{h \in \mathfrak{L}^2(\mathbf{R}^+)} \left| T_{P,\Theta,h,\varphi}(f)(z, z_{n+1}) \right|, \tag{3}$$

where

$$T_{P,\Theta,h,\varphi}(f)(z, z_{n+1}) = \int_{\mathbf{R}^n} e^{iP(y)} f(z - y, z_{n+1} - \varphi(\rho(y))) \frac{\Theta(y)h(\rho(y))}{\rho(y)^\alpha} dy, \tag{4}$$

$\mathfrak{L}^2(\mathbf{R}^+)$ $(\gamma \geq 1)$ is the set of all $h \in L^2(\mathbf{R}^+, \frac{d\rho}{\rho})$ with $\|h\|_{L^\gamma(\mathbf{R}^+, \frac{d\rho}{\rho})} \leq 1$, and $P : \mathbf{R}^n \to \mathbf{R}$ is a real-valued polynomial.

When $\alpha_1 = \cdots = \alpha_n = 1$, we have $\alpha = n$, $\rho(y) = |y|$ and $(\mathbf{R}^n, \rho) = (\mathbf{R}^n, |\cdot|)$, and hence, we denote $\mathfrak{M}_{P,\Theta,\varphi}$ by $\mathfrak{M}^c_{P,\Theta,\varphi}$. In addition, when $P(y) = 0$ and $\varphi(t) = t$, then $\mathfrak{M}^c_{P,\Theta,\varphi}$ is reduced to be the classical maximal operator $\mathfrak{M}^c_\Theta$, which was introduced by Chen and Lin in [2]. Subsequently, the L^p boundedness of $\mathfrak{M}^c_\Theta$ has received a wide amount of attention by many researchers. For instance, Al-Salman in [3] proved that the operator $\mathfrak{M}^c_\Theta$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $p \geq 2$ provided that $\Theta \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, and he also showed that the condition $\Theta \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is nearly optimal in the sense that $\mathfrak{M}^c_\Theta$ may not be bounded on L^p for any $p \geq 2$ whenever $\Theta \in L(\log L)^\nu(\mathbf{S}^{n-1})$ for some $0 < \nu < 1/2$. In [4], Al-Qassem established the L^p boundedness of $\mathfrak{M}^c_{0,\Theta,\varphi}$ for all $2 \leq p < +\infty$ provided that $\Theta \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ and φ is in $C^2([0, +\infty))$, an increasing and convex function with $\varphi(0) = 0$. For more information regarding the significance and the recent advances of the operators $\mathfrak{M}^c_{P,\Theta,\varphi}$, readers may consult [5–9], as well as the references therein.

Later on, the maximal operator $\mathfrak{M}^c_{P,\Theta,\varphi}$ was introduced in [10] in which the author proved the L^p $(p \geq 2)$ boundedness of $\mathfrak{M}^c_{P,\Theta,\varphi}$ under the conditions $\varphi(t) = t$ and $\Theta \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1}) \cup L(\log L)^{1/2}(\mathbf{S}^{n-1})$ for some $q > 1$. Recently, the result of [10] was improved in [11]. In fact, it was proven that $\mathfrak{M}^c_{P,\Theta,\varphi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $p \geq 2$ provided that $\Theta \in L(\log L)^{1/2}(\mathbf{S}^{n-1}) \cup B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ with $q > 1$ and $\varphi \in C^2(\mathbf{R}^+)$, a convex and increasing function with $\varphi(0) = 0$.

Although there are many problems concerning the L^p boundedness of $\mathfrak{M}^c_{P,\Theta,\varphi}$ that remain open, the investigation to establish the L^p boundedness of the parabolic maximal operators $\mathfrak{M}_{P,\Theta,\varphi}$ has attracted many mathematicians. For example, it was proven in [12] that the operator $\mathfrak{M}_{0,\Theta,\varphi}$ is of type (p, p) for all $p \geq 2$ if $\Theta \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1}) \cup L(\log L)^{1/2}(\mathbf{S}^{n-1})$ for some $q > 1$ and φ is a real-valued polynomial.

In view of the results of [11,12], the L^p boundedness of the maximal operator $\mathfrak{M}_{P,\Theta,\varphi}$ in the classical setting, as well as in the parabolic setting, a question arises naturally: Is the operator $\mathfrak{M}_{P,\Theta,\varphi}$ bounded on L^p under certain conditions on φ and $\Theta \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1}) \cup L(\log L)^{1/2}(\mathbf{S}^{n-1})$?

The main focus of the article is to answer the above question in the affirmative. Our conditions assumed on φ are those considered in [13]. More precisely, we say that φ satisfies the hypothesis **I** whenever φ is a nonnegative C^1 function on $(0, +\infty)$ such that $\varphi'(\rho)$ is monotone and φ is strictly increasing on $(0, +\infty)$; $\varphi(2\rho) \geq C_1\varphi(\rho)$ for some fixed $C_1 > 1$ and $\varphi(2\rho) \leq C_2\varphi(\rho)$ for some constant $C_2 \geq C_1$; $\rho\varphi'(\rho) \geq C_3\varphi(\rho)$ on $(0, +\infty)$

for some fixed $0 < C_3 < \log(C_2)$. We say that φ satisfies the hypothesis **D** whenever φ is a nonnegative C^1 function on $(0, +\infty)$ such that $\varphi'(\rho)$ is monotone and φ is strictly decreasing on $(0, +\infty)$; $\varphi(\rho) \geq C_1\varphi(2\rho)$ for some fixed $C_1 > 1$ and $\varphi(\rho) \leq C_2\varphi(2\rho)$ for some constant $C_2 \geq C_1$; $|\rho\varphi'(\rho)| \geq C_3\varphi(\rho)$ on $(0, +\infty)$ for some fixed $0 < C_3 < \log(C_2)$.

Sample functions for φ to satisfy the hypothesis **D** are $\varphi(\rho) = \rho^{-a}e^{-b\rho}$ for $a \geq 0$ and $b \geq 0$ and for the φ to satisfy the hypothesis **I** are $\varphi(\rho) = \rho^{a}e^{b\rho}$ for $a \geq 0$ and $b \geq 0$.

The main result of this paper is formulated as follows:

Theorem 1. *Let Θ satisfy the conditions (1) and (2) and belong to $L^q(\mathbf{S}^{n-1})$ for some $q > 1$ with $\|\Theta\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. Assume that $\mathfrak{M}_{P,\Theta,\varphi}$ is given by (3) and $\varphi(\cdot)$ satisfies the hypothesis **D** or **I**. Then, there exists a positive constant $C_{p,q}$ such that*

$$\left\|\mathfrak{M}_{P,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C_{p,q}\left(1 + \theta^{1/2}\right)\|f\|_{L^p(\mathbf{R}^{n+1})} \tag{5}$$

for all $2 \leq p < +\infty$, where $\theta = \log(e + \|\Theta\|_{L^q(\mathbf{S}^{n-1})})$, $C_{p,q} = \frac{2^{1/q'}}{2^{1/q'}-1}C_p$, and $C_p > 0$ is a constant that is independent of q, φ, Θ, and the coefficients of the polynomial P; however, it may depend on the degree of P.

Here and henceforth, the letter C refers to a positive constant whose value may vary at each occurrence, but independent of the fundamental variables.

2. Preparation

In this section, we give some preliminary lemmas that we shall need to prove Theorem 1.

Lemma 1. *Let Θ, φ, and θ be given as in Theorem 1. For $j \in \mathbf{Z}$, define $\mathcal{U}_{\varphi,j} : \mathbf{R}^{n+1} \to \mathbf{R}$ by*

$$\mathcal{U}_{\varphi,j}(\zeta, \zeta_{n+1}) = \int_1^{2^{2\theta}} \left| \int_{\mathbf{S}^{n-1}} \Theta(v)J(v)\mathcal{H}_{\varphi,j}(\rho, v, \zeta, \zeta_{n+1})d\sigma(v) \right|^2 \frac{d\rho}{\rho},$$

where

$$\mathcal{H}_{\varphi,j}(\rho, v, \zeta, \zeta_{n+1}) = e^{-i\left[(2^{-(j+1)\theta})D_\rho v\cdot\zeta + \varphi(2^{-(j+1)\theta}\rho)\zeta_{n+1}\right]}.$$

Then, a constant $C > 0$ exists such that

$$\mathcal{U}_{\varphi,j}(\zeta, \zeta_{n+1}) \leq C\theta \min\{\left|D_{2^{-(j+1)\theta}\rho}\zeta\right|^{-\frac{\kappa}{4s\theta}}, \left|D_{2^{-(j+1)\theta}\rho}\zeta\right|^{\frac{\kappa}{4s\theta}}\},$$

where s is denoted to be the distinct numbers of $\{\alpha_j\}$ and $0 < \kappa < 2s/q'$.

Proof. It is easy to see that

$$\mathcal{U}_{\varphi,j}(\zeta, \zeta_{n+1}) \leq C\theta. \tag{6}$$

Furthermore, by using [14], Lemma 2.2, it is easy to obtain

$$\begin{aligned} \left|\int_1^{2^{2\theta}} \mathcal{H}_{\varphi,j,}(\rho, v, \zeta, \zeta_{n+1})\overline{\mathcal{H}_{\varphi,j,}(\rho, w, \zeta, \zeta_{n+1})}\frac{d\rho}{\rho}\right| &\leq C\left|\{D_{2^{-(j+1)\theta}}(v - w)\cdot\zeta\}\right|^{-\frac{1}{4s}} \\ &\leq C\left(\left|(v - w)\cdot\xi\right|\left|D_{2^{-(j+1)\theta}}\zeta\right|\right)^{-\frac{1}{4s}}, \end{aligned} \tag{7}$$

where $\xi = \frac{D_{2^{-(j+1)\theta}}\zeta}{\left|D_{2^{-(j+1)\theta}}\zeta\right|}$. Combining (7) with the trivial estimate:

$$\left|\int_1^{2^{2\theta}} \mathcal{H}_{\varphi,j,}(\rho, v, \zeta, \zeta_{n+1})\overline{\mathcal{H}_{\varphi,j,}(\rho, w, \zeta, \zeta_{n+1})}\frac{d\rho}{\rho}\right| \leq C\theta \tag{8}$$

gives that, for any $\kappa \in (0,1]$,

$$\left| \int_1^{2^{2\theta}} \mathcal{H}_{\varphi,j,}(\rho,v,\zeta,\zeta_{n+1}) \overline{\mathcal{H}_{\varphi,j,}(\rho,w,\zeta,\zeta_{n+1})} \frac{d\rho}{\rho} \right| \le C\left(|(v-w)\cdot\xi||D_{2^{-(j+1)\theta}}\zeta|\right)^{-\frac{\kappa}{4s}} \theta^{1-\kappa}. \quad (9)$$

By Hölder's inequality, we obtain that

$$\begin{aligned} \left(\mathcal{U}_{\varphi,j}(\zeta,\zeta_{n+1})\right)^{q'} &\le C\|\Theta\|^{2q'}_{L^q(\mathbf{S}^{n-1})} \iint_{(\mathbf{S}^{n-1})^2} \left| \int_1^{2^{2\theta}} \mathcal{H}_{\varphi,j,}(\rho,v,\zeta,\zeta_{n+1}) \right. \\ &\times \left. \overline{\mathcal{H}_{\varphi,j,}(\rho,w,\zeta,\zeta_{n+1})} \frac{d\rho}{\rho} \right|^{q'} d\sigma(v) d\sigma(w). \end{aligned}$$

Now, choose $\kappa \in (0, 2s/q')$ to obtain that

$$\mathcal{U}_{\varphi,j}(\zeta,\zeta_{n+1}) \le C\left|D_{2^{-(j+1)\theta}}\zeta\right|^{-\frac{\kappa}{4s}} \|\Theta\|^2_{L^1(\mathbf{S}^{n-1})} \theta^{1-\kappa}.$$

By combining the last estimate with the estimate (6), we deduce that

$$\mathcal{U}_{\varphi,j}(\zeta,\zeta_{n+1}) \le C\theta\left|D_{2^{-(j+1)\theta}}\zeta\right|^{-\frac{\kappa}{4s\theta}}. \quad (10)$$

On the other hand, by using the cancellation condition (1), we obtain that

$$\begin{aligned} \left| \int_{\mathbf{S}^{n-1}} \Theta(v)\mathcal{H}_{\varphi,j}(\rho,v,\zeta,\zeta_{n+1}) J(v) d\sigma(v) \right| &\le C \int_{\mathbf{S}^{n-1}} \left| e^{-i2^{-(j+1)\theta} D_\rho v\cdot\zeta} - 1 \right| |\Theta(v)| d\sigma(v) \\ &\le C\|\Theta\|_{L^1(\mathbf{S}^{n-1})} \left| D_{2^{-(j+1)\theta}\rho}\zeta \right|, \end{aligned}$$

which when combined with the estimate

$$\left| \int_{\mathbf{S}^{n-1}} \Theta(v)\mathcal{H}_{\varphi,j}(\rho,v,\zeta,\zeta_{n+1}) J(v) d\sigma(v) \right| \le C\|\Theta\|_{L^1(\mathbf{S}^{n-1})}$$

leads to

$$\left| \int_{\mathbf{S}^{n-1}} \Theta(v)\mathcal{H}_{\varphi,j}(\rho,v,\zeta,\zeta_{n+1}) J(v) d\sigma(v) \right| \le C\|\Theta\|_{L^1(\mathbf{S}^{n-1})} \left| D_{2^{-(j+1)\theta}\rho}\zeta \right|^{\frac{\kappa}{4s\theta}}.$$

Therefore,

$$\mathcal{U}_{\varphi,j}(\zeta,\zeta_{n+1}) \le C\theta\left|D_{2^{-(j+1)\theta}\rho}\zeta\right|^{\frac{\kappa}{4s\theta}}. \quad (11)$$

Consequently, by (10) and (11), we finish the proof of the lemma. □

The following lemma is from [15]; it will play a significant role in the proof of Theorem 1.

Lemma 2. *Assume that α_j's and v_j's are fixed numbers and that $P_v : \mathbf{R}^+ \to \mathbf{R}^n$ is a function given by $P_v(\rho) = (v_1\rho^{\alpha_1}, \cdots, v_n\rho^{\alpha_n})$. Define the maximal function related to P_v by*

$$\mathfrak{M}_{P_v} f(z) = \sup_{v>0} \frac{1}{v} \int_0^v |f(z - P_v(\rho))| d\rho.$$

Then, there exists a constant $C_p > 0$ (independent of f and $v_j's$) such that

$$\|\mathfrak{M}_{P_v}(f)\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)}$$

for all $1 < p \le +\infty$.

The next lemma can be derived by employing a similar technique used in the proof of [16], Lemma 3.4; we omit the details.

Lemma 3. *Let φ satisfy the hypothesis **D** or **I**, and let M_φ be the maximal function defined on $\mathbf{R}$ by*

$$M_\varphi(f)(t) = \sup_{\mathbf{j}\in\mathbf{Z}} \left| \int_{2^j}^{2^{j+1}} f(t-\varphi(\rho)) \frac{d\rho}{\rho} \right|.$$

Then, for all $p \in (1, +\infty]$, there is a constant $C_p > 0$ such that

$$\|M_\varphi(f)\|_{L^p(\mathbf{R})} \le C_p \|f\|_{L^p(\mathbf{R})}.$$

Now, we are ready to prove the L^p boundedness of the maximal function, which is related to the operator $\mathfrak{M}_{P,\Theta,\varphi}$. Similar approaches utilized in [17], Lemma 3.6, lead to the following result.

Lemma 4. *Suppose that φ is given as in Theorem 1 and $M_{\varphi,v}$ is the maximal function defined on $\mathbf{R}^{n+1}$ by*

$$M_{\varphi,v} f(z, z_{n+1}) = \sup_{j\in\mathbf{Z}} \left| \int_{2^j}^{2^{j+1}} f(z - D_\rho v, z_{n+1} - \varphi(\rho)) \frac{d\rho}{\rho} \right|.$$

Then, for $f \in L^p(\mathbf{R^{n+1}})$ with $p \in (1, +\infty]$, we have

$$\|M_{\varphi,v}(f)\|_{L^p(\mathbf{R}^{n+1})} \le C_p \|f\|_{L^p(\mathbf{R}^{n+1})}.$$

Proof. Let $j \in \mathbf{Z}$, and let μ_j be the measure defined by

$$\widehat{\mu_j}(\zeta, \zeta_{n+1}) = \int_{2^j}^{2^{j+1}} e^{-i(\zeta\cdot D_\rho v + \zeta_{n+1}\varphi(\rho))} \frac{d\rho}{\rho}.$$

Then, we have that

$$M_{\varphi,v}(f)(z, z_{n+1}) = \sup_{\mathbf{j}\in\mathbf{Z}} |\mu_j * f(z, z_{n+1})|.$$

Let ψ be a smooth function with the properties $\widehat{\psi}(\zeta) = 0$ for $\rho(\zeta) \ge 1$ and $\widehat{\psi}(\zeta) = 1$ for $\rho(\zeta) \le \frac{1}{2}$. Let $\psi_\rho(\zeta) = \rho^{-n}\psi(D_{\rho^{-1}}\zeta)$. Define the sequence of measure η_j on $\mathbf{R}^n \times \mathbf{R}$ by

$$\widehat{\eta_j}(\zeta, \zeta_{n+1}) = \widehat{\mu_j}(\zeta, \zeta_{n+1}) - \widehat{\psi_{2^j}}(\zeta)\widehat{\mu_j}(0, \zeta_{n+1})$$

and its corresponding maximal function by

$$\eta^*(f)(z, z_{n+1}) = \sup_{\mathbf{j}\in\mathbf{Z}} |\eta_j * f(z, z_{n+1})|.$$

Therefore, it is easy to obtain that

$$|\widehat{\eta_j}(\zeta, \zeta_{n+1})| \le |A_{2^j} v \cdot \zeta|. \tag{12}$$

However, by using Lemma 1 (see also [18], Lemma 2.4), we obtain that for any $\varepsilon \in (0,1)$,

$$|\widehat{\eta}_j(\zeta,\zeta_{n+1})| \le |A_{2^j}v\cdot\zeta|^{-\varepsilon}. \tag{13}$$

It is clear that

$$\eta^*(f)(z,z_{n+1}) \le \left(\sum_j |\eta_j * f(z,z_{n+1})|^2\right)^{1/2} + C\big((\mathfrak{M}_{\mathcal{P}_v}\otimes id_{\mathbf{R}})\circ\mathcal{V}_\varphi\big)(f(z,z_{n+1})) \tag{14}$$

and

$$M_{\varphi,v}(f)(z,z_{n+1}) \le \left(\sum_j |\eta_j * f(z,z_{n+1})|^2\right)^{1/2} + 2C\big((\mathfrak{M}_{\mathcal{P}_v}\otimes id_{\mathbf{R}})\circ\mathcal{V}_\varphi\big)(f(z,z_{n+1})), \tag{15}$$

where

$$\mathcal{V}_\varphi f(z,z_{n+1}) = \sup_{j\in\mathbf{Z}}\left|\int_{2^j}^{2^{j+1}} f(z,z_{n+1}-\varphi(\rho))\frac{d\rho}{\rho}\right|.$$

The lemma is proven by (12) and (15), Lemmas 2 and 3, and following the bootstrapping argument employed in [17] (see also [19], Proposition 14). □

An important step toward proving Theorem 1 is to prove the following lemma.

Lemma 5. *Let Θ, φ, and θ be given as in Lemma 1. Then, the inequality:*

$$\|\mathfrak{M}_{0,\Theta,\varphi}(f)\|_{L^p(\mathbf{R}^{n+1})} \le C_{p,q}\left(1+\theta^{1/2}\right)\|f\|_{L^p(\mathbf{R}^{n+1})} \tag{16}$$

holds for $2 \le p < +\infty$.

Proof. By the duality,

$$\mathfrak{M}_{0,\Theta,\varphi}(f)(z,z+1) = \left(\int_0^{+\infty}\left|\int_{\mathbf{S}^{n-1}} J(v)f(z-D_\rho v, z_{n+1}-\varphi(\rho))\Theta(v)d\sigma(v)\right|^2\frac{d\rho}{\rho}\right)^{1/2}.$$

Let $\{\mu_j\}_{j\in\mathbf{Z}}$ be a collection of smooth functions on $(0,+\infty)$ satisfying the following:

$$\begin{aligned} \text{supp }\mu_j \quad &\subseteq \quad \mathcal{I}_{j,\theta} = \left[2^{-(j+1)\theta}, 2^{-(j-1)\theta}\right]; \quad \sum_{j\in\mathbf{Z}}\mu_j(\rho) = 1; \\ & 0 \le \mu_j \le 1; \quad \text{and} \quad \left|\frac{d^k\mu_j(\rho)}{d\rho^k}\right| \le \frac{C}{\rho^k}. \end{aligned}$$

Consider the multiplier operators Φ_j defined on $\mathbf{R}^{n+1}$ by

$$\widehat{(\Phi_j f)}(\zeta,\zeta_{n+1}) = \mu_j(\rho(\zeta))\hat{f}(\zeta,\zeta_{n+1}) \quad \textit{for} \quad (\zeta,\zeta_{n+1}) \in \mathbf{R}^n\times\mathbf{R}.$$

Therefore, by Minkowski's inequality, we obtain

$$\mathfrak{M}_{0,\Theta,\varphi}(f)(z,z_{n+1}) \le \sum_{j\in\mathbf{Z}} E_{\Theta,\varphi,j}(f)(z,z_{n+1}) \tag{17}$$

for any $f \in \mathcal{S}(\mathbf{R}^{n+1})$, where

$$E_{\Theta,\varphi,j}(f)(z,z_{n+1}) = \left(\sum_{k\in\mathbf{Z}}\int_{\mathcal{I}_{k,\theta}}\left|\mathcal{S}_{k+j,\rho}f(z,z_{n+1})\right|^2\frac{d\rho}{\rho}\right)^{1/2},$$

and

$$\mathcal{S}_{k,\rho}f(z,z_{n+1}) = \int\limits_{\mathbf{S}^{n-1}} J(v)(\Phi_k f)(z - D_\rho v, z_{n+1} - \varphi(\rho))\Theta(v)d\sigma(v).$$

Thus, to satisfy (16), it is enough to show that there exist positive constants C_p and ϵ so that

$$\|E_{\Theta,\varphi,j}(f)\|_{L^p(\mathbf{R}^{n+1})} \leq C_p\, 2^{-\epsilon|j|}\left(1+\theta^{1/2}\right)\|f\|_{L^p(\mathbf{R}^{n+1})} \tag{18}$$

for all $p \geq 2$. On the one hand, let us estimate the L^2-norm of $E_{\Theta,\varphi,j}(f)$ as follows:

$$\begin{aligned}
\|E_{\Theta,\varphi,j}(f)\|^2_{L^2(\mathbf{R}^{n+1})} &\leq \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}}\int_{\Gamma_{j+k}} \mathcal{U}_{\varphi,j}(\zeta,\zeta_{n+1})\left|\widehat{f}(\zeta,\zeta_{n+1})\right|^2 d\zeta d\zeta_{n+1} \\
&\leq C\,\theta\, 2^{\frac{-\epsilon\kappa|j|}{4s}}\sum_{j\in\mathbf{Z}}\int_{\mathbf{R}}\int_{\Gamma_{k+j}}\left|\widehat{f}(\zeta,\zeta_{n+1})\right|^2 d\zeta d\zeta_{n+1} \\
&\leq C\,\theta\, 2^{\frac{-\epsilon\kappa|j|}{4s}}\|f\|^2_{L^2(\mathbf{R}^{n+1})},
\end{aligned} \tag{19}$$

where $\Gamma_j = \{\zeta \in \mathbf{R}^n : \rho(\zeta) \in \mathcal{I}_{j,\theta}\}$. The last inequality is attained by using Plancherel's theorem, Fubini's theorem, and Lemma 1. Therefore, the inequality (18) is held for $p = 2$ once we choose ϵ small as much as we need.

On the other hand, the L^p-norm of $E_{\Theta,\varphi,j}(f)$ for $2 < p < +\infty$ is estimated as follows: By the duality, there exists $g \in L^{(p/2)'}(\mathbf{R}^{n+1})$ such that $\|g\|_{L^{(p/2)'}(\mathbf{R}^{n+1})} \leq 1$ and

$$\|E_{\Theta,\varphi,j}(f)\|^2_{L^p(\mathbf{R}^{n+1})} = \sum_{k\in\mathbf{Z}}\int\limits_{\mathbf{R}^{n+1}}\int_{\mathcal{I}_{k,\theta}}\left|\mathcal{S}_{k+j,\rho}f(z,z_{n+1})\right|^2\frac{d\rho}{\rho}|g(z,z_{n+1})|dzdz_{n+1}.$$

Hence, thanks to Hölder's inequality and Lemma 4, we obtain

$$\begin{aligned}
\|E_{\Theta,\varphi,j}(f)\|^2_{L^p(\mathbf{R}^{n+1})} &\leq \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^{n+1}}\int\limits_1^{2^{2\theta}}\int_{\mathbf{S}^{n-1}}|\Theta(z)|\left|\Phi_{k+j}f(z,z_{n+1})\right|^2 \\
&\times\left|g(z + D_{2^{-(j+k+1)\theta}\rho}v, z_{n+1} + \varphi(2^{-(j+k+1)\theta}\rho))\right|d\sigma(v)\frac{d\rho}{\rho}dzdz_{n+1} \\
&\leq C\|\Theta\|_{L^1(\mathbf{S}^{n-1})}\left\|M_{\varphi,2^{-(j+k+1)\theta}v}(\widetilde{g})\right\|_{L^{(p/2)'}(\mathbf{R}^{n+1})}\left\|\sum_{j\in\mathbf{Z}}\left|\Phi_{j+k}f\right|^2\right\|_{L^{(p/2)}(\mathbf{R}^{n+1})} \\
&\leq C_p\theta\|\widetilde{g}\|_{L^{(p/2)'}(\mathbf{R}^{n+1})}\left\|\sum_{j\in\mathbf{Z}}\left|\Phi_{j+k}f\right|^2\right\|_{L^{(p/2)}(\mathbf{R}^{n+1})},
\end{aligned}$$

where $\widetilde{g}(u,u_{n+1}) = g(-u,-u_{n+1})$. Thus, by the assumptions on g and the Littlewood-Paley theory, we deduce that

$$\|E_{\Theta,\varphi,j}(f)\|_{L^p(\mathbf{R}^{n+1})} \leq C_p\left(1+\theta^{1/2}\right)\|f\|_{L^p(\mathbf{R}^{n+1})},$$

which when combined with (19) gives that there exists $0 < \epsilon < 1$ such that for all $p \geq 2$,

$$\left\|E_{\Theta,\varphi,k}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C2^{-\epsilon|k|}\left(1+\theta^{1/2}\right)\|f\|_{L^p(\mathbf{R}^{n+1})}. \tag{20}$$

Consequently, by (17), (20), and taking ϵ small enough, we complete the proof of this lemma. □

We end this section with the following result.

Lemma 6. *Let θ, Θ, φ, s, and $\mathcal{H}_{\varphi,j}$ be given as in Lemma 1. Assume that $P(v) = \sum_{|\beta|\le k} \lambda_\beta v^\beta$ is a polynomial of degree $k > 1$ such that $\sum_{|\lambda|=k} |\lambda_\beta| = 1$ and $|v|^k$ is not one of its terms. For $j \in \mathbf{Z}$, we let*

$$\mathcal{G}^P_{\varphi,j}(\rho, v, \zeta, \zeta_{n+1}) = e^{-i\left[P\left((2^{-(j+1)\theta})D_\rho v\cdot\zeta\right)\right]}\mathcal{H}_{\varphi,j}(\rho, v, \zeta, \zeta_{n+1})$$

and

$$\mathcal{S}^P_{\varphi,j}(\zeta, \zeta_{n+1}) = \int_1^{2^{2\theta}} \left| \int_{\mathbf{S}^{n-1}} \Theta(v)J(v)\mathcal{G}^P_{\varphi,j}(\rho, v, \zeta, \zeta_{n+1})d\sigma(v) \right|^2 \frac{d\rho}{\rho}.$$

Then, there exists a constant $C > 0$, so that

$$\sup_{\zeta\times\zeta_{n+1}\in\mathbf{R}^{n+1}} \mathcal{S}^P_{\varphi,j}(\zeta, \zeta_{n+1}) \le C\theta 2^{\frac{(j+1)}{4\tau q'}}.$$

Proof. The proof of this lemma can be obtained by following the same technique employed in the proof of Lemma 1. Therefore, we shall only give a sketch of the proof of this lemma. One can easily deduce the trivial estimate

$$\left| \int_1^{2^{2\theta}} \mathcal{G}^P_{\varphi,j}(\rho, v, \zeta, \zeta_{n+1})\overline{\mathcal{G}^P_{\varphi,j}(\rho, u, \zeta, \zeta_{n+1})}\frac{d\rho}{\rho} \right| \le C\theta \tag{21}$$

and that

$$\begin{aligned} & P\left(2^{-(j+1)\theta}D_\rho v\cdot\zeta\right) + 2^{-(j+1)\theta}D_\rho v\cdot\zeta - P\left(2^{-(j+1)\theta}D_\rho\cdot u\right) - 2^{-(j+1)\theta}D_\rho u\cdot\zeta \\ = \;& 2^{-k(j+1)\theta}\rho^r\left(\sum_{|\beta|=k}\lambda_\beta\left(v^\beta - u^\beta\right)\cdot\zeta\right) + 2^{-(j+1)\theta}D_\rho(v-u)\cdot\zeta + \mathcal{R}, \end{aligned}$$

where $\frac{d^r}{d\rho^r}\mathcal{R} = 0$, $r = \tau k$, and $\tau = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Hence, we obtain that

$$\begin{aligned} & \left| \int_1^{2^{2\theta}} \mathcal{G}^P_{\varphi,j}(\rho, v, \zeta, \zeta_{n+1})\overline{\mathcal{G}^P_{\varphi,j}(\rho, u, \zeta, \zeta_{n+1})}\frac{d\rho}{\rho} \right| \\ \le \;& C_r\left|2^{-(j+1)k\theta}\{P(v) - P(u)\}\cdot\zeta\right|^{-1/r}. \end{aligned} \tag{22}$$

Therefore, we can reach the desired result by imitating the proof of Lemma 1. □

3. Proof of the Main Result

To prove Theorem 1, we follow the similar arguments to those appearing in the proof of [10], Theorem 1.1. Precisely, we use the induction on the degree of the polynomial P. It is clear that when the degree of P equals 0, then by the duality and Lemma 5, we obtain that

$$\left\|\mathfrak{M}_{P,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \le C_p\|f\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right) \tag{23}$$

for $p \ge 2$. Now, when the degree of P equals 1, that is $P(v) = a + \overrightarrow{b}\cdot v$, then set $\omega(v) = e^{-iP(v)}f(v)$. Therefore, by (23), we conclude

$$\begin{aligned} \left\|\mathfrak{M}_{P,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} &\le C_p\|\omega\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right) \\ &\le C_p\|f\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right). \end{aligned} \tag{24}$$

Next, we assume that our result is true for any polynomial whose degree is k or less with $k \geq 1$. Therefore, we need to prove that our result is still true whenever the degree of the polynomial is $k+1$. Without loss of generality, we may assume that $P(v) = \sum_{|\beta| \leq k+1} \lambda_\beta v^\beta$ is a polynomial of degree $k+1$ such that P does not contain $|v|^{k+1}$ as one of its terms and $\sum_{|\beta|=k+1} |\lambda_\beta| = 1$. Let $\{\Upsilon_j\}_{j\in\mathbf{Z}}$ is a collection of smooth functions defined on $(0, +\infty)$ with the following conditions:

$$\begin{aligned} \operatorname{supp} \Upsilon_j \quad &\subseteq \quad \mathcal{I}_{j,\theta} = \left[2^{-(j+1)\theta}, 2^{-(j-1)\theta}\right]; \\ &\sum_{j\in\mathbf{Z}} \Upsilon_j(\rho) = 1; \quad 0 \leq \Upsilon_j \leq 1; \quad \text{and} \quad \left|\frac{d^l \Upsilon_j(\rho)}{d\rho^l}\right| \leq \frac{C_l}{\rho^l}. \end{aligned}$$

Set

$$\Gamma^0(\rho) = \sum_{j=1}^{+\infty} \Upsilon_j(\rho) \quad \text{and} \quad \Gamma^{+\infty}(\rho) = \sum_{j=-\infty}^{0} \Upsilon_j(\rho).$$

Let

$$\mathcal{B}_{P,\Theta,\varphi}(f)(z, z_{n+1}) = \int_{\mathbf{S}^{n-1}} J(v)\, e^{iP(D_\rho v)} f(z - D_\rho v, z_{n+1} - \varphi(\rho(v)))\Theta(v) d\sigma(v),$$

$$\mathfrak{M}^0_{P,\Theta,\varphi}(f)(z, z_{n+1}) = \left(\int_0^1 \left|\Gamma^0(\rho)\mathcal{B}_{P,\Theta,\varphi}(f)(z, z_{n+1})\right|^2 \frac{d\rho}{\rho}\right)^{1/2}$$

and

$$\mathfrak{M}^{+\infty}_{P,\Theta,\varphi}(f)(z, z_{n+1}) = \left(\int_{2^{-\theta}}^{+\infty} \left|\Gamma^{+\infty}(\rho)\mathcal{B}_{P,\Theta,\varphi}(f)(z, z_{n+1})\right|^2 \frac{d\rho}{\rho}\right)^{1/2}.$$

Thanks to the generalized Minkowski's inequality, we obtain

$$\begin{aligned} \mathfrak{M}_{P,\Theta,\varphi}(f)(z, z_{n+1}) \quad &\leq \quad \mathfrak{M}^0_{P,\Theta,\varphi}(f)(z, z_{n+1}) + \mathfrak{M}^{+\infty}_{P,\Theta,\varphi}(f)(z, z_{n+1}) \\ &\leq \quad \mathfrak{M}^0_{P,\Theta,\varphi}(f)(z, z_{n+1}) + \sum_{j=-\infty}^{0} \mathfrak{M}^{+\infty}_{P,\Theta,\varphi,j}(f)(z, z_{n+1}), \end{aligned} \tag{25}$$

where

$$\mathfrak{M}^{+\infty}_{P,\Theta,\varphi,j}(f)(z, z_{n+1}) = \left(\int_{\mathcal{I}_{j,\theta}} \left|\mathcal{B}_{P,\Theta,\varphi}(f)(z, z_{n+1})\right|^2 \frac{d\rho}{\rho}\right)^{1/2}.$$

On one side, we estimate the L^2-norm of $\mathfrak{M}^{+\infty}_{P,\Theta,\varphi,j}(f)$ as follows: by Fubini's theorem, Plancherel's theorem, and Lemma 6, we deduce

$$\begin{aligned} \left\|\mathfrak{M}^{+\infty}_{P,\Theta,\varphi,j}(f)\right\|_{L^2(\mathbf{R}^{n+1})} \quad &= \quad \left(\int_{\mathbf{R}^{n+1}} \left|\hat{f}(\zeta, \zeta_{n+1})\right|^2 \mathcal{S}^P_{\varphi,j}(\zeta, \zeta_{n+1}) d\zeta d\zeta_{n+1}\right)^{1/2} \\ &\leq \quad C 2^{\frac{(j+1)}{8\tau q'}} \|f\|_{L^2(\mathbf{R}^{n+1})} \left(1 + \theta^{1/2}\right). \end{aligned} \tag{26}$$

On the other side, we estimate the L^p-norm of $\mathfrak{M}^{+\infty}_{P,\Theta,\varphi,j}(f)$ (for $p > 2$) as follows: by the duality, we obtain that there is a function ϕ that belongs to $L^{(p/2)'}(\mathbf{R}^{n+1})$ such that $\|\phi\|_{L^{(p/2)'}(\mathbf{R}^{n+1})} = 1$ and

$$\begin{aligned}\left\|\mathfrak{M}^{+\infty}_{P,\Theta,\varphi,j}(f)\right\|^2_{L^p(\mathbf{R}^{n+1})} &= \int_{\mathbf{R}^{n+1}} |\phi(u,u_{n+1})| \int_1^{2^{2\theta}} \left| \int_{\mathbf{S}^{n-1}} \mathcal{G}^P_{\varphi,j}(\rho,u,0,0)J(v) \right. \\ &\times \left. f(u - D_{2^{-(j+1)\theta}\rho}v, u_{n+1} - \varphi(2^{-(j+1)\theta}\rho))d\sigma(v)\right|^2 \frac{d\rho}{\rho} du du_{n+1}.\end{aligned}$$

By following the same steps used in estimating the L^p-norm of $E_{\Theta,\varphi,j}(f)$ in Lemma 5, we immediately obtain that

$$\left\|\mathfrak{M}^{+\infty}_{P,\Theta,\varphi,j}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right),$$

for which when combined with (26), we conclude that there exists $\iota \in (0,1)$ such that

$$\left\|\mathfrak{M}^{+\infty}_{P,\Theta,\varphi,j}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p 2^{\iota\frac{(j+1)}{8\tau q'}}\|f\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right) \tag{27}$$

for all $p \geq 2$. Now, let us estimate $\left\|\mathfrak{M}^{0}_{P,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})}$ with $p \geq 2$. Take $Q(v) = \sum_{|\beta|\leq k} \lambda_\beta v^\beta$. Therefore, by Minkowski's inequality, we obtain that

$$\mathfrak{M}^0_{P,\Theta,\varphi}(f)(z,z_{n+1}) \leq \mathfrak{M}^0_{Q,\Theta,\varphi}(f)(z,z_{n+1}) + \mathfrak{M}^0_{P,Q,\Theta,\phi}f(z,z_{n+1}), \tag{28}$$

where

$$\mathfrak{M}^0_{Q,\Theta,\varphi}(f)(z,z_{n+1}) = \left(\int_0^1 |\mathcal{B}_{Q,\Theta,\varphi}(f)(z,z_{n+1})|^2\frac{d\rho}{\rho}\right)^{1/2}$$

and

$$\mathfrak{M}^0_{P,Q,\Theta,\varphi}f(z,z_{n+1}) = \left(\int_0^1 |\mathcal{B}_{P,\Theta,\varphi}(f)(z,z_{n+1}) - \mathcal{B}_{Q,\Theta,\varphi}(f)(z,z_{n+1})|^2\frac{d\rho}{\rho}\right)^{1/2}.$$

Since the degree of the polynomial $Q(v)$ is less than or equal to k, then we have that

$$\left\|\mathfrak{M}^{0}_{Q,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right) \tag{29}$$

for all $p \geq 2$; since

$$\left|e^{iP(D_\rho v)} - e^{iQ(D_\rho v)}\right| \leq |P(D_\rho v) - Q(D_\rho v)| \leq \rho^{\tau(k+1)},$$

then by the Cauchy–Schwartz inequality, we obtain that

$$\begin{aligned}\mathfrak{M}^0_{P,Q,\Theta,\varphi}(f)(z,z_{n+1}) \leq \\ & C\left(\int_0^1 \int_{\mathbf{S}^{n-1}} \rho^{2\tau(k+1)}|\Theta(u)||f(z-D_\rho v, z_{n+1}-\varphi(\rho))|^2 d\sigma(v)\frac{d\rho}{\rho}\right)^{1/2} \\ \leq & \left(\sum_{j=1}^{+\infty} 2^{-j(2\tau(k+1))}\int_{2^{-j}}^{2^{-j+1}}\int_{\mathbf{S}^{n-1}} |\Theta(v)||f(z-D_\rho v, z_{n+1}-\varphi(\rho))|^2 d\sigma(v)\frac{d\rho}{\rho}\right)^{1/2} \\ \leq & C\left(\mathfrak{M}_{\varphi,v}\left(|f|^2\right)\right)^{1/2}.\end{aligned}$$

Hence, by Lemma 4, we obtain that

$$\left\|\mathfrak{M}^0_{P,Q,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p \left\||f|^2\right\|^{1/2}_{p/2} \leq C_p\|f\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right) \tag{30}$$

for all $p \geq 2$. Therefore, the inequalities (28)–(30) lead to

$$\left\|\mathfrak{M}^0_{P,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right). \tag{31}$$

Consequently, by (25) and (27) together with (31), we conclude that

$$\left\|\mathfrak{M}_{P,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p\|f\|_{L^p(\mathbf{R}^{n+1})}\left(1+\theta^{1/2}\right). \tag{32}$$

4. Further Results

For $\gamma \in [1,+\infty)$, let $\mathfrak{L}^\gamma(\mathbf{R}^+)$ be the class of all functions $h : \mathbf{R}^+ \to \mathbf{R}$, which are measurable and satisfy that

$$\|h\|_{L^\gamma(\mathbf{R}^+,\frac{d\rho}{\rho})} = \left(\int_0^{+\infty}|h(\rho)|^\gamma\frac{d\rho}{\rho}\right)^{1/\gamma} \leq 1,$$

and let $\mathfrak{L}^\infty(\mathbf{R}^+) = L^\infty(\mathbf{R}^+,\frac{d\rho}{\rho})$.

It is obvious that $\mathfrak{L}^{\gamma_1}(\mathbf{R}^+) \subseteq \mathfrak{L}^{\gamma_2}(\mathbf{R}^+)$ for $1 \leq \gamma_2 \leq \gamma_1 \leq +\infty$.

In this section, we establish some further results. Consider the maximal operator:

$$\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}(f)(z,z_{n+1}) = \sup_{h\in\mathfrak{L}^\gamma(\mathbf{R}^+)}\left|T_{P,\Theta,h,\varphi}(f)(z,z_{n+1})\right|.$$

The first result of this section is the following:

Theorem 2. *Let $\Theta \in L^q(\mathbf{S}^{n-1})$, $q > 1$ and satisfy the conditions (1) and (2) with $\|\Theta\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. Let P and φ be given as in Theorem 1. Then,*

$$\left\|\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \leq C_{p,q}\left(1+\theta^{1/\gamma'}\right)\|f\|_{L^p(\mathbf{R}^{n+1})} \tag{33}$$

for all $p \in [\gamma',+\infty)$ with $\gamma \in (1,2]$, and

$$\left\|\mathfrak{M}^{(1)}_{P,\Theta,\varphi}(f)\right\|_{L^\infty(\mathbf{R}^{n+1})} \leq C\|f\|_{L^\infty(\mathbf{R}^{n+1})}. \tag{34}$$

Proof. Notice that when $\gamma = 2$, we have $\mathfrak{M}^{(2)}_{P,\Theta,\varphi} = \mathfrak{M}_{P,\Theta,\varphi}$. Hence, by Theorem 1, the inequality (33) is satisfied for all $2 \leq p < +\infty$. Next, when $\gamma = 1$, we have $h \in L^1(\mathbf{R}^+,\frac{d\rho}{\rho})$ and $f \in L^\infty(\mathbf{R}^{n+1})$. This gives that

$$\left|\int_{\mathbf{R}^+}\mathcal{B}_{P,\Theta,\varphi}(f)(z,z_{n+1})h(\rho)\frac{d\rho}{\rho}\right| \leq C\|f\|_{L^\infty(\mathbf{R}^{n+1})}\|h\|_{L^1(\mathbf{R}^+,\frac{d\rho}{\rho})}$$

for all $(z,z_{n+1}) \in \mathbf{R}^{n+1}$. Thus, when we take the supermom on both sides over all h with $\|h\|_{L^1(\mathbf{R}^+,\frac{d\rho}{\rho})} \leq 1$, we obtain that

$$\mathfrak{M}^{(1)}_{P,\Theta,\varphi}(f)(z,z_{n+1}) \leq C\|f\|_{L^\infty(\mathbf{R}^{n+1})}$$

for almost everywhere $(z, z_{n+1}) \in \mathbf{R}^n \times \mathbf{R}$. Therefore,

$$\left\| \mathfrak{M}^{(1)}_{P,\Theta,\varphi}(f) \right\|_{L^\infty(\mathbf{R}^{n+1})} \leq C\|f\|_{L^\infty(\mathbf{R}^{n+1})}. \tag{35}$$

Finally, when $\gamma \in (1,2)$, we obtain by the duality that

$$\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}(f)(z, z_{n+1}) = \left(\int_{\mathbf{R}^+} \left| \mathcal{B}_{P,\Theta,\varphi}(f)(z, z_{n+1}) \right|^{\gamma'} \frac{d\rho}{\rho} \right)^{1/\gamma'}.$$

Therefore,

$$\left\| \mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}(f) \right\|_{L^p(\mathbf{R}^{n+1})} = \left\| \mathcal{B}_{P,\Theta,\varphi}(f) \right\|_{L^p(L^{\gamma'}(\mathbf{R}^+, \frac{d\rho}{\rho}), \mathbf{R}^{n+1})}.$$

Consequently, by utilizing the interpolation theorem for the Lebesgue mixed normed spaces to the inequalities (5) and (35), we instantly acquire (33) and (34). The proof is complete. □

Again, when $\alpha_1 = \cdots = \alpha_n = 1$, we denote $\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}$ by $\mathfrak{M}^{(\gamma),c}_{P,\Theta,\varphi}$, and when $P(y) \equiv 0$ and $\varphi(t) = t$, we denote $\mathfrak{M}^{(\gamma),c}_{P,\Theta,\phi}$ by $\mathfrak{M}^{(\gamma),c}_{\Theta}$. Let us recall some results related to these operators. Historically, the investigation to obtain the L^p boundedness of $\mathfrak{M}^{(\gamma),c}_{\Theta}$ was started in [2], in which the authors proved that if $\Theta \in \mathcal{C}(\mathbf{S}^{n-1})$ and $h \in \mathfrak{L}^\gamma(\mathbf{R}^+)$ with $1 \leq \gamma \leq 2$, then the L^p boundedness of $\mathfrak{M}^{(\gamma),c}_{\Theta}$ holds for $(n\gamma)' < p < +\infty$. Later on, Al-Qassem improved this result in [4], who showed that if $\Theta \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ and φ is $C^2([0, +\infty))$, an increasing and convex function with $\varphi(0) = 0$, then $\mathfrak{M}^{(\gamma),c}_{0,\Theta,\varphi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for any $p \in [\gamma', +\infty)$ with $\gamma \in (1,2]$ and bounded on $L^\infty(\mathbf{R}^{n+1})$ for $\gamma = 1$. Very recently, Ali and Al-Mohammed in [11] established the $L^p(\mathbf{R}^{n+1})$ boundedness of $\mathfrak{M}^{(\gamma),c}_{P,\Theta,\varphi}$ for any $p \geq \gamma'$ with $\gamma \in (1,2]$ provided that Θ is in the space $L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1}) \cup B^{(0,-1/\gamma)}_q(\mathbf{S}^{n-1})$ with $q > 1$ and φ is given as in [4].

On the other side, the investigation of the L^p boundedness of the parabolic maximal operators $\mathfrak{M}^{(\gamma)}_{P,\Theta,\phi}$ was started in [20]. In fact, the authors of [20] obtained that the operator $\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $p \in [\gamma', +\infty)$ with $\gamma \in (1,2]$ whenever $\varphi(t) = t$ and Θ belongs to the space $B^{(0,-1/\gamma)}_q(\mathbf{S}^{n-1})$ or belongs to the space $L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$. Afterward, the L^p boundedness of $\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}$ under varied conditions on the kernels has received attention by many authors. For recent advances on the study of such operators, the readers are referred to [12,18,21] and the references therein.

By using the conclusion of Theorem 2 and employing Yano's extrapolation argument (see also [10,11]), we obtain the following theorem, which improves and extends the results cited above.

Theorem 3. *Assume that P, φ, and θ are given as in Theorem 1. Let Θ be in $L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ or in $B^{(0,-1/\gamma)}_q(\mathbf{S}^{n-1})$ with $q \in (1, +\infty]$. Then, $\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $p \in [\gamma', +\infty)$ with $\gamma \in (1,2]$; it is bounded on $L^\infty(\mathbf{R}^{n+1})$ for $\gamma = 1$.*

In this article, we are also interested in studying the L^p boundedness of the parabolic singular integral operator $T_{P,\Theta,h,\varphi}$ under certain conditions on the kernels. This operator was first studied by Fabes and Rivière in [1], who showed that if $\Theta \in C^1(\mathbf{S}^{n-1})$, then $T_{0,\Theta,1,\rho}$ ($\varphi(t) = t$, $h \equiv 1$, and $P(y) = 0$) is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < +\infty$. Later on, Nagel Rivière and Wainger improved this result in [22]. In fact, they proved that $T_{0,\Theta,1,\rho}$ is still bounded on $L^p(\mathbf{R}^n)$ for $1 < p < +\infty$ whenever the assumption $\Theta \in C^1(\mathbf{S}^{n-1})$ is replaced by a weaker condition $\Theta \in L(\log L)(\mathbf{S}^{n-1})$. Under certain conditions on the kernels, a considerable amount of research has been performed to prove the L^p boundedness of $T_{P,\Theta,h,\varphi}$; we refer the readers to see [12,17,19,23–25], among others.

By Theorem 3, we obtain the L^p boundedness of the integral operator $T_{P,\Theta,h,\varphi}$, where the range of p is the full range $(1,+\infty)$ whenever $1<\gamma\leq 2$. This result is formulated as follows:

Theorem 4. *Let Θ, φ, and P be given as in Theorem 3. Suppose that $h\in\mathfrak{L}^{\gamma}(\mathbf{R}^{+})$ with $\gamma\in(1,2]$. Then, the singular integral operator $T_{P,\Theta,h,\varphi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $1<p<+\infty$.*

Proof. As an immediate consequence of Theorem 3 and the fact:

$$\left|T_{P,\Theta,h,\varphi}(f)(z,z_{n+1})\right|\leq\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}(f)(z,z_{n+1})\|h\|_{L^{\gamma}(\mathbf{R}^{+},\frac{d\rho}{\rho})},\tag{36}$$

we conclude that $T_{P,\Theta,h,\varphi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for $\gamma'\leq p<+\infty$ with $\gamma\in(1,2]$. Furthermore, by a standard duality argument, one can easily establish the L^p boundedness of $T_{P,\Theta,h,\varphi}$ for $1<p\leq\gamma$ with $\gamma\in(1,2]$. Hence, whenever $\gamma=2$, we are done. However, when $\gamma\in(1,2)$, then the real interpolation theorem gives that $T_{P,\Theta,h,\varphi}$ is bounded on L^p $(\gamma<p<\gamma')$. This completes the proof. □

Let us present a new rough integral operator, which is related to the maximal operator $\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}$; it is the generalized parabolic Marcinkiewicz integral operator given by

$$\begin{aligned}&\mu^{(\gamma)}_{P,\Theta,\varphi}(f)(z,z_{n+1})\\ =&\left(\int_{\mathbf{R}^{+}}\left|\frac{1}{\delta}\int_{\rho(y)\leq\delta}e^{iP(y)}f(z-w,z_{n+1}-\varphi(\rho(w)))\Theta(w)(\rho(w))^{-\alpha+1}dw\right|^{\gamma'}d\delta\right)^{1/\gamma'}.\end{aligned}\tag{37}$$

Since, for any $1\leq\gamma\leq 2$,

$$\mu^{(\gamma)}_{P,\Theta,\varphi}(f)(z,z_{n+1})\leq C\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}(f)(z,z_{n+1}),$$

we obtain the following result.

Theorem 5. *Suppose that Θ, φ, and P are given as in Theorem 3, and suppose that $\mu^{(\gamma)}_{P,\Theta,\varphi}$ is given as in (37) for some $\gamma\in[1,2]$. Then, the integral operator $\mu^{(\gamma)}_{P,\Theta,\varphi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $p\in[\gamma',+\infty)$ with $\gamma\in(1,2]$, and it is bounded in $L^{\infty}(\mathbf{R}^{n+1})$ for $\gamma=1$.*

We point out that under some specific constraints, the operator $\mu^{(\gamma)}_{P,\Theta,\varphi}$ was investigated in [11,12,26–30].

5. Conclusions

In this paper, we established appropriate L^p estimates for the parabolic operator $\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}$ whenever Θ is in $L^q(\mathbf{S}^{n-1})$. These estimates were used with Yano's extrapolation argument to satisfy the L^p boundedness of $\mathfrak{M}^{(\gamma)}_{P,\Theta,\varphi}$ under weaker conditions imposed on the integral kernels. Then, we presented some results that came from this result. Precisely, we obtained the boundedness of the parabolic singular integral $T_{P,\Theta,h,\varphi}$, as well as the generalized parabolic Marcinkiewicz integral operator $\mu^{(\gamma)}_{P,\Theta,\varphi}$ under very weak assumptions on the kernels.

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