IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received September 27, 2018, accepted November 27, 2018, date of publication December 4, 2018,
date of current version December 31, 2018.

Digital Object Identifier 10.1109/ACCESS.2018.2885014

Bayesian Monitoring of Linear Profiles Using DEWMA Control Structures With Random *X*

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The work of S. A. Abbasi was supported by the Qatar University under Project QUST-1-CAS-2018-41.

ABSTRACT The process structures of manufacturing industry are efficiently modeled using linear profiles. Classical and Bayesian set-ups are two well-appreciated schemes for designing control charts for the monitoring of process structures. Mostly in profiles monitoring the independent variables along with the process parameters are assumed fixed. There are manufacturing processes where these conditions may not hold. The advancement in technology and day-to-day changes in process structures caused the parametric uncertainty along with variability in explanatory variables. This paper considered the case of random *X* and assumes different conjugate and non-conjugate priors to handle parametric uncertainty using double exponentially weighted moving average (DEWMA) control charts. Three univariate DEWMA charts are designed for the monitoring of *Y*-intercepts, slopes, and error variances. The average run length criterion has been used to evaluate the proposed and competing charts. The wide spread relative study identifies that the proposed Bayesian DEWMA control charts are better than the competing charts based on early detection of out-of-control profiles, particularly for smaller value shifts. The Bayesian DEWMA charts using conjugate priors are the quickest in all as they take less sample points to show out-of-control profile. A case study has been considered to further justify the superiority of Bayesian DEWMA charts over competing charts.

INDEX TERMS DEWMA, linear profiles, priors, posteriors, run length measures.

I. INTRODUCTION

Statistical Process Control (SPC) community has made significant contributions to developed scientifically equipped statistical models to enhance the efficiency of process structures in industrial engineering. The growing scientific innovations required from statisticians with more assorted and complex problems to emphases on better quality of statistical models by extracting more and more information about product or process Stoumbos *et al.* [1]. They described in detail that with more emphasis on better quality and by accessing more and more information about process parameters, we may face with more complex and diverse models. The SPC community is continuously working and producing sophisticated process structures for these complex models. The process structures derived to model the quality characteristics of interest as well as constructing the corresponding control charts to check the stability of the process parameters. The quality characteristics of interest can be defined either by probability distribution or by using profiles model. In many practical situations of manufacturing industry the quality characteristics are categorized with the association between dependent and one or set of independent variables, this is called the profile function of quality characteristics Kim *et al.* [2]. Noorossana *et al.* [3] described situation of profile function when thickness of tape is measured at randomly selected locations. Abbas *et al.* [4] defined process structures by using linear profiles model when Outlet Concentration of the Product (CA in Kmole/m^3) is associated with Inlet Concentration of Solvent Flow (CAS in Kmole/m^3). Linear profiles are also used in electrical engineering such as in photo-voltaic systems which are an effective source to gain the energy from sun; the monitoring of the system voltage is inversely proportional to capacitance Riaz *et al.* [5].

In the literature, researchers are trying to produce different techniques by using different control charting structures for the monitoring of linear profiles under phase I or II. Kim *et al.* [2] suggested the use of three separate univariate Exponentially Weighted Moving Average (EWMA) control charts after taking average deviation of explanatory variable of simple linear profiles model to overcome the problem of interpretability of shifts in process parameters. Noorossana *et al.* [6] constructed a new method by using jointly the Multivariate Cumulative Sum (MCUSUM) and R control charts to monitor the simple linear profiles. Zou *et al.* [7] and Mahmoud *et al.* [8] developed change point method and used EWMA control charts for linear profiles monitoring when the process parameters of Y-intercepts, slopes and errors variances are assumed to be unknown. A method based on CUSUM control charts after taking deviation from mean of explanatory variable of simple linear profiles has been proposed by Saghaei *et al.* [9]. Noorossana *et al.* [10], [11] considered multivariate simple linear profiles using Multivariate EWMA (MEWMA) control charts in phase I and II for the monitoring of multivariate profiles function. Mahmoud *et al.* [12] considered the case when each profile cannot exceed a sample size of two and later Yeh and Zerehsaz [13] monitored the simple linear profiles in phase I when each profile consists single observation. The case of within profile correlation for the monitoring of simple linear profiles studied by Zhang *et al.* [14] by using the normal distribution models. Chen *et al.* [15] checked the deleterious impact of false phase I estimation of process parameters on phase II performance evaluation while considering the case of simple linear profiles. Zhang *et al.* [16] proposed EWMA control charting structures based on score test to detect pre specified quadratic changes while monitoring the simple linear profiles. With all these parametric approaches, a nonparametric approach is widely acknowledged in literature where less weight is given to the distributions of process parameters. The nonparametric profile monitoring is useful when it is too complicated to define the relationship parametrically (cf. Qiu and Zou [17]; Qiu *et al.* [18]; Zhang *et al.* [19]; Pacella *et al.* [20]).

In almost all the literature of earlier studies the explanatory variables are not random which reflect a case of specified values for it. They also assumed fixed process parameters which mean that its estimated values are unique. However, there are process structures where these two conditions may not hold i.e., the example provided by Noorossana *et al.* [3]. In this case we observed the thickness of tape at four random locations to define a profile function, while we may assume different process parameter values with the possible changes in process structures to meet day to day requirements. Bayesian approach is used to handle this parametric uncertainty and provide more realistic finding and conclusions. The main advantage of Bayesian approach is that it assists the investigator to represent and take full description of the uncertainties related to parameter values. In contrast, the decision based on maximum likelihood estimation involves fixing the values of parameters. This may be an important bearing on final results of the analysis when there is considerable uncertainty in parameters. Under these circumstances the Bayesian techniques provide novel diagnostic tool for under study parameters. This Bayesian approach is advanced and elegant statistical tool which described the under study parameters as uncertain Zellnar [21] and probability approach is best way to express uncertainty. In this study the parametric uncertainty of process structures are resolved by constructing Bayesian control charts. This parametric uncertainty is expressed by using prior distributions while its parameters are called as hyperparameters. The SPC community regularly using Bayesian control charts to adjust this parametric uncertainty and comes up with flexible and demanding control structures to produce reliable products. The literature on Bayesian control charts widely available (cf. Hamada [22]; Triantafyllopoulos [23]; Marcellus [24]; Marcellus [25]; Hassan *et al.* [26]; Pan and Rigdon [27]; Demirhan and Hamurkaroglu [28]; Ali *et al.* [29]; Raubenheimer and Van der Merwe [30]).

In this article, we have evaluated the parametric uncertainty by designing Bayesian Double EWMA (DEWMA) control charts while explanatory variables of linear profiles models are not fixed. The parametric uncertainty is expressed by using different conjugate and non-conjugate priors. Three separate univariate DEWMA control charts are designed after taking average deviation from each value of independent variables. Due to the varying nature of explanatory variables we have standardized the slope estimators to incorporate this variability.

The remaining article is organized as follows: Section 2 presents the simple linear profiles model in deviation form and its estimation procedures. Section 3 presents the Bayesian control charting structure of DEWMA control charts. Section 4 describes the performance measures and simulation settings. The elicitation and sensitivity analysis of hyper-parameters is performed in section 5. Section 6 demonstrates the evaluation of proposed charts and comparison with competing charts. Section 7 comes up with case study while section 8 presents the conclusions.

II. BAYESIAN ESTIMATION OF SIMPLE LINEAR PROFILES MODEL

In this section the linear profiles model and its classical and Bayesian estimation procedures are described. Let us consider a simple linear profiles model as:

$$Y_{ij} = \alpha_{0j} + \alpha_{1j} X_{ij} + \varepsilon_{ij}, \quad i = 1, 2, 3, \ldots, n \tag{1}$$

Here α_{0j} and α_{1j} are specified in-control process parameters for j^{th} profile. The explanatory variables X_{ij} are generated from normal distribution and error terms ε_{ij} are from standard normal distribution for j^{th} profile. Let us consider deviation from mean of explanatory variables for model in Equation (1) to defined separate EWMA control charts for process parameters Kim *et al.* [2].

$$Y_{ij} = \beta_{0j} + \beta_{1j} X^*_{ij} + \varepsilon_{ij}, \tag{2}$$

Where $\beta_{0j} = \alpha_{0j} + \alpha_{1j}\bar{X}_j$, $\beta_{1j} = \alpha_{1j}$, and $X^*_{ij} = (X_{ij} - \bar{X}_j)$. Then the least square estimators of profiles model in Equation (2) follows normal distribution with estimates as: $\hat{\beta}_{1j} = S_{x(j)y(j)}/S_{x(j)x(j)}$, and $\hat{\beta}_{0j} = \bar{Y}_j$ while variances as: $Var(\hat{\beta}_{1j}) = \sigma_j^2/S_{x(j)x(j)}$, and $Var(\hat{\beta}_{0j}) = \sigma_j^2/n_j$. Here, $\bar{Y}_j = \sum_{i=1}^{n_j} Y_{ij}/n_j$, $S_{x(j)y(j)} = \sum_{i=1}^{n_j}(X_{ij}-\bar{X}_j)Y_{ij}$, and $S_{x(j)x(j)} = \sum_{i=1}^{n_j}(X_{ij}-\bar{X}_j)^2$. The mean square error (MSE) which is an unbiased estimator of the σ_j^2 for the j^{th} profile defined as: $MSE_j = \sum_{i=1}^{n_j}\hat{\varepsilon}_{ij}^2/n_j - 2$, where $\hat{\varepsilon}_{ij} = (Y_{ij} - \hat{\beta}_{0j} - \hat{\beta}_{1j}X_{ij}^*)$.

The results obtained using fixed random explanatory variables are also applicable for the case random explanatory variables Neter *et al.* [31] and Montgomery *et al.* [32].

A. BAYESIAN ESTIMATION USING IMPROPER PRIORS

The selection of prior is vital part of Bayesian approach. There are three ways to select a prior. The first is subjective approach; second the objective approach; third the pragmatic approach. We have incorporated these approaches in this study.

Let us consider independent improper priors for the process parameters of simple linear profiles model in Equation (2) for the j^{th} profile.

$$p\left(\beta_{0j}, \beta_{1j}\right) \propto 1; \quad p\left(\sigma_j^2\right) \propto \sigma_j^{-2}, \tag{3}$$

where $-\infty < \beta_{0j}, \beta_{1j} < \infty; \sigma^2 > 0$

The integration results of improper priors are undefined. But the resultant posterior distributions are still proper. The posterior estimates using improper priors are similar with classical estimates of the process parameters of profile model in Equation (2) Abbas *et al.* [4].

B. BAYESIAN ESTIMATION USING NON-CONJUGATE PRIORS

This subsection considers non-conjugate priors of Bramwell, Holdsworth, Pinton (BHP) Bramwell *et al.* [33] for Y-intercepts and slopes and for errors variances as Levy distribution Jonathan and Roger [34]. The BHP probability distribution with infinite range is famous for the monitoring of rare fluctuations in processes. Another non-conjugate prior of Levy distribution is the continuous probability distribution stable for nonnegative variables i.e., variance. Let us consider Y-intercepts and slopes priors as: $\beta_{0j} \sim f(\mu_0, \tau_0)$ and $\beta_{1j} \sim f(\mu_1, \tau_1)$, respectively while errors variances prior defined as $\sigma_j^2 \sim f(\varphi_0)$. The posterior distribution for Y-intercepts and slopes are normal with posterior means and variances for the j^{th} profile as:

$$\beta_{onj} = \frac{\left(n\bar{Y}_j\mu_0 + \frac{\pi}{2}\sigma_j^2\right)\mu_0^3\exp\left(\frac{\tau_0}{\mu_0}\right) + \left(\tau_0 - \mu_0^2\right)\sigma_j^2}{n\mu_0^4\exp\left(\frac{\tau_0}{\mu_0}\right) + \sigma_j^2},$$
$$\text{and } \beta_{1nj} = \frac{\left(\mu_1 S_{x(j)y(j)} + \frac{\pi}{2}\sigma_j^2\right)\mu_1^3\exp\left(\frac{\tau_1}{\mu_1}\right) + \left(\tau_1 - \mu_1^2\right)\sigma_j^2}{S_{x(j)x(j)}\mu_1^4\exp\left(\frac{\tau_1}{\mu_1}\right) + \sigma_j^2} \tag{4}$$

And variances as:

$$\sigma_{onj}^2 = \frac{\sigma_j^2\mu_0^4\exp\left(\frac{\tau_0}{\mu_0}\right)}{n\mu_0^4\exp\left(\frac{\tau_0}{\mu_0}\right) + \sigma_j^2},$$
$$\text{and } \sigma_{1nj}^2 = \frac{\sigma_j^2\mu_1^4\exp\left(\frac{\tau_1}{\mu_1}\right)}{S_{x(j)x(j)}\mu_1^4\exp\left(\frac{\tau_1}{\mu_1}\right) + \sigma_j^2}. \tag{5}$$

While the posterior distribution for errors variances is inverse gamma for the j^{th} profile with posterior estimates as:

$$\eta_{nj} = (n+1)/2, \quad \pi_{nj} = \varphi_0/2 + (n-2)\sigma_{0j}^2/2, \tag{6}$$

where $\sigma_{0j}^2 = \sum_{i=1}^{n}(Y_{ij} - \beta_{0j} - \beta_{1j}X_{ij}^*)^2/n - 2$.

C. BAYESIAN ESTIMATION USING CONJUGATE PRIORS

Let us consider conjugate priors of normal (i.e., $\beta_{0j} \sim N(\theta_0, \delta_0)$ and $\beta_1 \sim N(\theta_1, \delta_1)$) for the Y-intercepts and slopes and inverse gamma prior (i.e., $\sigma_j^2 \sim IG(\nu_0, \psi_0)$) for errors variances. The posterior distributions for Y-intercepts and slopes are normal while the posterior distribution for errors variances is inverse gamma. The means and variances for the posterior distributions of Y-intercept and slopes for the j^{th} sample given as:

$$\beta'_{onj} = \frac{n\bar{Y}_j\delta_0 + \theta_0\sigma_j^2}{n\delta_0 + \sigma_j^2},$$
$$\text{and } \beta'_{1nj} = \frac{S_{x(j)y(j)}\delta_1 + \theta_1\sigma_j^2}{S_{x(j)x(j)}\delta_1 + \sigma_j^2}. \tag{7}$$

And variances as:

$$\sigma_{onj}'^2 = \frac{\delta_0\sigma_j^2}{n\delta_0 + \sigma_j^2}, \quad \text{and } \sigma_{1nj}'^2 = \frac{\delta_1\sigma_j^2}{S_{x(j)x(j)}\delta_1 + \sigma_j^2}. \tag{8}$$

The posterior estimates of inverse gamma distribution for the j^{th} sample given as:

$$\alpha'_{nj} = \nu_0 + n/2, \quad \text{and } \beta'_{nj} = \psi_0 + (n-2)\sigma_{0j}^2/2. \tag{9}$$

III. PROPOSED BAYESIAN DEWMA STRUCTURES

This section presents designed structure of three univariate Bayesian DEWMA charts for the monitoring of process parameters of simple linear profiles. We first designed the DEWMA control chart to monitor the Y-intercepts, then DEWMA control chart for slopes and at the end DEWMA control chart to monitor errors variances. We presented here Bayesian DEWMA control charting structure under conjugate priors, while for non-conjugate priors can constructed on similar lines.

The posterior estimates of the Y-intercept (β_0), slopes (β_1) and errors variances are used to construct the DEWMA statistics defined as:

$$EWMA_{\beta_I}(j) = \kappa_1\beta'_{0nj} + \kappa_2 EWMA_{\beta I}(j-1),$$
$$\vee j = 1, 2, 3, \ldots$$

$$
\begin{aligned}
EWMA_{\beta I}(0) &= \beta_0,\\
DEWMA_{\beta I}(j) &= \kappa_3 EWMA_{\beta I}(j) + \kappa_4 DEWMA_{\beta I}(j-1),\\
&\quad \vee j = 1,2,3,\ldots\\
DEWMA_{\beta I}(0) &= \beta_0.\\
EWMA_{\beta_S}(j) &= \kappa_1 S\left(\beta'_{1nj}\right) + \kappa_2 EWMA_{\beta S}(j-1),\\
&\quad \vee j = 1,2,3,\ldots\\
EWMA_{\beta S}(0) &= 0,\\
DEWMA_{\beta S}(j) &= \kappa_3 EWMA_{\beta S}(j) + \kappa_4 DEWMA_{\beta S}(j-1),\\
&\quad \vee j = 1,2,3,\ldots\\
DEWMA_{\beta S}(0) &= 0.\\
S\left(\beta'_{1nj}\right) &= (\beta'_{1nj} - \beta_{1j})/\sqrt{(\sigma'^2_{1nj})}\\
EWMA_{\beta E}(j) &= \max\left\{\kappa_1 \ln\left(MSE_{\beta j}\right)\right.\\
&\quad \left. + \kappa_2 EWMA_{\beta E}(j-1), \ln\left(\sigma^2_{0j}\right)\right\},\\
&\quad \vee j = 1,2,3,\ldots\\
EWMA_{\beta E}(0) &= \ln\left(\sigma^2_{0j}\right),\\
DEWMA_{\beta E}(j) &= \max\left\{\kappa_3 EWMA_{\beta E}(j)\right.\\
&\quad \left. + \kappa_4 DEWMA_{\beta E}(j-1)\right\},\\
&\quad \vee j = 1,2,3,\ldots\\
DEWMA_{\beta E}(0) &= \ln\left(\sigma^2_{0j}\right).
\end{aligned}
\tag{10}
$$

The posterior estimates of slopes are standardized as the control limits under Bayesian frame involves X-values, this mean control limits may not be the same at each sample taken. This problem is fixed after standardization and standardized posterior estimates of slope are used to compute DEWMA statistic. We have designed the Bayesian DEWMA statistic after incorporating approximated formula for MSE_j the unbiased estimator of errors variances derived by Abbas *et al.* [4] given as:

$$
Var[\ln(\mathrm{MSE}_j)] = \frac{2}{Q_1} + \frac{2}{Q_2} + \frac{4}{3Q_3} - \frac{16}{15Q_4}. \tag{11}
$$

Then the corresponding control limits are given as (12), shown at the bottom of this page. The signal is out-of-control whenever the DEWMA statistic fall outside of lower or upper control limits.

It can be observed that the DEWMA statistic for errors variance and corresponding upper control limit does not changed in case of random explanatory variable. This means that the simulative results may not vary much from the case of fixed explanatory variable while monitoring the errors variances.

IV. PERFORMANCE MEASURES AND SIMULATION STEPS

The performance evaluation of proposed Bayesian DEWMA control charts for the monitoring of linear profiles based on different individual performance measure of Average Run Length (ARL), Standard Deviation of Run Length (SDRL) and Median of Run Length (MDRL). The ARL values are most widely used as an evaluation measure at specific value shift, the mean of run length (RL) distribution. It is interpreted as the average number of values required to a control structure to identify first out-of-control signal or signal false alarm. Whenever the process is in-control the ARL represented by ARL_0 and for out-of-control situation the ARL represented by ARL_1. We have also computed the SDRL values as an additional indicator to further assess distributional spread of run length and the MDRL values to evaluate the skewness of the run length distribution.

There are different simulation procedures available in literature to evaluate the designed structures. This study used the Monte Carlo simulations to obtain the values of performance measures. Following are the procedural steps

$$
\begin{aligned}
LCL_{\beta I}[i] &= \alpha_{0j} + \alpha_{1j}\mu_x - L_I \sqrt{\left(\frac{\alpha^2_{1j}\sigma^2_x}{n} + \sigma'^2_{onj}\right) \frac{\kappa_1^4\left(1+\kappa_2^2 - \left(i^2+2i+1\right)\kappa_2^{2i} + \left(2i^2+2i-1\right)\kappa_2^{2i+2} - i^2\kappa_2^{2i+4}\right)}{\left(1-\kappa_2^2\right)^3}}\\
UCL_{\beta I}[i] &= \alpha_{0j} + \alpha_{1j}\mu_x + L_I \sqrt{\left(\frac{\alpha^2_{1j}\sigma^2_x}{n} + \sigma'^2_{onj}\right) \frac{\kappa_1^4\left(1+\kappa_2^2 - \left(i^2+2i+1\right)\kappa_2^{2i} + \left(2i^2+2i-1\right)\kappa_2^{2i+2} - i^2\kappa_2^{2i+4}\right)}{\left(1-\kappa_2^2\right)^3}}\\
LCL_{\beta S}[i] &= -L_S \sqrt{\frac{\kappa_1^4\left(1+\kappa_2^2 - \left(i^2+2i+1\right)\kappa_2^{2i} + \left(2i^2+2i-1\right)\kappa_2^{2i+2} - i^2\kappa_2^{2i+4}\right)}{\left(1-\kappa_2^2\right)^3}}\\
UCL_{\beta S}[i] &= +L_S \sqrt{\frac{\kappa_1^4\left(1+\kappa_2^2 - \left(i^2+2i+1\right)\kappa_2^{2i} + \left(2i^2+2i-1\right)\kappa_2^{2i+2} - i^2\kappa_2^{2i+4}\right)}{\left(1-\kappa_2^2\right)^3}}\\
UCL_{\beta E}[i] &= L_E \sqrt{\mathrm{Var}\left[\ln\left(MSE_{Bj}\right)\right] \frac{\kappa_1^4\left(1+\kappa_2^2 - \left(i^2+2i+1\right)\kappa_2^{2i} + \left(2i^2+2i-1\right)\kappa_2^{2i+2} - i^2\kappa_2^{2i+4}\right)}{\left(1-\kappa_2^2\right)^3}}
\end{aligned}
\tag{12}
$$

for Monte Carlo (MC) simulation of ARL calculation for Bayesian DEWMA control charts while monitoring the process parameters of profiles model when independent variables are not fixed.

(i) Construct the Bayesian DEWMA statistics where Bayesian EWMA statistics are used as input values for process parameters with corresponding control limits.
(ii) Decide about the sample size (n) of each profile and the values of smoothing constants (k_1, k_2, k_3, k_4).
(iii) Generate error terms from standard normal distribution and X-values from normal distribution with specified values of mean and variance and then compute the values of response variable Y.
(iv) Specified the in-control values of the process parameters of B_0 and B_1.
(v) Elicitate the hyperparameters values of the prior distributions of process parameters.
(vi) Decide about the initial values of control limits coefficients (L_I, L_S, L_E).
(vii) Based on the variables in (iii) and hyperparameters in (v), compute the values of posterior estimates.
(viii) Start computing the Bayesian DEWMA statistics for each univariate control chart for the j^{th} profile.
(ix) RL (run length) increases one unit with in-control DEWMA statistics. Record RL at first out-of-control signal.
(x) Repeat the process (say 10,000 times) and compute in-control ARL (ARL_0), if ARL_0 equals specified ARL_0 i.e., 200 then record these control limits coefficients and move to next step, otherwise readjust them and run the program again.
(xi) Compute out-of-control ARL(ARL_1) at each shift i.e., Y-intercept (B_0 to $B_0 + \delta_I\sigma$), slope (B_1 to $B_1 + \delta_S\sigma$), and error variance (σ to $\delta_E\sigma$).
(xii) Repeat the process to a point which declares out-of-control and record ARL_1.
(xiii) Repeat the above steps for sufficient number of times (10,000 say) to obtain out-of-control ARL.

V. ELICITATION AND SENSITIVITY ANALYSIS

The elicitation process is the recognized way of counting expert opinion into probability distribution. The inclusion of prior information is one of the prime prospects of Bayesian philosophy. This study considers method of elicitation introduced by Garthwaite *et al.* [35] where they used a piecewise linear model to construct relationship between response and explanatory variables. Let us consider the explanatory variable X follows normal distribution with mean = 5 and variance = 5/3 then the values of response variable Y are calculated by using the model $Y_{ij} = 13 + 2X_{ij} + e_{ij}$, while 13 and 2 are the in-control values of the process parameters. The errors terms are assumed to follow the standard normal distribution. The values of smoothing constants (K_1, K_2, K_3, K_4) in DEWMA statistics for Y-intercepts, slopes and errors variances are selected as (0.2, 0.8, 0.2, 0.8), respectively. After ending up with assessment procedure the hyper-parameters values obtained as: For Y-intercepts prior mean 15 and variance 25, for slopes prior mean 2.5 and variance 9, and for errors variances prior the hyper-parameters are 0.05 and 0.35 (for detailed procedure see Abbas *et al.* [4]).

After the elicitation the sensitivity analyses are another essential aspect in Bayesian approach to obtained refined and optimum values of hyperparameters. The elicitated hyperparameters values describe locations and scales point of the different conjugate and non-conjugate prior distributions of process parameters. We assumed different location and scale values to evaluate the performance of proposed control charts. These assessments are based on individual performance measures values.

The sensitivity analyses are conducted for the hyper-parameters to optimize performance of the proposed Bayesian DEWMA control charting structure. The values of response variable are generated using model in Equation (2), while random explanatory variable, errors terms, and in-control process parameters values are mentioned above in elicitation phase. The sample size ($n = 4$) is selected, while the values of control limit coefficients are selected in such way that individual in-control ARL is approximately 590, while the overall $ARL_0 = 200$. The total numbers of 10,000 simulations are performed to obtain the values of performance measures, while considering step shifts in the process parameters of profiles function. The resultant values of performance measures are provided in Tables 1-4 for different combinations of locations and scales under conjugate and non-conjugate priors. The hyperparameters values of inverse gamma prior have negligible impact of proposed control structures so the ARL values are not reported here. Following lines described the impact of different combinations of hyper-parameters values on the proposed Bayesian DEWMA structures while taking shifts in process parameters:

- Different values of location hyperparameters of non-conjugate priors for Y-intercepts and slopes have negligible impact on the performance of proposed control charting structures of process parameters (see Table 1).
- The scale hyperparameters values of non-conjugate priors for Y-intercepts and slopes have some impact on the performance of Bayesian DEWMA charts (i.e., decrease in scale hyperparameters values decreases the ARL, SDRL, MDRL values while monitoring shifts in process parameters; cf. Table 2).
- The location hyperparameters of conjugate priors show significant impact on the performance of proposed DEWMA charts for the monitoring of process parameters as increasing location hyperparameters values decreases the ARL, SDRL, MDRL values (cf. Table 2).
- The values of scale hyperparameters of conjugate priors also influence on the performance of Bayesian DEWMA charts while monitoring process parameters. Decreasing in scale hyperparameters values decreases the values of ARL, SDRL, and MDRL (cf. Table 2).

We have selected the values of hyperparameters for non-conjugate and conjugate priors which show most efficient

TABLE 1. ARL, SDRL and MDRL Values for Sensitivity Analysis of Location Hyperparameters of BHPL Priors at $ARL_0 = 200$, under DEWMA Control Chart.

	Sensitivity Analysis Considering Y-intercept Shifts											
	DEWMA-BNCP$_X$-1			DEWMA-BNCP$_X$-2			DEWMA-BNCP$_X$-3			DEWMA -BNCP$_X$-4		
	$\mu_0 = 10, \mu_1 = 1.5$			$\mu_0 = 15, \mu_1 = 2.5$			$\mu_0 = 25, \mu_1 = 6.5$			$\mu_0 = 35, \mu_1 = 10.5$		
δ_I	ARL	SDRL	MDRL	ARL	SDRL	MDRL	ARL	SDRL	MDRL	ARL	SDRL	MDRL
0.0	199.59	208.53	134.00	199.51	208.53	134.00	199.23	208.52	134.00	199.26	208.40	134.00
0.2	142.30	148.90	97.00	142.20	148.85	97.00	141.98	148.55	97.00	142.11	148.52	97.00
0.4	68.98	68.21	49.00	68.94	68.16	49.00	69.05	68.22	49.00	68.95	68.19	49.00
0.6	36.30	32.86	27.00	36.33	32.85	27.00	36.29	32.82	27.00	36.32	32.81	27.00
0.8	21.62	18.27	17.00	21.53	18.25	17.00	21.50	18.22	17.00	21.48	18.19	17.00
1.0	14.28	10.64	12.00	14.32	10.65	12.00	14.36	10.67	12.00	14.36	10.69	12.00
1.2	10.29	7.07	9.00	10.28	7.08	9.00	10.26	7.02	9.00	10.27	7.06	9.00
1.4	7.93	5.27	7.00	7.91	5.23	7.00	7.93	5.27	7.00	7.93	5.25	7.00
1.6	6.40	4.05	6.00	6.40	4.05	6.00	6.39	4.04	6.00	6.39	4.04	6.00
1.8	5.15	3.16	5.00	5.16	3.18	5.00	5.17	3.19	5.00	5.17	3.19	5.00
2.0	4.43	2.65	4.00	4.41	2.64	4.00	4.43	2.64	4.00	4.42	2.64	4.00
δ_S	Sensitivity Analysis Considering Slopes Shifts											
0.000	199.59	208.53	134.00	199.51	208.53	134.00	199.23	208.52	134.00	199.26	208.40	134.00
0.025	145.90	152.45	100.00	145.87	152.21	100.00	145.75	152.01	100.50	146.01	152.30	100.00
0.050	91.28	95.09	62.00	91.22	95.12	62.00	91.01	94.91	62.00	91.31	95.10	62.00
0.075	56.81	57.60	39.00	56.75	57.49	39.00	56.64	57.47	39.00	56.83	57.56	39.00
0.100	36.53	33.87	27.00	36.52	33.91	27.00	36.58	34.01	27.00	36.62	33.95	27.00
0.125	25.95	22.75	20.00	25.95	22.76	20.00	25.94	22.79	20.00	25.99	22.75	20.00
0.150	19.54	16.16	16.00	19.53	16.17	15.50	19.43	16.09	15.00	19.57	16.23	16.00
0.175	14.59	11.79	12.00	14.63	11.82	12.00	14.64	11.76	12.00	14.53	11.73	12.00
0.200	11.77	8.85	10.00	11.74	8.82	10.00	11.76	8.87	10.00	11.81	8.86	10.00
0.225	9.66	7.00	8.00	9.67	7.00	8.00	9.69	7.01	8.00	9.66	6.95	8.00
0.250	8.13	5.63	7.00	8.13	5.62	7.00	8.10	5.62	7.00	8.11	5.61	7.00
δ_E	Sensitivity Analysis Considering Errors Variances Shifts											
1.00	199.51	208.53	134.00	199.23	208.52	134.00	199.23	208.52	134.00	199.26	208.40	134.00
1.20	32.00	32.25	22.00	31.97	33.25	23.00	31.97	35.25	23.00	32.01	35.27	23.00
1.40	10.90	11.81	7.00	10.90	11.81	7.00	10.90	10.81	7.00	10.89	11.80	7.00
1.60	5.51	5.80	3.00	5.52	5.80	3.00	5.52	5.60	3.00	5.51	5.80	3.00
1.80	3.58	3.59	2.00	3.59	3.59	2.00	3.59	3.59	2.00	3.58	3.59	2.00
2.00	2.73	2.58	2.00	2.73	2.58	2.00	2.73	2.58	2.00	2.73	2.59	2.00
2.20	2.14	1.84	1.00	2.14	1.84	1.00	2.14	1.84	1.00	2.14	1.83	1.00
2.40	1.84	1.52	1.00	1.84	1.52	1.00	1.84	1.52	1.00	1.85	1.52	1.00
2.60	1.62	1.20	1.00	1.62	1.20	1.00	1.62	1.20	1.00	1.62	1.21	1.00
2.80	1.46	1.00	1.00	1.46	1.00	1.00	1.46	1.00	1.00	1.46	0.99	1.00
3.00	1.37	0.83	1.00	1.37	0.83	1.00	1.37	0.83	1.00	1.37	0.84	1.00

Note: BHPL= Bramwell, Holdsworth, Pinton, Levy, BNCP=Bayesian Non-conjugate Prior

performance in terms of individual performance measures. The BHP prior for Y-intercepts has location value of 25 and scale value of 10, while the BHP prior for slopes has location value of 6.5 and scale value of 3. The Levy prior for errors variance has hyperparameters value as 0.5. Now, the normal prior of Y-intercepts has location value of 35 and

TABLE 2. ARL, SDRL and MDRL Values for Sensitivity Analysis of Scale Hyperparameters of BHPL Priors at $ARL_0 = 200$, under DEWMA Control Charts.

	Sensitivity Analysis Considering *Y*-intercept Shifts											
	DEWMA-BNCP$_X$-1			DEWMA-BNCP$_X$-2			DEWMA-BNCP$_X$-3			DEWMA -BNCP$_X$-4		
	$\tau_0 = 30, \tau_1 = 10$			$\tau_0 = 20, \tau_1 = 6$			$\tau_0 = 10, \tau_1 = 3$			$\tau_0 = 5, \tau_1 = 1.5$		
δ_I	ARL	SDRL	MDRL	ARL	SDRL	MDRL	ARL	SDRL	MDRL	ARL	SDRL	MDRL
0.0	199.51	208.53	134.00	200.29	207.57	138.00	199.23	205.35	138.00	201.68	202.44	143.00
0.2	142.20	148.85	97.00	139.44	142.04	97.00	131.82	132.89	92.00	130.44	122.08	90.00
0.4	68.94	68.16	49.00	67.78	66.11	48.00	64.82	62.02	47.00	64.79	56.43	45.00
0.6	36.33	32.85	27.00	35.61	31.76	27.00	34.12	30.55	25.00	33.99	28.07	25.00
0.8	21.53	18.25	17.00	21.33	17.70	17.00	20.44	16.52	16.00	20.26	14.90	16.00
1.0	14.32	10.65	12.00	14.27	10.44	12.00	13.95	10.30	12.00	13.87	9.73	11.00
1.2	10.28	7.08	9.00	10.30	7.21	9.00	10.14	6.81	9.00	10.12	6.62	9.00
1.4	7.91	5.23	7.00	7.87	5.12	7.00	7.88	5.20	7.00	7.87	4.78	7.00
1.6	6.40	4.05	6.00	6.31	4.03	6.00	6.28	3.90	6.00	6.30	3.75	6.00
1.8	5.16	3.18	5.00	5.25	3.21	5.00	5.15	3.10	5.00	5.23	3.12	5.00
2.0	4.41	2.64	4.00	4.36	2.58	4.00	4.36	2.60	4.00	4.39	2.51	4.00
δ_S	Sensitivity Analysis Considering Slopes Shifts											
0.000	199.51	208.53	134.00	200.29	207.57	138.00	199.23	205.35	138.00	201.68	202.44	143.00
0.025	145.87	152.21	100.00	142.60	148.01	98.00	137.57	141.62	95.00	138.20	133.55	97.00
0.050	91.22	95.12	62.00	87.77	90.40	61.00	84.96	86.21	59.00	85.00	80.53	60.00
0.075	56.75	57.49	39.00	54.50	54.36	38.00	53.83	51.81	38.00	53.77	48.32	39.00
0.100	36.52	33.91	27.00	36.09	32.81	27.00	35.44	31.29	27.00	35.81	30.48	27.00
0.125	25.95	22.76	20.00	25.85	22.69	20.00	25.49	21.42	20.00	25.98	20.77	21.00
0.150	19.53	16.17	15.50	18.67	15.35	15.00	18.97	15.17	16.00	19.46	14.66	16.00
0.175	14.63	11.82	12.00	15.03	11.70	13.00	15.04	11.20	13.00	15.46	11.10	13.00
0.200	11.74	8.82	10.00	11.88	8.75	10.00	12.23	8.87	11.00	12.68	8.73	11.00
0.225	9.67	7.00	8.00	9.73	7.00	8.00	9.91	6.84	9.00	10.37	6.99	9.00
0.250	8.13	5.62	7.00	8.27	5.72	7.00	8.42	5.68	7.00	8.76	5.65	8.00
δ_E	Sensitivity Analysis Considering Errors Variances Shifts											
1.00	199.51	208.53	134.00	200.29	207.57	138.00	199.23	205.35	138.00	201.68	202.44	143.00
1.20	33.00	34.25	23.00	33.18	35.38	24.00	33.43	36.52	25.00	34.01	37.12	26.00
1.40	10.90	11.81	7.00	11.33	12.12	8.00	12.03	12.19	9.00	12.19	12.33	10.00
1.60	5.51	5.80	3.00	5.80	5.97	4.00	6.52	6.39	4.00	7.15	6.67	5.00
1.80	3.58	3.59	2.00	3.90	3.81	2.00	4.24	3.92	3.00	4.33	4.38	3.00
2.00	2.73	2.58	2.00	2.81	2.56	2.00	3.16	2.87	2.00	3.43	3.12	3.00
2.20	2.14	1.84	1.00	2.29	1.98	1.00	2.44	2.06	2.00	2.92	2.19	2.00
2.40	1.84	1.52	1.00	1.90	1.55	1.00	2.12	1.75	1.00	2.26	1.95	2.00
2.60	1.62	1.20	1.00	1.66	1.24	1.00	1.84	1.43	1.00	2.04	1.68	2.00
2.80	1.46	1.00	1.00	1.52	1.05	1.00	1.61	1.15	1.00	1.69	1.36	1.00
3.00	1.37	0.83	1.00	1.42	0.92	1.00	1.50	1.00	1.00	1.59	1.03	1.00

scale value of 15, while normal prior of slopes has location value of 6.5 and scale value of 6. The errors variances prior of inverse gamma have hyperparameters values of 0.05 and 0.3.

VI. EVALUATION AND COMPARISONS

We now evaluate our proposed scheme of Bayesian DEWMA control charts when explanatory variables are assumed to be random while process parameters are uncertain.

TABLE 3. ARL, SDRL and MDRL Values for Sensitivity Analysis of Location Hyperparameters of NIG Priors at $ARL_0 = 200$, under DEWMA Control Charts.

	Sensitivity Analysis Considering *Y*-intercept Shifts											
	DEWMA-BCP$_X$-1			DEWMA-BCP$_X$-2			DEWMA-BCP$_X$-3			DEWMA -BCP$_X$-4		
	$\theta_0 = 15, \theta_1 = 2.5$			$\theta_0 = 25, \theta_1 = 4.5$			$\theta_0 = 35, \theta_1 = 6.5$			$\theta_0 = 45, \theta_1 = 8.5$		
δ_I	ARL	SDRL	MDRL	ARL	SDRL	MDRL	ARL	SDRL	MDRL	ARL	SDRL	MDRL
0.0	199.53	208.14	135.50	199.74	207.18	136.00	199.61	201.60	139.00	200.99	197.73	142.00
0.2	139.47	145.35	95.00	119.96	122.88	83.00	112.31	109.84	79.00	111.91	105.83	79.50
0.4	68.02	67.44	47.00	59.10	56.15	42.00	56.41	52.73	41.00	56.39	50.96	41.00
0.6	35.50	32.23	26.00	31.17	27.93	23.00	29.99	25.63	23.00	30.37	25.08	23.00
0.8	21.26	17.86	16.00	19.41	15.03	16.00	18.99	14.04	16.00	19.32	13.77	16.00
1.0	14.25	10.67	12.00	12.90	9.52	11.00	13.05	8.99	11.00	13.43	8.77	12.00
1.2	10.29	7.28	9.00	9.81	6.68	9.00	9.73	6.29	9.00	10.09	6.20	9.00
1.4	7.84	5.14	7.00	7.60	4.85	7.00	7.67	4.66	7.00	8.00	4.63	7.00
1.6	6.29	4.06	6.00	6.16	3.77	6.00	6.32	3.71	6.00	6.68	3.68	6.00
1.8	5.23	3.22	5.00	5.15	3.02	5.00	5.23	3.01	5.00	5.41	2.91	5.00
2.0	4.34	2.58	4.00	4.29	2.50	4.00	4.46	2.52	4.00	4.73	2.53	4.00
δ_S	Sensitivity Analysis Considering Slopes Shifts											
0.000	199.53	208.14	135.50	199.74	207.18	136.00	199.61	201.60	139.00	200.99	197.73	142.00
0.025	152.65	162.63	103.00	131.25	134.15	92.00	124.74	123.43	88.00	124.09	120.74	88.00
0.050	95.67	99.05	66.00	77.88	78.16	54.00	74.38	71.49	53.00	74.60	68.71	54.00
0.075	58.94	60.06	40.00	48.90	47.82	34.00	47.08	43.30	34.00	47.51	41.53	35.00
0.100	38.18	36.40	27.00	33.07	30.15	24.00	32.47	28.23	25.00	33.35	28.14	25.00
0.125	26.15	23.17	20.00	22.95	19.28	18.00	22.71	18.19	18.00	23.47	17.99	19.00
0.150	19.37	16.73	15.00	17.41	14.08	14.00	17.35	13.05	14.00	18.12	12.88	15.00
0.175	14.78	11.65	12.00	13.95	10.60	12.00	14.05	9.94	12.00	14.73	9.71	13.00
0.200	11.94	9.17	10.00	11.18	8.04	10.00	11.59	7.89	10.00	12.14	7.82	11.00
0.225	9.63	6.99	8.00	9.42	6.49	8.00	9.59	6.37	9.00	10.04	6.16	9.00
0.250	8.17	5.80	7.00	8.14	5.64	7.00	8.27	5.22	7.00	8.81	5.28	8.00
δ_E	Sensitivity Analysis Considering Errors Variances Shifts											
1.00	199.53	208.14	135.50	199.74	207.18	136.00	199.61	201.60	139.00	200.99	197.73	142.00
1.20	33.80	36.22	22.00	33.96	37.08	23.50	33.59	36.28	26.00	34.39	35.00	27.00
1.40	10.63	11.69	7.00	11.01	11.98	7.00	11.07	12.04	8.00	11.76	11.58	10.00
1.60	5.41	5.62	3.00	5.59	5.96	4.00	5.59	6.13	4.00	6.01	6.26	5.00
1.80	3.48	3.57	2.00	3.54	3.74	2.00	3.66	3.83	3.00	4.32	3.89	4.00
2.00	2.60	2.44	2.00	2.69	2.52	2.00	2.80	2.76	2.00	3.20	2.85	3.00
2.20	2.11	1.88	1.00	2.14	1.96	1.00	2.21	2.03	2.00	2.55	2.29	2.00
2.40	1.78	1.41	1.00	1.89	1.51	1.00	2.00	1.66	1.00	2.28	1.81	2.00
2.60	1.59	1.18	1.00	1.65	1.25	1.00	1.72	1.38	1.00	1.96	1.47	1.00
2.80	1.46	0.97	1.00	1.51	1.03	1.00	1.60	1.14	1.00	1.74	1.28	1.00
3.00	1.34	0.83	1.00	1.40	0.86	1.00	1.47	0.97	1.00	1.59	1.05	1.00

Note: NIG=Normal and Inverse Gamma, BCP=Bayesian Conjugate Prior

The three univariate classical EWMA charts represented by EWMA-C_X while Bayesian EWMA charts by EWMA-$BNCP_X$ and EWMA-BCP_X for non-conjugate and conjugate priors. The three univariate classical DEWMA charts represented by DEWMA-C_X while Bayesian DEWMA charts by DEWMA-$BNCP_X$ and DEWMA-BCP_X for

TABLE 4. **ARL, SDRL and MDRL Values for Sensitivity Analysis of Scale Hyperparameters of NIG Priors at $ARL_0 = 200$, under DEWMA Control Charts.**

	Sensitivity Analysis Considering *Y*-intercept Shifts											
	DEWMA-BCP$_X$-1			DEWMA-BCP$_X$-2			DEWMA-BCP$_X$-3			DEWMA -BCP$_X$-4		
	$\delta_0 = 35, \delta_1 = 12$			$\delta_0 = 25, \delta_1 = 9$			$\delta_0 = 15, \delta_1 = 6$			$\delta_0 = 5, \delta_1 = 3$		
δ_I	ARL	SDRL	MDRL	ARL	SDRL	MDRL	ARL	SDRL	MDRL	ARL	SDRL	MDRL
0.0	199.63	207.84	134.50	201.19	206.87	137.00	200.28	208.76	137.00	199.16	200.02	139.00
0.2	136.03	142.86	93.00	134.09	139.43	92.00	119.41	123.69	82.00	111.96	109.93	79.00
0.4	66.69	66.23	47.00	65.45	64.22	46.00	58.64	56.15	42.00	56.36	51.94	40.00
0.6	35.14	31.74	26.00	34.33	31.46	25.00	31.01	27.48	23.00	30.09	25.14	23.00
0.8	20.83	17.50	16.00	20.54	16.97	16.00	19.28	14.82	16.00	19.18	13.91	16.00
1.0	14.12	10.48	12.00	13.79	10.36	12.00	12.99	9.44	11.00	13.22	8.81	11.00
1.2	10.07	7.27	9.00	10.17	7.10	9.00	9.76	6.65	9.00	9.94	6.23	9.00
1.4	7.80	5.09	7.00	7.85	5.19	7.00	7.69	4.89	7.00	7.84	4.62	7.00
1.6	6.28	3.98	6.00	6.25	3.88	6.00	6.17	3.75	6.00	6.54	3.67	6.00
1.8	5.16	3.20	5.00	5.16	3.22	5.00	5.16	2.99	5.00	5.31	2.94	5.00
2.0	4.39	2.65	4.00	4.34	2.61	4.00	4.36	2.52	4.00	4.65	2.53	4.00
δ_S	Sensitivity Analysis Considering Slopes Shifts											
0.000	199.63	207.84	134.50	201.19	206.87	137.00	200.28	208.76	137.00	199.16	200.02	139.00
0.025	149.51	159.10	102.00	145.77	154.93	101.00	131.80	135.47	91.00	123.37	122.04	86.00
0.050	92.28	95.67	63.00	88.62	92.00	61.00	77.82	78.26	54.00	73.80	69.99	52.00
0.075	57.10	58.39	39.00	55.19	56.25	38.00	48.62	47.57	34.00	45.95	41.42	33.00
0.100	36.92	34.77	27.00	35.78	33.07	27.00	33.03	29.90	24.00	32.27	27.44	25.00
0.125	25.73	22.88	20.00	25.49	22.82	19.00	22.92	19.33	18.00	22.79	18.01	18.00
0.150	19.30	16.45	15.00	18.65	15.18	15.00	17.42	13.90	14.00	17.37	12.82	15.00
0.175	14.46	11.70	12.00	14.53	11.64	12.00	13.90	10.61	11.00	13.80	9.43	12.00
0.200	11.63	8.88	10.00	11.52	8.72	10.00	11.13	8.00	10.00	11.55	7.77	10.00
0.225	9.62	7.04	8.00	9.47	6.78	8.00	9.35	6.46	8.00	9.64	6.31	9.00
0.250	8.05	5.70	7.00	8.13	5.77	7.00	8.14	5.60	7.00	8.19	5.09	7.00
δ_E	Sensitivity Analysis Considering Errors Variances Shifts											
1.00	199.63	207.84	134.50	201.19	206.87	137.00	200.28	208.76	137.00	199.16	200.02	139.00
1.20	34.00	36.39	23.00	34.56	36.51	23.00	34.67	36.16	24.00	34.76	33.73	24.00
1.40	10.67	11.70	7.00	10.75	11.66	7.00	11.05	11.45	7.00	11.73	11.24	9.00
1.60	5.44	5.68	3.00	5.46	5.71	3.00	5.62	5.70	4.00	6.16	5.67	4.00
1.80	3.52	3.58	2.00	3.55	3.54	2.00	3.64	3.49	2.00	4.12	3.68	3.00
2.00	2.64	2.47	2.00	2.71	2.52	2.00	2.78	2.56	2.00	3.04	2.59	2.00
2.20	2.12	1.85	1.00	2.11	1.81	1.00	2.20	1.85	1.00	2.45	2.01	2.00
2.40	1.79	1.42	1.00	1.81	1.45	1.00	1.90	1.55	1.00	2.04	1.60	1.00
2.60	1.59	1.18	1.00	1.60	1.18	1.00	1.64	1.20	1.00	1.78	1.27	1.00
2.80	1.46	1.00	1.00	1.46	1.00	1.00	1.48	0.99	1.00	1.63	1.12	1.00
3.00	1.34	0.81	1.00	1.36	0.81	1.00	1.41	0.88	1.00	1.48	0.96	1.00

non-conjugate and conjugate priors. The combination of three univariate Bayesian DEWMA control charts are designed to have individual in-control ARL of approximately 590 and overall in-control ARL of 200 which is parallel to competing charts in this study. The ARL values are computed after simulation of 10,000 while the shifts in process parameters are incorporated in term of standard deviation.

TABLE 5. ARL Comparisons for Y-intercepts Parameter Shifts β_0 to $\beta_0 + \delta_I \sigma$.

Charts	State of X	δ_I										
		0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
EWMA-C_X	Random x	201	157.0	94.6	52.2	30.3	19.5	13.9	10.6	8.4	7.0	5.9
	Fixed x	200	59.9	16.2	7.9	5.2	3.8	3.1	2.6	2.3	2.1	1.9
EWMA-$BNCP_X$	Random x	201	131	73.4	41.4	26.1	17.2	12.9	10.3	8.3	7.0	5.7
	Fixed x	200	50.6	13.7	6.5	4.1	2.9	2.2	1.8	1.5	1.3	1.2
EWMA-BCP_X	Random x	200	125.3	69.3	40.1	25.1	16.9	12.7	10.1	8.3	7.0	6.1
	Fixed x	200	38.7	12.4	6.5	4.2	3.0	2.4	1.9	1.6	1.4	1.2
DEWMA-C_X	Random x	199	146.8	72.7	37.1	22.1	14.4	10.5	8.0	6.4	5.2	4.5
	Fixed x	200	44.1	12.1	6.0	3.8	2.7	2.0	1.6	1.4	1.2	1.1
DEWMA-$BNCP_X$	Random x	199	121.2	60.1	32.4	20.1	13.6	10.1	7.8	6.3	5.2	4.4
	Fixed x	199	39.7	11.4	5.8	3.7	2.6	2.0	1.6	1.3	1.2	1.1
DEWMA-BCP_X	Random x	199	109.8	55.4	32.1	20.0	14.5	11.0	8.7	7.3	6.0	5.2
	Fixed x	200	33.4	11.5	6.3	4.2	3.0	2.3	1.9	1.6	1.4	1.2

TABLE 6. ARL Comparisons for Slopes Parameter Shifts β_1 to $\beta_1 + \delta_S \sigma$.

Charts	State of X	δ_S										
		0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
EWMA-C_X	Random x	201	168.0	122.7	80.6	54.4	37.3	26.9	20.1	16.0	13.0	11.0
	Fixed x	200	101.6	36.5	17.0	10.3	7.2	5.5	4.5	3.8	3.3	2.9
EWMA-$BNCP_X$	Random x	201	139.1	92.9	61.7	41.8	30.6	23.6	18.0	14.8	12.4	10.5
	Fixed x	200	81.0	30.7	14.5	8.5	5.8	4.3	3.3	2.7	2.3	2.0
EWMA-BCP_X	Random x	200	133.5	88.9	59.2	40.4	30.1	23.1	17.9	14.6	12.3	10.6
	Fixed x	200	67.6	26.6	13.8	8.5	6.0	4.6	3.6	3.0	2.5	2.2
DEWMA-C_X	Random x	199	162.3	105.2	63.7	41.4	27.5	20.2	15.2	12.2	10.0	8.5
	Fixed x	200	80.7	26.3	12.5	7.8	5.4	4.0	3.1	2.5	2.1	1.8
DEWMA-$BNCP_X$	Random x	199	133.8	82.5	53.7	35.6	25.6	19.1	15.2	12.2	10.4	8.9
	Fixed x	199	69.9	24.2	12.1	7.5	5.1	3.9	3.1	2.5	2.1	1.8
DEWMA-BCP_X	Random x	199	122.7	75.1	49.0	34.1	25.0	19.5	15.6	12.8	11.0	9.5
	Fixed x	199	29.5	9.6	5.0	3.3	2.4	1.9	1.6	1.5	1.4	1.3

Table 5 provides the ARL values for shifts in Y-intercept in terms of standard deviation. The proposed Bayesian DEWMA charts show superiority over the competing charts chart in terms of ARL values. The performance of DEWMA-BCP_X chart is superior to all charts in this study for the cases of random and fixed explanatory variables. This superiority is highly significant for smaller value shifts and slightly better for larger shifts. The performance of DEWMA-$BNCP_X$ chart is better than DEWMA-C_X chart while it shows almost similar performance with EWMA-$BNCP_X$ chart. This pattern is observed for both random and fixed explanatory variable cases for monitoring shifts in Y-intercepts. The proposed DEWMA-BCP_X chart outperforms the competing charts in this study.

Table 6 presents the results of ARL values at ARL_0 of 200 while monitoring the shifts in the slope parameter of profile functions under classical and Bayesian EWMA and DEWMA schemes of charts. The resultant ARL values indicate that DEWMA-$BNCP_X$ and DEWMA-BCP_X charts surpass the DEWMA-C_X chart while monitoring shifts in slopes. DEWMA-BCP_X chart shows better performance than EWMA-BCP_X chart while DEWMA-$BNCP_X$ comes up with similar performance with EWMA-$BNCP_X$ chart. Overall the proposed DEWMA-BCP_X chart showed superiority over the other charts in this study.

Table 7 gives the ARL values of EWMA and DEWMA charts under classical and Bayesian setups for shifts in the errors variance. It is observed that the performance of different charts under study does not reflect significant different. Our proposed Bayesian scheme does not work well while detecting shifts in errors variances because the standard deviation shifts are usually detected by R chart. As the

TABLE 7. ARL Comparisons for Process Standard Deviation Parameter Shifts σ to $\delta_E\sigma$.

Charts	State of X	δ_E										
		0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
EWMA-C_X	Random x	201	35.5	13.1	7.0	5.0	3.9	3.1	2.4	2.3	2.1	2.0
	Fixed x	200	33.5	12.7	7.2	5.1	3.9	3.2	2.8	2.5	2.3	2.1
EWMA-$BNCP_X$	Random x	202	33.7	12.9	7.0	5.5	4.7	4.0	3.4	3.3	2.9	2.7
	Fixed x	199	32.2	10.5	5.4	3.4	2.5	2.0	1.7	1.5	1.4	1.3
EWMA-BCP_X	Random x	200	32.4	11.1	6.9	5.2	4.8	4.5	4.1	3.9	3.5	3.3
	Fixed x	200	32.1	10.7	5.7	3.7	2.7	2.1	1.8	1.6	1.4	1.4
DEWMA-C_X	Random x	199	34.1	10.6	5.4	3.5	2.6	2.1	1.8	1.6	1.5	1.3
	Fixed x	200	31.2	10.3	5.4	3.3	2.4	2.0	1.7	1.5	1.4	1.1
DEWMA-$BNCP_X$	Random x	199	33.7	12.9	7.0	5.3	3.9	3.1	2.6	2.2	2.0	1.8
	Fixed x	199	30.7	9.2	4.8	3.0	2.2	1.8	1.6	1.4	1.3	1.2
DEWMA-BCP_X	Random x	201	34.0	10.3	5.4	4.7	4.3	3.4	2.8	2.4	2.2	1.9
	Fixed x	199	29.5	9.6	5.0	3.2	2.6	2.0	1.7	1.5	1.3	1.3

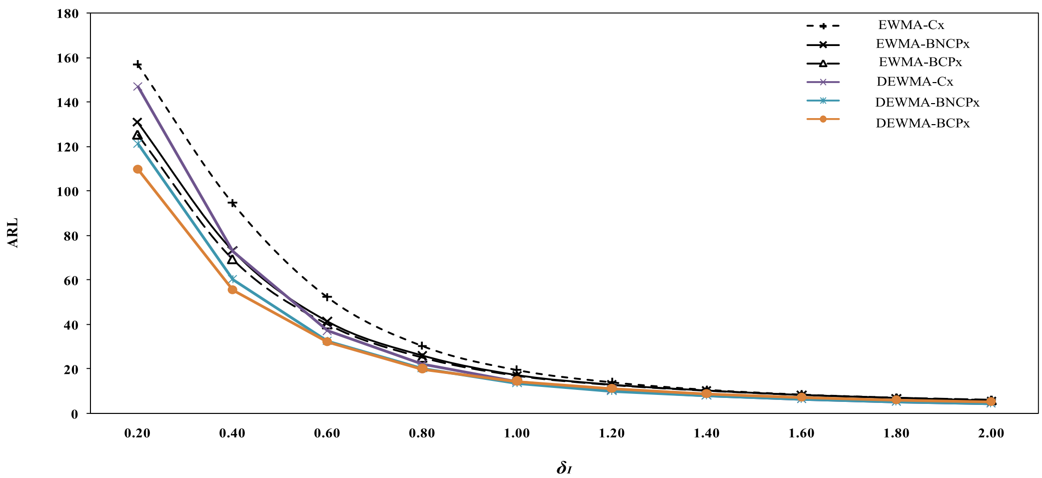

FIGURE 1. ARL Curve for EWMA and DEWMA Charts under Classical and Bayesian Set-up for Y-intercept Shifts.

shifts are incorporated in terms standard deviations, it means increase in shift values will increase the standard deviation of the estimators of the Y-intercepts and slopes. The effort to smooth these estimators over time with the help of EWMA or DEWMA charts is not helping.

Figures 1-3 display the results of ARL values at ARL_0 of 200 while monitoring the shifts in the Y-intercepts, slopes and errors variances of simple linear profiles model in Equation (2). The curves of DEWMA-BCP_X charts are at the lower side for monitoring the Y-intercept and slope shifts (cf. Figs. 1-2). This reflects the faster detection ability of proposed chart. The magnitude of difference in ARL values of proposed and competing chart is high for small shifts and smaller for large shifts. This reflects the sensitivity of DEWMA-BCP_X chart for small shifts. The graphical display of different control charts of this study for the shifts in errors variance have no significant different which indicate similar performance pattern for proposed and competing charts.

The above findings conclude that the proposed method has faster detection potential of sustainable shifts in the Y-intercepts and slopes, while similar in case errors variances monitoring. Our proposed method outperforms the competing methods for small-to-moderate shifts, although for larger values of shifts, its performance is roughly the same. It is observed that for process structures where parametric uncertainty is unavoidable the incorporation of Bayesian schemes of control charts seems wise decision. The inclusion of prior information weights the posterior estimates and enhances its potential for better performance. The tangible benefits obtained by using Bayesian charts should be tradeoff with cost on such information.

This study with its all benefits has some limitations as the selection of priors mostly subjective because Bayesian does not tell much about it. It requires sound knowledge and skills about understudy problem to convert subjective prior belief into mathematical form of priors. Posterior distributions

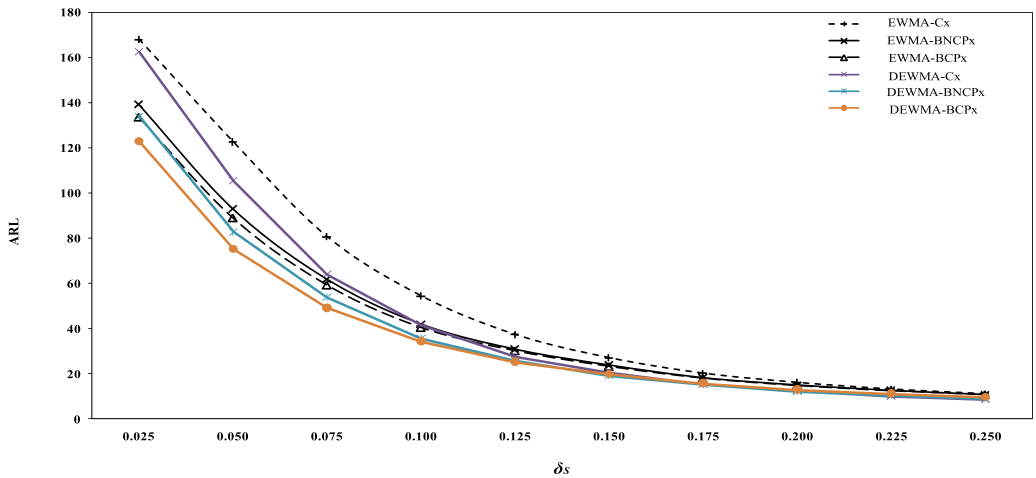

FIGURE 2. ARL Curve for EWMA and DEWMA Charts under Classical and Bayesian Set-up for Slope Shifts.

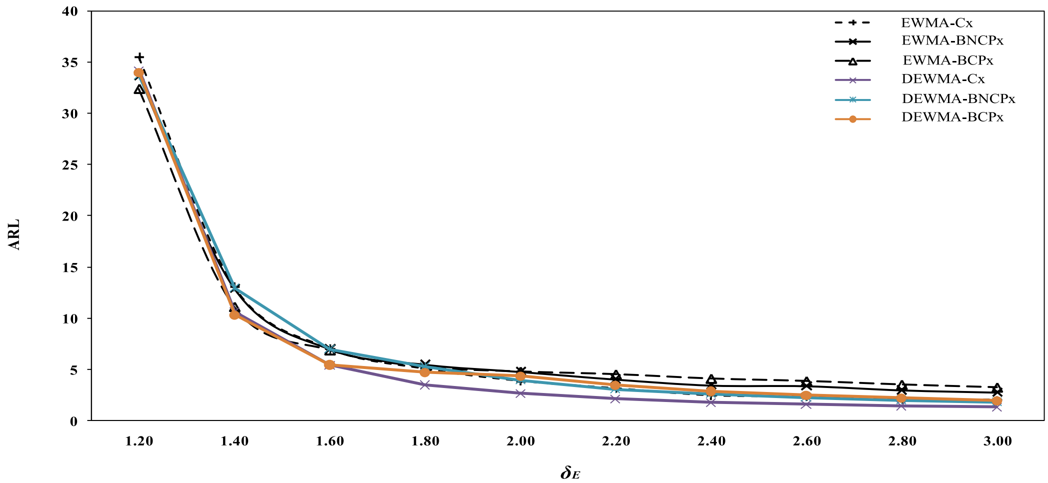

FIGURE 3. ARL Curve for EWMA and DEWMA Charts under Classical and Bayesian Set-up for Errors Variance Shifts.

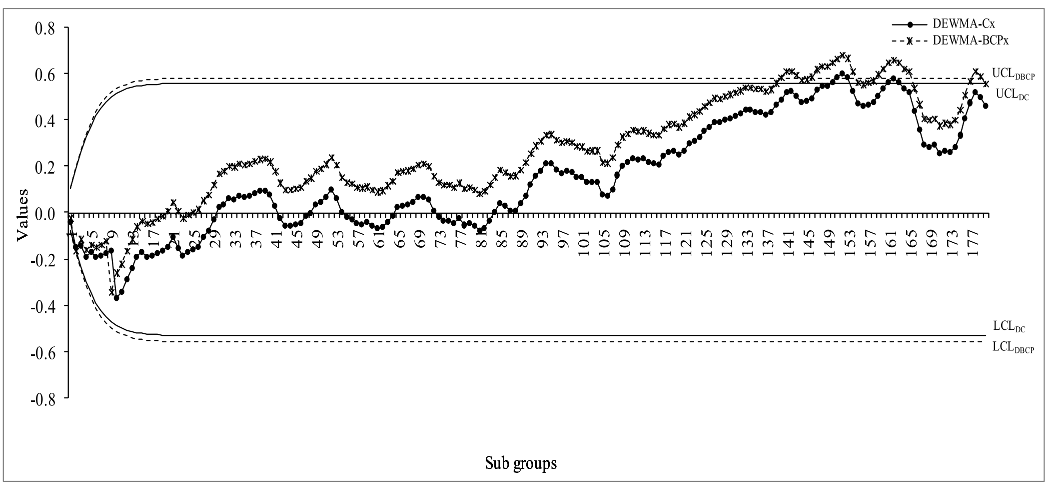

FIGURE 4. Control chart display of DEWMA-Cx & DEWMA-BCPx charts for shift in *Y*-intercept.

heavily depend upon prior distribution which required careful attention while selecting priors as well as its hyper-parameter values. So from practical view point some time it's really difficult to convince subject expert on the valid selection of priors. Bayesian results often come with high computational cost particularly for large numbers of parameters.

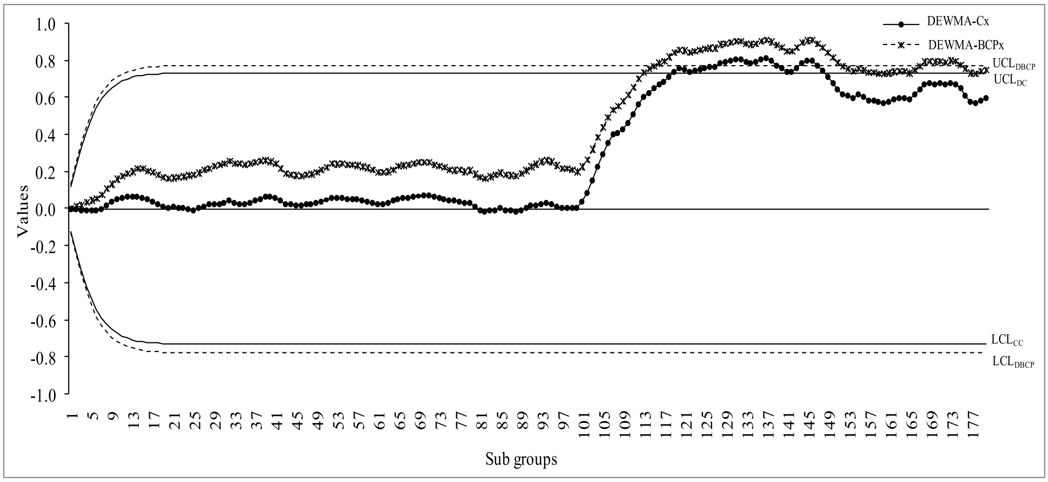

FIGURE 5. Control chart display of DEWMA-Cx & DEWMA-BCPx charts for shift in Slope.

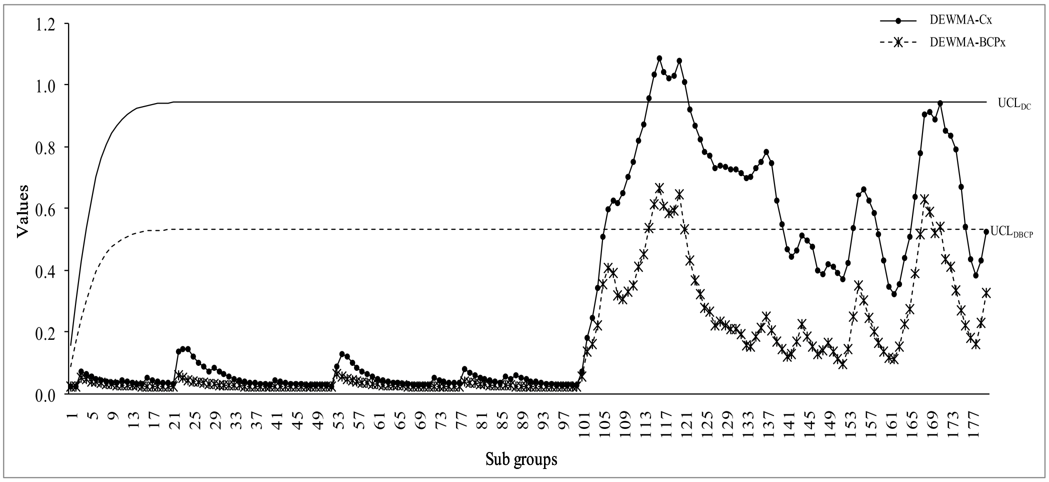

FIGURE 6. Control chart display of DEWMA-Cx & DEWMA-BCPx charts for shift in Errors Variance.

The simulation results may provide slightly different results if same random seed is not used.

VII. CASE STUDY

This section consider a real data application to further justify the importance of proposed Bayesian DEWMA charts when explanatory variables are not fixed and the process parameters are uncertain. This will give some insight into the computer implementation of proposed Bayesian approach. One can follow accordingly to apply proposed DEWMA charts to monitor production line.

Let us consider a data set of Typhoon predication for Xiamen city in the Fujian region of P. R. China for the year 2014. The data set composed in the library of College of Oceanography and Environmental Science, Xiamen University. An area with low atmosphere pressure is called Typhoon i.e., low pressure weather systems, while the air rotates anti-clockwise around the center point of low pressure area. The observation of high intensity of low pressure means high wind speed around the center point. The categories of the Typhoon are usually described by using the Saffir-Simpson hurricane wind scale (SSHWS) to rank the hurricane. Usually the tropical cyclone must have wind speed of at least 33 (m/s) to rank the storm in category-1, while the highest category-5 reserved for the hurricane with wind speed over 70 (m/s) and results in catastrophic damages. The high speed of wind means more dangerous and destructive nature of hurricane. The response variable of Atmosphere Pressure (Y) measured in hecto pascal (hpa) for the explanatory variable of Wind Speed (X) measured in meter per second (m/s) is observed for this study.

We consider the six month data set for the intense period of Typhoon in this region. The each observation of data set is measured with a sampling interval of six hour i.e., four measurements are recorded per day at 6am, 12pm, 18pm, and 12am, respectively. We have recorded a total of $N = 720$ observations. The data set is standardized to attain consistency in data and to observe smaller shifts in process parameters. We now defined each day as one profile with a sampling interval of 24 hours, results in total of $m = 180$ profiles.

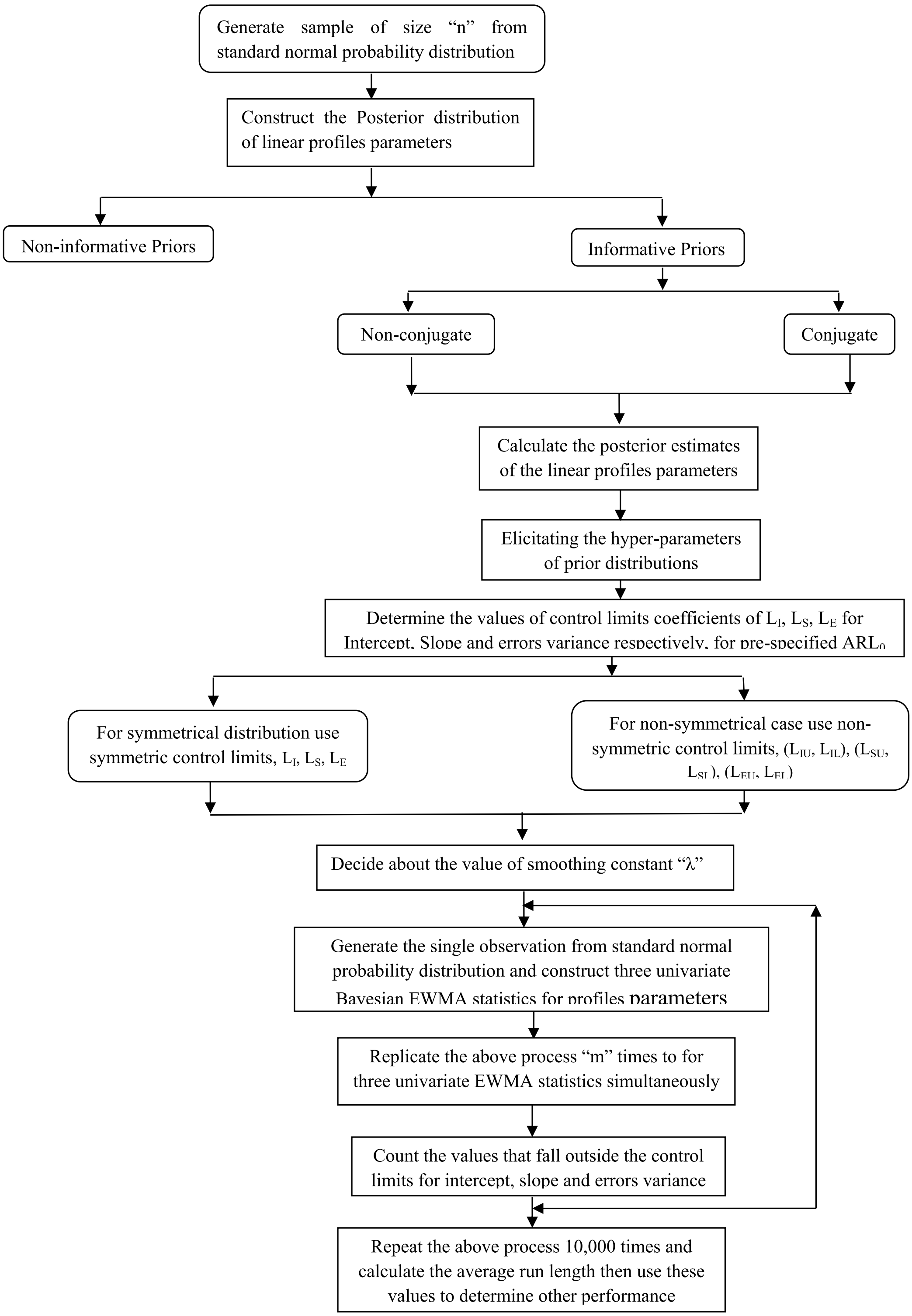

FIGURE 7. **Flow Chart: Bayesian DEWMA charts when random X.**

The values of response variable within profiles are random similar to the case in our study. We now divide the data set into in-control and out-of-control states. The first $N_0 = 400$ observations or $m_0 = 100$ profiles are considered for in-control state, while $N_1 = 320$ observations or $m_1 = 80$ profiles are considered for out-of-control state. For the in-control situation the shift value for the Y-intercept and slope are 0, while for the errors variance it is 1. The profile model with in-control process parameters are $Y_{ij} = 0.02 - 0.99X_{ij} + e_{ij}$. Under the in-control values of process parameter the control limit coefficients are adjusted for ARL_0 of 200. Now based on the real data sets the hyper-parameter values are elicitated as: The mean and variance of normal prior for the Y-intercept are 1.3 and 1.95, and the mean and variance of normal prior for slope are -0.23 and 2.92. The hyper-parameters for inverse gamma prior of errors variance are 1.35 and 0.5.

For the out-of-control scenario the shift values for Y-intercept as 0.4, for slope as 0.7, and for errors variance as 1.2. The figures 4-6 present the DEWMA-C_X and DEWMA-BCP_X statistics after incorporating aforesaid information. The values of DEWMA control chart under classical and Bayesian setups are plotted at Y-axis and the profiles are scaled at X-axis. The solid lines reflect the statistics and control limits of DEWMA-C_X, while the dotted lines represent the statistics and control limits of DEWMA-BCP_X. Figures 4-6 indicate that DEWMA-BCP_X detect 15, 14, and one more out-of-control signals than that of DEWMA-C_X control chart for shifts at Y-intercept, slope and errors variance, respectively. This mean the profiles shown in-control by DEWMA-C_X are actually out-of-control and are identified by DEWMA-BCP_X control charts. These findings are in accordance with the simulated results of section 5. This concludes the study that Bayesian schemes with prior information enhance perform of control charts and comes up with substantial benefits in manufacturing processes.

The procedural flowchart for the implementation of proposed methodology for efficient process monitoring is to given in Fig. 7.

VIII. CONCLUSIONS AND SUGGESTIONS

In this paper we have investigated the case of random explanatory variables for the monitoring of linear profiles and process parameters are assumed uncertain. The Bayesian DEWMA control structures are presented using non-conjugate and conjugate priors. Bayesian control charts incorporated the parametric uncertainty in the form of prior distributions that required more knowledge and keen observation of manufacturing process. There are situations in manufacturing processes where parametric uncertainty is unavoidable then Bayesian setups efficiently handle such situation which is not the case with classical approach.

Based on ARL values it is observed that proposed Bayesian DEWMA charts perform efficiently than that of classical DEWMA charts. This indicates that the inclusion of prior knowledge improves the efficiency of DEWMA control charts. It is observed that DEWMA-BCP_x control charts outperform the competing control charts for monitoring the Y-intercepts and slopes, while almost similar performance for errors variances monitoring. For the monitoring of linear profiles using Bayesian setup, conjugate priors are more effective than non-conjugate priors. This reflects that choice of priors and corresponding hyperparameters values should be selected carefully.

The proposed Bayesian schemes consider the simple case that can be extended to multivariate schemes of charts. Further, the simple linear case can also be extended and investigated for non-linear functions of profiles.

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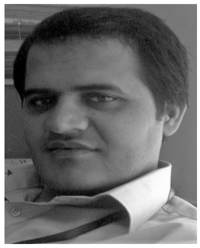

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