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Some New Results on Universal Metric Spaces

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Abstract : In this paper we establish some common fixed point results for two self-mappings f and g on a universal metric space of dimension n. To prove our results we assume that f is a weakly U-contraction mapping of types A_u and B_u with respect to g. Also we introduce a new concept Γ-distance on a complete partially ordered U_n-metric space and prove a fixed point theorem.

Keywords : universal metric spaces; altering distance function; weakly U-contraction mapping of types A_u (or B_u).
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1 Introduction and Preliminaries

Mustafa and Sims [1] introduced the concept of G-metric spaces in the year 2004 as a generalization of the metric spaces. In this type of spaces a non-negative real number is assigned to every triplet of elements. After that, many papers relating different "G-metric spaces" have been published by authors (see [2–17]).

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In the present work, we introduce a new notion of generalized G-metric space called universal metric space of dimension n and study some fixed point results for two self-mappings f and g on U_n-metric spaces. For similar results in this paper in G-metric space; see [18, 19, 20].

For $n \geq 2$, let X^n denotes the cartesian product $X \times \cdots \times X$ and $\mathbb{R}_+ = [0, +\infty)$. We begin with the following definition.

Definition 1.1. Let X be a non-empty set. Let $U_n : X^n \longrightarrow \mathbb{R}_+$ be a function that satisfies the following conditions:

(U1) $U_n(x_1, \ldots, x_n) = 0$ if $x_1 = \cdots = x_n$,

(U2) $U_n(x_1, \ldots, x_n) > 0$ for all $x_1, \ldots, x_n$ with $x_i \neq x_j$, for some $i, j \in \{1, \ldots, n\}$,

(U3) $U_n(x_1, \ldots, x_n) = U_n(x_{\pi_1}, \ldots, x_{\pi_n})$, for every permutation $(\pi_{(1)}, \ldots, \pi_{(n)})$ of $(1, 2, \ldots, n)$,

(U4) $U_n(x_1, x_2, \ldots, x_{n-1}, x_{n-1}) \leq U_n(x_1, x_2, \ldots, x_{n-1}, x_n)$ for all $x_1, \ldots, x_n \in X$,

(U5) $U_n(x_1, x_2, \ldots, x_n) \leq c(U_n(x_1, a, \ldots, a) + U_n(a, x_2, \ldots, x_n))$, for all $x_1, \ldots, x_n$, $a \in X$, $0 < c \leq 1$.

The function U_n is called a *universal metric of dimension* n, or *more specifically a* U_n*-metric* on X, and the pair (X, U_n) is called a U_n-metric space.

In the sequel, for simplicity we assume that $c = 1$. The following useful properties of a U_n-metric are easily derived from the axioms.

Proposition 1.2. *Let* (X, U_n) *be a* U_n*-metric space, then for any* $x_1, ..., x_n, a \in X$ *it follows that:*

(1) If $U_n(x_1, \ldots, x_n) = 0$*, then* $x_1 = \cdots = x_n$*,*

(2) $U_n(x_1, \ldots, x_n) \leq \sum_{j=2}^n U_n(x_1, \ldots, x_1, x_j)$*,*

(3) $U_n(x_1, \ldots, x_n) \leq \sum_{j=1}^n U_n(x_j, a, \ldots, a)$*,*

(4) $U_n(x_1, x_2, \ldots, x_2) \leq (n-1)U_n(x_1, \ldots, x_1, x_2)$*.*

The following are relevant examples of U_n-metric spaces. Note that most of them come from combing all pairwise ordinary distances in a some way.

I) Let (X, d) be a usual metric space, then (X, S_n) and (X, M_n) are U_n-metric spaces, where

$$S_n(x_1, \ldots, x_n) = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} d(x_i, x_j),$$

$$M_n(x_1, \ldots, x_n) = \max\{d(x_i, x_j) : 1 \leq i < j \leq n\}.$$

II) Let ϕ be a non-decreasing and concave function with $\phi(0) = 0$. If (X, d) is a usual metric space, then (X, ϕ_n) defined by

$$\phi_n(x_1, \ldots, x_n) = \phi^{-1}\Bigg(\sum_{1 \leq i < j \leq n} \phi(d(x_i, x_j)) \Bigg)$$

is a U_n-metric.

III) Let $X = C([0, T])$ be the set of all continuous functions defined on $[0, T]$. Defined $I_n : X^n \longrightarrow \mathbb{R}^+$ by

$$I_n(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} \sup_{t \in [0,T]} |x_i(t) - x_j(t)|.$$

(X, I_n) is a U_n-metric space.

The above examples show that from any metric on X we can construct a U_n-metric. Conversely, for any U_n-metric U_n on X,

$$d_U(x, y) = U_n(x, y, \ldots, y) + U_n(x, \ldots, x, y),$$

defines a metric on X.

Definition 1.3. Let (X, U_n) be a U_n-metric space, then for $x_0 \in X$, $r > 0$, the U_n-ball with center x_0 and radius r is

$$B_U(x_0, r) = \{y \in X : U_n(x_0, y, \ldots, y) < r\}.$$

Proposition 1.4. *Let (X, U_n) be a U_n-metric space, then for $x_0 \in X$, $r > 0$,*

(1) If $U_n(x_0, x_1, \ldots, x_n) < r$, then $x_1, \ldots, x_n \in B_U(x_0, r)$,

(2) If $y \in B_U(x_0, r)$, then there exists, $\delta > 0$ such that $B_U(y, \delta) \subseteq B_U(x_0, r)$,

(3) $B_U(x_0, \frac{1}{n}) \subseteq B_{d_U}(x_0, r) \subseteq B_U(x_0, r)$.

Remark 1.5.

(i) It follows from (2) of the above proposition that the family of all U_n-balls, $\mathcal{B} = \{B_U(x, r) : x \in X, r > 0\}$, is the base of a topology $\mathcal{T}(U)$ on X, the U_n-metric topology.

(ii) It follows from (3) of Proposition 1.4, that U_n-metric topology $\mathcal{T}(U)$ coincides with the metric topology arising from d_U. Thus while 'isometrically' distinct, every U_n-metric space is topologically equivalent to a metric space. This allows us to readily transport many concepts and results from metric spaces into U_n-metric space setting.

Definition 1.6. Let (X, U_n) be a U_n-metric space. The sequence $\{x_k\} \subseteq X$ is *U_n-convergent to x* if its converges to x in the U_n-metric topology, $\mathcal{T}(U)$.

Proposition 1.7. *Let (X, U_n) be a U-metric space. Then for a sequence $\{x_k\} \subseteq X$, and a point $x \in X$ the following are equivalent:*

(1) $\{x_k\}$ *is U_n-convergent to x.*

(2) $d_U(x_k, x) \to 0$, *as* $k \to \infty$.

(3) $U(x_k, \ldots, x_k, x) \to 0$, *as* $k \to \infty$.

(4) $U(x_k, x, \ldots, x) \to 0$, *as* $k \to \infty$.

(5) $U(x_m, x_k, \ldots, x_l, x) \to 0$, *as* $m, k, \ldots, l \to \infty$.

Definition 1.8. Let (X, U_n), (Y, V_m) be Universal metric spaces of dimension n, m respectively, a function $f : X \longrightarrow Y$ is $U_{n,m}$-continuous at point $x_0 \in X$ if $f^{-1}(B_{V_m}(f(x_0), r)) \in \mathcal{T}(U)$, for all $r > 0$. We say f is $U_{n,m}$-continuous if it is $U_{n,m}$-continuous at all points of X; that is, continuous as a function from X with the $\mathcal{T}(U)$-topology to Y with the $\mathcal{T}(V)$-topology.

In the sequel, for simplicity we have assume that $n = m$. Since U_n-metric topologies are metric topologies we have:

Proposition 1.9. *Let (X, U_n), (Y, V_n) be U_n-metric spaces, a function $f : X \longrightarrow Y$ is U_n-continuous at point $x \in X$ if and only if it is U_n-sequentially continuous at x; that is, whenever $\{x_k\}$ is U_n-convergent to x we have $(f(x_k))$ is U_n-convergent to $f(x)$.*

Proposition 1.10. *Let (X, U_n) be a U_n-metric space. Then the function $U_n(z_1, z_2, \ldots, z_n)$ is jointly continuous in all n of its variables.*

Now we discuss about concept completeness of U_n-metric spaces

Definition 1.11. Let (X, U_n) be a U_n-metric space, then a sequence $\{x_k\} \subseteq X$ is said to be *U_n-Cauchy* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $U_n(x_k, x_m, \ldots, x_l) < \varepsilon$ for all $k, m, \ldots, l \geq N$.

The next proposition follow directly from the definitions.

Proposition 1.12. *In a U_n-metric space, (X, U_n), the following are equivalent.*

(1) The sequence $\{x_k\}$ is U_n-Cauchy.

(2) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $U_n(x_m, \ldots, x_m, x_k) < \varepsilon$, for all $k, m \geq N$.

(3) $\{x_k\}$ is a Cauchy sequence in the metric space (X, d_U).

2 Main Results

In metric fixed point theory, the concept of altering distance function has been used by many authors in a number of works on fixed points. An altering distance function is actually a control function which alters the distance between two points in a metric space. This concept was introduced by Khan et al. in 1984 in their well known paper [21] in which addressed a new category of metric fixed point problems by use of such functions.

Definition 2.1. The function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called an *altering distance function* if the following properties are satisfied:

(a) ψ is continuous and increasing;

(b) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.2. Let (X, U_n) be a U_n-metric space and $f, g : X \longrightarrow X$ be two mappings. We say that f is a weakly U-contraction mapping of type B_u with respect to g if for all $z_1, \ldots, z_n \in X$, the following inequality holds:

$$\begin{aligned}\psi\Big(U_n(fz_1, fz_2, \ldots, fz_n)\Big) &\leq \psi\left(\frac{1}{n}\left(\sum_{i=1}^{n-1} U_n(gz_i, \ldots, gz_i, fz_{i+1}) + U_n(gz_n, \ldots, gz_n, fz_1)\right)\right) \\ &\quad - \phi\left(\sum_{i=1}^{n-1} U_n(gz_i, \ldots, gz_i, fz_{i+1})e_i + U_n(gz_n, \ldots, gz_n, fz_1)e_n\right) \quad (2.1)\end{aligned}$$

where

(a) ψ is an altering distance function;

(b) $\phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ is continuous function with $\phi(u_1, \ldots, u_n) = 0$ if and only if $u_1 = \cdots = u_n = 0$.

Theorem 2.3. *Let (X, U_n) be a U_n-metric space and $f, g : X \longrightarrow X$ be two mappings such that f is a weakly U-contraction mapping of type B_u with respect to g. Assume that*

(i) $f(X) \subseteq g(X)$,

(ii) $g(X)$ is a complete subset of (X, U_n),

(iii) The pair $\{f, g\}$is weakly compatible.

Then f and g have a unique common fixed point.

Proof. By the fact that $f(X) \subseteq g(X)$, we can construct a sequence $\{x_k\}$ in X such that $gx_{k+1} = fx_k$ for any $k \in \mathbb{N}$. If for some k, $gx_{k+1} = gx_k$, then $gx_k = fx_k$, that is, f and g have a common fixed point. Thus, we may assume that $gx_{k+1} \neq gx_k$ for any $k \in \mathbb{N}$. For $k \in \mathbb{N}$, then by (2.1) and (U_5), we get

$$\begin{aligned}
&\psi\Big(U_n(gx_k,\dots,gx_k,gx_{k+1})\Big)\\
&\quad= \psi\Big(U_n(fx_{k-1},\dots,fx_{k-1},fx_k)\Big)\\
&\quad\le \psi\Big(\frac{1}{n}((n-2)U_n(gx_{k-1},\dots,gx_{k-1},gx_k)+U_n(gx_{k-1},\dots,gx_{k-1},gx_{k+1})\\
&\qquad\quad + U_n(gx_k,gx_k,\dots,gx_k))\Big) - \phi\Big(\sum_{i=1}^{n-2} U_n(gx_{k-1},\dots,gx_{k-1},gx_k)e_i\\
&\qquad\quad + U_n(gx_{k-1},\dots,gx_{k-1},gx_{k+1})e_{n-1} + U_n(gx_k,gx_k,\dots,gx_k)e_n\Big)\\
&\quad\le \psi\Big(\frac{1}{n}((n-2)U_n(gx_{k-1},\dots,gx_{k-1},gx_k)+U_n(gx_{k-1},\dots,gx_{k-1},gx_{k+1}))\Big)\\
&\quad\le \psi\Big(\frac{n-1}{n}U_n(gx_{k-1},\dots,gx_{k-1},gx_k)+\frac{1}{n}U_n(gx_k,\dots,gx_k,gx_{k+1})\Big). \qquad (2.2)
\end{aligned}$$

Since ψ is increasing, by (2.2), we have

$$\begin{aligned}
&U_n(gx_k,\dots,gx_k,gx_{k+1})\\
&\qquad \le \frac{1}{n}((n-2)U_n(gx_{k-1},\dots,gx_{k-1},gx_k)+U_n(gx_{k-1},\dots,gx_{k-1},gx_{k+1}))\\
&\qquad \le \frac{n-1}{n}U_n(gx_{k-1},\dots,gx_{k-1},gx_k)+\frac{1}{n}U_n(gx_k,\dots,gx_k,gx_{k+1}). \qquad (2.3)
\end{aligned}$$

Then, it follows easily that

$$U_n(gx_k,\dots,gx_k,gx_{k+1}) \le U_n(gx_{k-1},\dots,gx_{k-1},gx_k) \text{ for any } \quad k \ge 1. \qquad (2.4)$$

Therefore $\{U_n(gx_k,\dots,gx_k,gx_{k+1}),\ k \in \mathbb{N}\}$ is a decreasing sequence. Hence there exists $r \ge 0$ such that

$$\lim_{k\to+\infty} U_n(gx_k,\dots,gx_k,gx_{k+1}) = r. \qquad (2.5)$$

Letting $k \to +\infty$ in (2.3), we get

$$r \le \frac{n-2}{n}r + \frac{1}{n}\lim_{k\to+\infty} U_n(gx_{k-1},\dots,gx_{k-1},gx_{k+1}) \le \frac{n-1}{n}r + \frac{1}{n}r = r,$$

which implies that

$$\lim_{k\to+\infty} U_n(gx_{k-1},\dots,gx_{k-1},gx_{k+1}) = 2r. \qquad (2.6)$$

Again, from (2.2) we have

$$\begin{aligned}\psi\Big(U_n(gx_k,\dots,gx_k,gx_{k+1})\Big)\\ &\le \psi\Big(\frac{1}{n}((n-2)U_n(gx_{k-1},\dots,gx_{k-1},gx_k)+U_n(gx_{k-1},\dots,gx_{k-1},gx_{k+1}))\Big)\\ &\quad -\phi\Big(\sum_{i=1}^{n-2}U_n(gx_{k-1},\dots,gx_{k-1},gx_k)e_i+(gx_{k-1},\dots,gx_{k-1},gx_{k+1})e_{n-1}\Big).\end{aligned}$$

Letting $k\to+\infty$ and using (2.5), (2.6) and from continuities of ψ and ϕ, we get

$$\psi(r)\le\psi(r)-\phi(r,\dots,r,2r,0),$$

and hence $\phi(r,\dots,r,2r,0)=0$. By a property of ϕ, we deduce that $r=0$, that is,

$$\lim_{k\to+\infty}U_n(gx_k,\dots,gx_k,gx_{k+1})=0. \tag{2.7}$$

Next, we will show that $\{gx_k\}$ is a U_n-Cauchy sequence. Suppose, on the contrary, that $\{gx_k\}$ is not a U_n-Cauchy sequence, that is,

$$\lim_{m,k\to+\infty}U_n(gx_m,\dots,gx_m,gx_k)\ne 0.$$

Then, there exists $\varepsilon>0$ for which we can find two subsequences $\{gx_{m_i}\}$ and $\{gx_{n_i}\}$ of $\{x_k\}$ such that n_i is the smallest index for which

$$n_i>m_i>i,\qquad U_n(gx_{m_i},\dots,gx_{m_i},gx_{n_i})\ge\varepsilon. \tag{2.8}$$

This means that

$$U_n(gx_{m_i},\dots,gx_{m_i},gx_{n_i-1})<\varepsilon. \tag{2.9}$$

Now, from (2.8), (2.9), (U_5) and item (4) of Proposition 1.2, we have that

$$\begin{aligned}\varepsilon &\le U_n(gx_{m_i},\dots,gx_{m_i},gx_{n_i})\\ &\le U_n(gx_{m_i},\dots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i+1},\dots,gx_{m_i+1},gx_{n_i})\\ &\le U_n(gx_{m_i},\dots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i+1},\dots,gx_{m_i+1},gx_{n_i-1})\\ &\quad +U_n(gx_{n_i-1},\dots,gx_{n_i-1},gx_{n_i})\\ &\le n\,U_n(gx_{m_i},\dots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i},\dots,gx_{m_i},gx_{n_i-1})\\ &\quad +U_n(gx_{n_i-1},\dots,gx_{n_i-1},gx_{n_i})\\ &< n\,U_n(gx_{m_i},\dots,gx_{m_i},gx_{m_i+1})+\varepsilon+U_n(gx_{n_i-1},\dots,gx_{n_i-1},gx_{n_i}).\end{aligned}$$

Letting $i\to+\infty$ in the above inequalities and using (2.7), we get that

$$\begin{aligned}\lim_{i\to+\infty}U_n(gx_{m_i},\dots,gx_{m_i},gx_{n_i}) &= \lim_{i\to+\infty}U_n(gx_{m_i+1},\dots,gx_{m_i+1},gx_{n_i})\\ &= \lim_{i\to+\infty}U_n(gx_{m_i},\dots,gx_{m_i},gx_{n_i-1})\\ &= \varepsilon. \end{aligned}\tag{2.10}$$

By (2.1), we have

$$\begin{aligned}
&\psi\Big(U_n(gx_{m_i+1},\ldots,gx_{m_i+1},gx_{n_i})\Big)\\
&\quad=\psi\Big(U_n(fx_{m_i},\ldots,fx_{m_i},fx_{n_i-1})\Big)\\
&\quad\le\psi\Big(\frac{1}{n}((n-2)U_n(gx_{m_i},\ldots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i},\ldots,gx_{m_i},gx_{n_i})\\
&\qquad+U_n(gx_{n_i-1},\ldots,gx_{n_i-1},gx_{m_i+1}))\Big)-\phi\Big(\sum_{i=1}^{n-2}U_n(gx_{m_i},\ldots,gx_{m_i},gx_{m_i+1})e_i\\
&\qquad+U_n(gx_{m_i},\ldots,gx_{m_i},gx_{n_i})e_{n-1}+U_n(gx_{n_i-1},\ldots,gx_{n_i-1},gx_{m_i+1})e_n\Big)\\
&\quad\le\psi\Big(\frac{1}{n}((n-2)U_n(gx_{m_i},\ldots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i},\ldots,gx_{m_i},gx_{n_i})\\
&\qquad+U_n(gx_{n_i-1},\ldots,gx_{n_i-1},gx_{m_i+1}))\Big).
\end{aligned}\tag{2.11}$$

Once again, since ψ is increasing, we get

$$\begin{aligned}
&U_n(gx_{m_i+1},\ldots,gx_{m_i+1},gx_{n_i})\\
&\qquad\le\frac{1}{n}((n-2)U_n(gx_{m_i},\ldots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i},\ldots,gx_{m_i},gx_{n_i})\\
&\qquad\quad+U_n(gx_{n_i-1},\ldots,gx_{n_i-1},gx_{m_i+1})).
\end{aligned}$$

Then, by (U_5) and Proposition 1.2, we have

$$\begin{aligned}
&U_n(gx_{m_i+1},\ldots,gx_{m_i+1},gx_{n_i})\\
&\quad\le\frac{1}{n}((n-2)U_n(gx_{m_i},\ldots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i},\ldots,gx_{m_i},gx_{n_i})\\
&\qquad+U_n(gx_{n_i-1},\ldots,gx_{n_i-1},gx_{m_i+1}))\\
&\quad\le\frac{1}{n}((n-2)U_n(gx_{m_i},\ldots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i},\ldots,gx_{m_i},gx_{n_i})\\
&\qquad+(n-1)U_n(gx_{n_i-1},gx_{m_i+1},\ldots,gx_{m_i+1}))\\
&\quad\le\frac{1}{n}((n-2)U_n(gx_{m_i},\ldots,gx_{m_i},gx_{m_i+1})+U_n(gx_{m_i},\ldots,gx_{m_i},gx_{n_i})\\
&\qquad+(n-1)\Big(U_n(gx_{m_i},\ldots,gx_{m_i},gx_{n_i-1})+U_n(gx_{m_i},gx_{m_i+1},\ldots,gx_{m_i+1})\Big)\Big).
\end{aligned}$$

Letting $i\to+\infty$ in the above inequalities, and using (2.7) and (2.10), we get that

$$U_n(gx_{n_i-1},\ldots,gx_{n_i-1},gx_{m_i+1})=(n-1)\varepsilon.\tag{2.12}$$

Now, letting $i \to +\infty$ in (2.11) and using (2.7), (2.10), (2.12) and the continuities of ψ and ϕ, we have

$$\psi(\varepsilon) \leq \psi\bigg(\frac{1}{n}(\varepsilon + (n-1)\varepsilon)\bigg) - \phi(0, \ldots, 0, \varepsilon, (n-1)\varepsilon).$$

Therefore, we get $\phi(0, \ldots, 0, \varepsilon, (n-1)\varepsilon) = 0$ and hence, by property of ϕ, we deduce $\varepsilon = 0$, a contradiction. Thus $\{gx_k\}$ is a U_n-Cauchy sequence in $g(X)$. Since $(g(X), U_n)$ is complete, there exist $t, u \in X$ such that $\{gx_k\}$ converges to $t = gu$, that is,

$$\lim_{k\to+\infty} U_n(gx_k, gu, \ldots, gu) = \lim_{k\to+\infty} U_n(gx_k, \ldots, gx_k, gu) = 0. \tag{2.13}$$

Then by Proposition 1.10, we have

$$\lim_{k\to+\infty} U_n(gx_k, fu, \ldots, fu) = U_n(gu, fu, \ldots, fu), \tag{2.14}$$

and

$$\lim_{k\to+\infty} U_n(gx_k, \ldots, gx_k, fu) = U_n(gu, \ldots, gu, fu). \tag{2.15}$$

Let us show that $fu = t$. By (2.1), we have

$$\begin{aligned} &\psi\Big(U_n(gx_{k+1}, \ldots, gx_{k+1}, fu)\Big) \\ &\qquad \leq \psi\bigg(\frac{1}{n}((n-2)U_n(gx_k, \ldots, gx_k, gx_{k+1}) + U_n(gx_k, \ldots, gx_k, fu) \\ &\qquad\quad + U_n(gu, \ldots, gu, gx_{k+1}))\bigg) - \phi\bigg(\sum_{i=1}^{n-2} U_n(gx_k, \ldots, gx_k, gx_{k+1})e_i \\ &\qquad\quad + U_n(gx_k, \ldots, gx_k, fu)e_{n-1} + U_n(gu, \ldots, gu, gx_{k+1})e_n\bigg). \end{aligned}$$

Letting $k \to +\infty$ and using (2.7), (2.13) and (2.14) and the continuities of ψ and ϕ, we get

$$\begin{aligned} &\psi\Big(U_n(gu, \ldots, gu, fu)\Big) \\ &\qquad \leq \psi\bigg(\frac{1}{n}U_n(gu, \ldots, gu, fu)\bigg) - \phi\Big(0, \ldots, 0, U_n(gu, \ldots, gu, fu), 0\Big). \end{aligned} \tag{2.16}$$

Since ψ is increasing therefore $U_n(gu, \ldots, gu, fu) = 0$ and hence $fu = gu = t$. Then, u is a coincidence point of f and g, and since the pair $\{f, g\}$ is weakly compatible, we have $ft = gt$. Now we prove that $ft = gt = t$. By (2.1), we have

$$\begin{aligned}
&\psi\Big(U_n(gt,\dots,gt,gx_{k+1})\Big)\\
&\qquad \le \psi\Big(\frac{1}{n}((n-2)U_n(gt,\dots,gt,ft) + U_n(gt,\dots,gt,gx_{k+1})\\
&\qquad\qquad + U_n(gx_k,\dots,gx_k,ft))\Big) - \phi\Big(\sum_{i=1}^{n-2} U_n(gt,\dots,gt,ft)e_i\\
&\qquad\qquad + U_n(gt,\dots,gt,gx_{k+1})e_{n-1} + U_n(gx_k,\dots,gx_k,ft)e_n\Big).
\end{aligned}$$

Letting $k \to +\infty$, we get

$$\begin{aligned}
\psi\Big(U_n(gt,\dots,gt,gu)\Big) &\le \psi\Big(\frac{1}{n}(U_n(gt,\dots,gt,gu) + U_n(gu,\dots,gu,gt))\Big)\\
&\qquad - \phi\Big(U_n(gt,\dots,gt,gu)e_{n-1} + U_n(gu,\dots,gu,gt)e_n\Big)\\
&\le \psi\Big(\frac{1}{n}U_n(gt,\dots,gt,gu) + \frac{n-1}{n}U_n(gt,\dots,gt,gu))\Big)\\
&\qquad - \phi\Big(U_n(gt,\dots,gt,gu)e_{n-1} + U_n(gu,\dots,gu,gt)e_n\Big)\\
&= \psi\Big(U_n(gt,\dots,gt,gu)\Big)\\
&\qquad - \phi\Big(U_n(gt,\dots,gt,gu)e_{n-1} + U_n(gu,\dots,gu,gt)e_n\Big),
\end{aligned}$$

Which is true if $\phi\Big(U_n(gt,\dots,gt,gu)e_{n-1} + U_n(gu,\dots,gu,gt)e_n\Big) = 0$, that is, $gt = gu = t$. We conclude that $t = gt = ft$, and so t is a common fixed point of f and g. To prove the uniqueness, let t' be another common fixed point of f and g. By (2.1), we have

$$\begin{aligned}
&\psi\Big(U_n(t,t',\dots,t')\Big)\\
&\qquad = \psi\Big(U_n(ft,ft',\dots,ft')\Big)\\
&\qquad \le \psi\Big(\frac{1}{n}(U_n(gt,\dots,gt,ft') + (n-2)U_n(gt',\dots,gt',ft')\\
&\qquad\quad + U_n(gt',\dots,gt',ft))\Big) - \phi\Big(U_n(gt,\dots,gt,ft')e_1\\
&\qquad\quad + \sum_{i=2}^{n-1} U_n(gt',\dots,gt',ft')e_i + U_n(gt',\dots,gt',ft)e_n\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \psi\bigg(\frac{1}{n}(U_n(gt,\ldots,gt,ft') + U_n(gt',\ldots,gt',ft))\bigg)\\
&\qquad - \phi\bigg(U_n(gt,\ldots,gt,ft')e_1 + U_n(gt',\ldots,gt',ft)e_n\bigg)\\
&\leq \psi\bigg(\frac{1}{n}((n-1)U_n(t,t',\ldots,t') + U_n(t',\ldots,t',t))\bigg)\\
&\qquad - \phi\bigg(U_n(t,\ldots,t,t')e_1 + U_n(t',\ldots,t',t)e_n\bigg)\\
&= \psi\bigg(U_n(t,t',\ldots,t')\bigg) - \phi\bigg(U_n(t,\ldots,t,t')e_1 + U_n(t',\ldots,t',t)e_n\bigg).
\end{aligned}
$$

Therefore, $\phi\bigg(U_n(t,\ldots,t,t')e_1+U_n(t',\ldots,t',t)e_n\bigg) = 0$ and hence $U_n(t,\ldots,t,t') = U_n(t',\ldots,t',t) = 0$. Thus $t = t'$. □

Corollary 2.4. *Let (X, U_n) be a U_n-metric space and $f, g : X \longrightarrow X$ be two mappings such that:*

$$
U_n(fz_1, fz_2, \ldots, fz_n) \leq \alpha\left(\sum_{i=1}^{n-1} U_n(gz_i,\ldots,gz_i,fz_{i+1}) + U_n(gz_n,\ldots,gz_n,fz_1)\right),
$$

where $\alpha \in [0, \frac{1}{n})$. Assume that

(i) $f(X) \subseteq g(X)$,

(ii) $g(X)$ is a complete subspace of (X, U_n),

(iii) The pair $\{f, g\}$is weakly compatible.

Then f and g have a unique common fixed point.

Proof. It suffices to take $\psi(t) = t$ and $\phi(\sum_{i=1}^n u_i\, e_i) = (\frac{1}{n} - \alpha)(\sum_{i=1}^n u_i)$ in Theorem 2.3. □

Corollary 2.5. *Let (X, U_n) be a complete U_n-metric space and $f : X \longrightarrow X$ be such that:*

$$
\begin{aligned}
&\psi\bigg(U_n(fz_1, fz_2, \ldots, fz_n)\bigg)\\
&\qquad\qquad \leq \psi\bigg(\frac{1}{n}(\sum_{i=1}^{n-1} U_n(z_i,\ldots,z_i,fz_{i+1}) + U_n(z_n,\ldots,z_n,fz_1))\bigg)\\
&\qquad\qquad - \phi\bigg(\sum_{i=1}^{n-1} U_n(z_i,\ldots,z_i,fz_{i+1})e_i + U_n(z_n,\ldots,z_n,fz_1)e_n\bigg)
\end{aligned}
$$

where

(i) ψ *is an altering distance function;*

(ii) $\phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ *is continuous function with* $\phi(u_1, \ldots, u_n) = 0$ *if and only if* $u_1 = \cdots = u_n = 0$.

Then f *has a unique fixed point.*

Proof. It suffices to take $g = Id_x$, the identity mapping on X in Theorem 2.3. □

Corollary 2.6. *Let* (X, U_n) *be a complete* U_n*-metric space and* $f : X \longrightarrow X$ *be two mappings such that:*

$$U_n(fz_1, fz_2, \ldots, fz_n) \leq \frac{1}{n}\Big(\sum_{i=1}^{n-1} U_n(z_i, \ldots, z_i, fz_{i+1}) + U_n(z_n, \ldots, z_n, fz_1)\Big) - \phi\Big(\sum_{i=1}^{n-1} U_n(z_i, \ldots, z_i, fz_{i+1})e_i + U_n(z_n, \ldots, z_n, fz_1)e_n\Big)$$

where $\phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ *is continuous function with* $\phi(u_1, \ldots, u_n) = 0$ *if and only if* $u_1 = \cdots = u_n = 0$. *Then* f *has a unique fixed point.*

Proof. It follows by taking $\psi(t) = t$ in Corollary 2.5. □

Definition 2.7. Let (X, U_n) be a U_n-metric space and $f, g : X \longrightarrow X$ be two mappings. We say that f is a weakly U-contraction mapping of type A_u with respect to g if for all $z_1, \ldots, z_n \in X$, the following in equality holds:

$$\psi\Big(U_n(fz_1, fz_2, \ldots, fz_n)\Big) \leq \psi\Big(\frac{1}{n}\Big(\sum_{i=1}^{n-1} U_n(gz_i, fz_{i+1}, \ldots, fz_{i+1}) + U_n(gz_n, fz_1, \ldots, fz_1)\Big)\Big) - \phi\Big(\sum_{i=1}^{n-1} U_n(gz_i, fz_{i+1}, \ldots, fz_{i+1})e_i + U_n(gz_n, fz_1, \ldots, fz_1)e_n\Big)$$

where

(a) ψ is an altering distance function;

(b) $\phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ is continuous function with $\phi(u_1, \ldots, u_n) = 0$ if and only if $u_1 = \cdots = u_n = 0$.

Using arguments similar to those in Theorem 2.3, we can prove the following theorem.

Theorem 2.8. *Let* (X, U_n) *be a* U_n*-metric space and* $f, g : X \longrightarrow X$ *be two mappings such that* f *is a weakly* U*-contraction mapping of type* A_u *with respect to* g. *Assume that* *(i)* $f(X) \subseteq g(X)$,

(ii) $g(X)$ *is a complete subspace of* (X, U_n),

(iii) *The pair* $\{f, g\}$*is weakly compatible.*

Then f *and* g *have a unique common fixed point.*

As in the case of Theorem 2.3, we can deduce various corollaries from Theorem 2.8.

Corollary 2.9. *Let* (X, U_n) *be a* U_n*-metric space and* $f, g : X \longrightarrow X$ *be two mappings such that:*

$$U_n(fz_1, fz_2, \ldots, fz_n) \leq \alpha\biggl(\sum_{i=1}^{n-1} U_n(gz_i, fz_{i+1}, \ldots, fz_{i+1}) + U_n(gz_n, fz_1, \ldots, fz_1)\biggr),$$

where $\alpha \in [0, \frac{1}{n})$. *Assume that*

(i) $f(X) \subseteq g(X)$,

(ii) $g(X)$ *is a complete subspace of* (X, U_n),

(iii) *the pair* $\{f, g\}$*is weakly compatible.*

Then f *and* g *have a unique common fixed point.*

Corollary 2.10. *Let* (X, U_n) *be a complete* U_n*-metric space and* $f : X \longrightarrow X$ *be such that:*

$$\begin{aligned}\psi\Bigl(U_n(fz_1,fz_2, \ldots, fz_n)\Bigr) &\\ \leq \psi\biggl(\frac{1}{n}\biggl(\sum_{i=1}^{n-1} U_n(z_i, fz_{i+1}, \ldots, fz_{i+1}) + U_n(z_n, fz_1, \ldots, fz_1)\biggr)\biggr) &\\ - \phi\biggl(\sum_{i=1}^{n-1} U_n(z_i, fz_{i+1}, \ldots, fz_{i+1})e_i + U_n(z_n, fz_1, \ldots, fz_1)e_n\biggr)&\end{aligned}$$

where

(i) ψ *is an altering distance function,*

(ii) $\phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ *is continuous function with* $\phi(u_1, \ldots, u_n) = 0$ *if and only if* $u_1 = \cdots = u_n = 0$.

Then f *has a unique fixed point.*

Corollary 2.11. *Let (X, U_n) be a complete U_n-metric space and $f : X \longrightarrow X$ be such that:*

$$\begin{aligned} &U_n(fz_1, fz_2, \dots, fz_n) \\ &\quad\le \frac{1}{n}\Big(\sum_{i=1}^{n-1} U_n(z_i, fz_{i+1}, \dots, fz_{i+1}) + U_n(z_n, fz_1, \dots, fz_1)\Big) \\ &\qquad - \phi\Big(\sum_{i=1}^{n-1} U_n(z_i, fz_{i+1}, \dots, fz_{i+1})e_i + U_n(z_n, fz_1, \dots, fz_1)e_n\Big) \end{aligned}$$

where $\phi : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ is continuous function with $\phi(u_1, \dots, u_n) = 0$ if and only if $u_1 = \cdots = u_n = 0$. Then f has a unique fixed point.

Remark 2.12. *Using arguments similar to those in corollaries 2.10, 2.5 with weak condition we can prove the following theorem.*

Theorem 2.13. *Let X be a complete U_n-metric space. Suppose the map $f : X \longrightarrow X$ satisfies for all $z_1, z_2, \dots, z_n \in X$*

$$\psi(U_n(fz_1, fz_2, \dots, fz_n)) \le \psi(U_n(z_1, z_2, \dots, z_n)) - \phi(U_n(z_1, z_2, \dots, z_n)), \quad (2.17)$$

where ψ and ϕ are altering distance functions. Then f has a unique fixed point (say u) and f is U_n-continuous at u.

Proof. Let x_0 be an arbitrary point in X, and let $x_{k+1} = fx_k$ for any $k \in \mathbb{N}$. Assume $x_k \ne x_{k-1}$. For $k \in \mathbb{N}$, we use (2.17) and definition of ϕ

$$\begin{aligned} \psi(U_n(x_k, x_{k+1}, \dots, x_{k+1})) &= \psi(U_n(fx_{k-1}, fx_k, \dots, fx_k)) \\ &\le \psi(U_n(x_{k-1}, x_k, \dots, x_k)) - \phi(U_n(x_{k-1}, x_k, \dots, x_k)) \\ &\le \psi(U_n(x_{k-1}, x_k, \dots, x_k)). \quad (2.18) \end{aligned}$$

Since ψ is non-decreasing, we get that

$$U_n(x_k, x_{k+1}, \dots, x_{k+1}) \le U_n(x_{k-1}, x_k, \dots, x_k). \quad (2.19)$$

If we take $t_k = U_n(x_k, x_{k+1}, \dots, x_{k+1})$, then from (2.19), we get $0 \le t_k \le t_{k-1}$, so the sequence $\{t_k\}$ is non-increasing, hence it converges to some $r \ge 0$. Letting this in (2.18), then as $k \to +\infty$

$$\psi(r) \le \psi(r) - \phi(r),$$

using the continuity of ψ and ϕ. Then, we find $\phi(r) = 0$, hence by a property of ϕ, we have $r = 0$. We rewrite this as

$$\lim_{k\to+\infty} U_n(x_k, x_{k+1}, \dots, x_{k+1}) = 0. \quad (2.20)$$

Next, we prove that $\{x_k\}$ is a U_n-Cauchy sequence. We argue by contradiction. Assume that $\{x_k\}$ is not a U_n-Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_k\}$ with $n_i > m_i > i$ such that

$$U_n(x_{n_i}, x_{m_i}, \dots, x_{m_i}) \geq \varepsilon. \tag{2.21}$$

Further, corresponding to m_i, we can choose n_i in such a way that it is the smallest integer with $n_i > m_i$ and satisfying (2.21). Then

$$U_n(x_{n_i-1}, x_{m_i}, \dots, x_{m_i}) < \varepsilon. \tag{2.22}$$

We have, using (2.22) and the condition (U_5), that

$$\begin{aligned}\varepsilon &\leq U_n(x_{n_i}, x_{m_i}, \dots, x_{m_i})\\ &\leq U_n(x_{n_i}, x_{n_i-1}, \dots, x_{n_i-1}) + U_n(x_{n_i-1}, x_{m_i}, \dots, x_{m_i})\\ &< \varepsilon + U_n(x_{n_i}, x_{n_i-1}, \dots, x_{n_i-1}).\end{aligned} \tag{2.23}$$

In other words, we have:

$$\begin{aligned}0 &\leq U_n(x_{n_i}, x_{n_i-1}, \dots, x_{n_i-1})\\ &= U_n(x_{n_i-1}, \dots, x_{n_i-1}, x_{n_i})\\ &\leq (n-1)U_n(x_{n_i-1}, x_{n_i}, \dots, x_{n_i}).\end{aligned}$$

Letting $i \to +\infty$, and using (2.20), we find $U_n(x_{n_i}, x_{n_i-1}, \dots, x_{n_i-1}) \to 0$. We take this in (2.23)

$$\lim_{i\to+\infty} U_n(x_{n_i}, x_{m_i}, \dots, x_{m_i}) = \varepsilon. \tag{2.24}$$

Moreover, we have:

$$\begin{aligned}U_n(x_{n_i}, x_{m_i}, \dots, x_{m_i}) \leq{}& U_n(x_{n_i}, x_{n_i-1}, \dots, x_{n_i-1}) + U_n(x_{n_i-1}, x_{m_i-1}, \dots, x_{m_i-1})\\ &+ U_n(x_{m_i-1}, x_{m_i}, \dots, x_{m_i}),\end{aligned}$$

and

$$\begin{aligned}U_n(x_{n_i-1}, x_{m_i-1}, \dots, x_{m_i-1}) \leq{}& U_n(x_{n_i-1}, x_{n_i}, \dots, x_{n_i}) + U_n(x_{n_i}, x_{m_i}, \dots, x_{m_i})\\ &+ U_n(x_{m_i}, x_{m_i-1}, \dots, x_{m_i-1}).\end{aligned}$$

Letting $i \to +\infty$ in the above two inequalities and using (2.20)-(2.24)

$$\lim_{i\to+\infty} U_n(x_{n_i-1}, x_{m_i-1}, \dots, x_{m_i-1}) = \varepsilon. \tag{2.25}$$

Setting $z_1 = x_{n_i-1}$ and $z_2 = \cdots = z_n = x_{m_i-1}$ in (2.17) and using (2.21), we obtain thanks to the fact that ψ is increasing

$$\begin{aligned}\psi(\varepsilon) &\leq \psi(U_n(x_{n_i}, x_{m_i}, \dots, x_{m_i})) = \psi(U_n(fx_{n_i-1}, fx_{m_i-1}, \dots, fx_{m_i-1}))\\ &\leq \psi(U_n(x_{n_i-1}, x_{m_i-1}, \dots, x_{m_i-1})) - \phi(U_n(x_{n_i-1}, x_{m_i-1}, \dots, x_{m_i-1})).\end{aligned}$$

Letting $i \to +\infty$, then using (2.25) and the continuity of ψ and ϕ, we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon),$$

yielding that $\phi(\varepsilon) = 0$, which is a contradiction since $\varepsilon > 0$. This show that $\{x_k\}$ is a U_n-convergent to some $u \in X$, then

$$\lim_{k\to+\infty} U_n(x_k, \dots, x_k, u) = U_n(x_k, u, \dots, u) = 0. \tag{2.26}$$

We show now that u is a fixed point of the map f. From (2.17),

$$\begin{aligned}\psi(U_n(x_{k+1}, \dots, x_{k+1}, fu)) &= \psi(U_n(fx_k, \dots, fx_k, fu))\\ &\leq \psi(U_n(x_k, \dots, x_k, u)) - \phi(U_n(x_k, \dots, x_k, u)).\end{aligned}$$

Thanks to (2.26) and the continuity of ψ and ϕ, we find

$$\lim_{k\to+\infty} U_n(x_{k+1}, \dots, x_{k+1}, fu) = 0. \tag{2.27}$$

Again, using the condition (U_4) and (U_5), one can write

$$U_n(u, \dots, u, fu) \leq U_n(u, \dots, u, x_{k+1}) + U_n(x_{k+1}, \dots, x_{k+1}, fu).$$

Letting $k \to +\infty$ in the above inequality and having in mind (2.26) and (2.27), one finds $U_n(u, \dots, u, fu) = 0$, and then $fu = u$. Hence u is a fixed point of f. Let us to show its uniqueness. Let v be another fixed point of f, then

$$\begin{aligned}\psi(U_n(u, \dots, u, v)) &= \psi(U_n(fu, \dots, fu, fv))\\ &\leq \psi(U_n(u, \dots, u, v)) - \phi(U_n(u, \dots, u, v)).\end{aligned}$$

It follows that $\phi(U_n(u, \dots, u, v)) = 0$, and then $U_n(u, \dots, u, v) = 0$, yielding that $u = v$. Following Proposition 1.9, to show that f is U_n-continuous at u, let $\{y_k\}$ be any sequence in X such that $\{y_k\}$ is U_n-convergent to u. For $k \in \mathbb{N}$, we have

$$\begin{aligned}\psi(U_n(u, \dots, u, fy_k)) &= \psi(U_n(fu, \dots, fu, fy_k))\\ &\leq \psi(U_n(u, \dots, u, y_k)) - \phi(U_n(u, \dots, u, y_k)).\end{aligned}$$

Letting $k \to +\infty$ and using again the continuity of ψ and ϕ, the right-hand side of the above inequality tends to 0, then we obtain

$$\lim_{k\to+\infty} U_n(u, \dots, u, fy_k) = 0.$$

Hence $\{fy_k\}$ is U_n-convergent to $u = fu$, so f is U_n-continuous at u. □

Corollary 2.14. *Let X be a complete U_n-metric space. Suppose the map $f : X \longrightarrow X$ satisfies for $m \in \mathbb{N}$ and $z_1, z_2, \dots, z_n \in X$*

$$\psi(U_n(f^m z_1, f^m z_2, \dots, f^m z_n)) \leq \psi(U_n(z_1, z_2, \dots, z_n)) - \phi(U_n(z_1, z_2, \dots, z_n)),$$

where ψ and ϕ are altering distance functions. Then f has a unique fixed point (say u), and f is U_n-continuous at u.

Proof. From Theorem 2.13, we conclude that f^m has a unique fixed point say u. Since

$$fu = f(f^m u) = f^{m+1}u = f^m(fu),$$

we have that fu is also a fixed point to f^m. By uniques of u, we get $fu = u$. □

Corollary 2.15. *Let X be a complete U_n-metric space. Suppose the map $f : X \longrightarrow X$ satisfies for all $z_1, z_2, \dots, z_n \in X$*

$$U_n(fz_1, fz_2, \dots, fz_n) \le \alpha\, U_n(z_1, z_2, \dots, z_n),$$

where $\alpha \in [0, 1)$, then f has a unique fixed point (say u), and f is U_n-continuous at u.

Proof. It suffices to take in Theorem 2.13, $\psi(t) = t$ and $(1-\alpha)t$ for $\alpha \in [0, 1)$. □

To continue we define a Γ-distance on a complete U_n-metric space which is a generalization of the concept of ω-distance due to Kada et al. [22] and prove fixed point theorem in partially ordered U_n-metric space.

Definition 2.16. Let (X, U_n) be a U_n-metric space. Then a function $\Gamma : X^n \longrightarrow \mathbb{R}_+$ is called an Γ-distance on X if the following conditions are satisfied:

(a) $\Gamma(x_1, x_2, \dots, x_n) \le \Gamma(x_1, a, \dots, a) + \Gamma(a, x_2, \dots, x_n)$ for all $x_1, \dots, x_n, a \in X$,

(b) for any $x_1, \dots, x_{n-1} \in X$, $\Gamma(x_1, \dots, x_{n-1}, .) : X \longrightarrow \mathbb{R}_+$ is lower semi continuous,

(c) for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Gamma(x_1, a, \dots, a) \le \delta$ and $\Gamma(a, x_2, \dots, x_n) \le \delta$ imply $U_n(x_1, x_2, \dots, x_n) \le \varepsilon$.

Also, X is said to be Γ-bounded if there is a constant $M > 0$ such that $\Gamma(x_1, \dots, x_n) \le M$ for all $x_1, \dots, x_n \in X$.

Example 2.17. *Let (X, d) be a metric space and $U_n : X^n \longrightarrow \mathbb{R}_+$ defined by*

$$U_n(x_1, \dots, x_n) = \max\{d(x_i, x_j) : 1 \le i < j \le n\},$$

for all $x_1, \dots, x_n \in X$. Then $\Gamma = U_n$ is a Γ-distance on X.

Proof. (a) and (b) are immediate. We show (c). Let $\varepsilon > 0$ be given and put $\delta = \varepsilon/2$. If $U_n(x_1, a, \dots, a) \le \delta$ and $U_n(a, x_2, \dots, x_n) \le \delta$ then $d(x_1, a) \le \delta$, $d(a, x_i) \le \delta$ for $i = 2, \dots, n$ and $d(x_i, x_j) \le \delta$ for $2 \le i < j \le n$, which implies that $U_n(x_1, \dots, x_n) \le 2\delta = \varepsilon$. □

Example 2.18. *Let (X, d) be a usual metric space, then (X, U_n) is U_n-metric space, where*

$$U_n(x_1, \dots, x_n) = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} d(x_i, x_j)$$

Then the function $\Gamma : X^n \longrightarrow \mathbb{R}_+$ *defined by*

$$\Gamma(x_1,\dots,x_n) = \frac{2}{n(n-1)}\sum_{i=2}^{n} d(x_1,x_i)$$

for all $x_1,\dots,x_n \in X$ *is a* Γ*-distance on* X*.*

Proof. The proofs of (a) and (b) are obvious. we show (c). Let $\varepsilon > 0$ be given and put $\delta = \frac{\varepsilon}{n^2-2n+2}$. If $\Gamma(x,a,\dots,a) \leq \delta$ and $\Gamma(a,x_2,\dots,x_n) \leq \delta$, we have, respectively, $d(x_1,a) \leq \frac{n\delta}{2}$ and $d(a,x_i) \leq \frac{n(n-1)\delta}{2}$ for $i = 2,\dots,n$. which imply that $U_n(x_1,\dots,x_n) \leq (n^2-2n+2)\delta = \varepsilon$. □

Lemma 2.19. *Let* (X,U_n) *be* U_n*-metric space and* Γ *be a* Γ*-distance on* X*. Let* $\{x_k\}$*,* $\{y_k\}$ *be sequence in* X*,* $\{\delta_k\}$ *and* $\{\beta_k\}$ *be sequences in* $\mathbb{R}_+$ *converging to zero and let* $z_1,\dots,z_n,a \in X$*. Then we have the following:*

(1) *If* $\Gamma(y_k,x_k,\dots,x_k) \leq \delta_k$ *and* $\Gamma(x_k,y_m,\dots,y_l,z_n) \leq \beta_k$ *for any* $l \geq \cdots \geq m \geq k \in \mathbb{N}$*, then* $U_n(y_k,y_m,\dots,y_l,z_n) \to 0$ *and hence* $y_k \to z_n$*.*

(2) *If* $\Gamma(x_k,x_m,\dots,x_l) \leq \delta_k$ *for any* $l \geq \cdots \geq m \geq k \in \mathbb{N}$*, then* $\{x_k\}$ *is an* U_n*-Cauchy sequence.*

(3) *If* $\Gamma(x_k,a,\dots,a) \leq \delta_k$ *for any* $k \in \mathbb{N}$*, then* $\{x_k\}$ *is an* U_n*-Cauchy sequence.*

Proof. From definition of Γ-distance, there exists a $\delta > 0$ such that $\Gamma(v_1,a,\dots,a) \leq \delta$ and $\Gamma(a,v_2,\dots,v_{n-1},z_n) \leq \delta$ imply $U_n(v_1,\dots,v_{n-1},z_n) \leq \varepsilon$. Choose $N \in \mathbb{N}$ such that $\delta_k \leq \delta$ and $\beta_k \leq \delta$ for every $k \geq N$. Then we have, for any $l \geq \cdots \geq m \geq k \geq N$,

$$\Gamma(y_k,x_k,\dots,x_k) \leq \delta_k \leq \delta, \qquad \Gamma(x_k,y_m,\dots,y_l,z_n) \leq \beta_k \leq \delta$$

and hence $U_n(y_k,y_m,\dots,y_l,z_n) \leq \varepsilon$, so that $\{y_k\}$ converges to z_n.

Now we prove (2). Let $\varepsilon > 0$ be given. As in the proof of (1), choose $\delta > 0$ and then $N \in \mathbb{N}$. Then, for any $l \geq \cdots \geq m \geq k \geq N$,

$$\Gamma(x_k,x_{k+1},\dots,x_{k+1}) \leq \delta_k \leq \delta, \qquad \Gamma(x_{k+1},x_m,\dots,x_l) \leq \delta_{k+1} \leq \delta,$$

and hence $U_n(x_k,x_m,\dots,x_l) \leq \varepsilon$. This implies that $\{x_k\}$ is an U_n-Cauchy sequence. Condition (3) is a special case of (2). □

Definition 2.20. Suppose $(X,\leq)$ is a partially ordered set and $F : X \longrightarrow X$ is a mapping of X into itself. We say that F is non-decreasing if $x \leq y$ implies $F(x) \leq F(y)$ for $x,y \in X$.

Theorem 2.21. *Suppose* $(X,\leq)$ *is a partially ordered set. Suppose that there exists a* U_n*-metric on* X *such that* (X,U_n) *is a complete* U_n*-metric space and* Γ *is an* Γ*-distance on* X *and* F *is a non-decreasing mapping from* X *into itself. Let* X *be* Γ*-bounded. Suppose that there exists* $r \in [0,1)$ *such that*

$$\Gamma(Fx,F^2x,\dots,F^2x,Fv) \leq r\Gamma(x,Fx,\dots,Fx,v)$$

for $x \leq Fx$ and $v \in X$. Also for every $x \in X$

$$\inf\{\Gamma(x, F^k x, F^m x, \ldots, F^l x, y) : x \leq Fx\} > 0 \quad \textit{for } l > \cdots > m > k \in \mathbb{N} \quad (2.28)$$

for every $y \in X$ with $Fy \neq y$. If there exists an $x_0 \in X$ with $x_0 \leq Fx_0$, then F has a fixed point. Moreover, if $Fw = w$, then $\Gamma(w, \ldots, w) = 0$.

Proof. If $Fx_0 = x_0$, then the proof is finished. Suppose that $Fx_0 \neq x_0$. Since $x_0 \leq Fx_0$ and F is non-decreasing, we obtain

$$x_0 \leq Fx_0 \leq F^2 x_0 \leq \cdots \leq F^{k+1} x_0 \leq \cdots .$$

For all $k \in \mathbb{N}$ and $t \geq 0$,

$$\begin{aligned}\Gamma(F^k x_0, F^{k+1} x_0, \ldots, F^{k+1} x_0, F^{k+t} x_0) &\leq r\Gamma(F^{k-1} x_0, F^k x_0, \ldots, F^k x_0, F^{k+t-1} x_0) \\ &\leq \cdots \leq r^k \Gamma(x_0, Fx_0, \ldots, Fx_0, F^t x_0).\end{aligned}$$

Thus, for any $l_1 > l_2 > \cdots > l_{k-1} > l_k$ in which $l_i + l_i' = l_{i-1}$, $l_i' \in \mathbb{N}\, i = 2, \ldots, k$, we have

$$\begin{aligned}\Gamma(F^{l_k} x_0, &F^{l_{k-1}} x_0, \ldots, F^{l_1} x_0) \\ &\leq \Gamma(F^{l_k} x_0, F^{l_k+1} x_0, \ldots, F^{l_k+1} x_0) + \Gamma(F^{l_k+1} x_0, F^{l_{k-1}} x_0, \ldots, F^{l_1} x_0) \\ &\leq \Gamma(F^{l_k} x_0, F^{l_k+1} x_0, \ldots, F^{l_k+1} x_0) + \Gamma(F^{l_k+1} x_0, F^{l_k+2} x_0, \ldots, F^{l_k+2} x_0) \\ &\quad + \Gamma(F^{l_k+2} x_0, F^{l_{k-1}} x_0, \ldots, F^{l_1} x_0) \\ &\leq \Gamma(F^{l_k} x_0, F^{l_k+1} x_0, \ldots, F^{l_k+1} x_0) + \Gamma(F^{l_k+1} x_0, F^{l_k+2} x_0, \ldots, F^{l_k+2} x_0) \\ &\quad + \cdots + \Gamma(F^{l_{k-1}-2} x_0, F^{l_{k-1}-1} x_0, \ldots, F^{l_{k-1}-1} x_0) \\ &\quad + \Gamma(F^{l_{k-1}-1} x_0, F^{l_{k-1}} x_0, F^{l_{k-2}} x_0, \ldots, F^{l_1} x_0) \\ &\leq \sum_{i=l_k}^{l_{k-1}-1} M r^i \\ &\leq \frac{r^{l_k}}{1-r} M.\end{aligned}$$

By (2) of Lemma 2.19, $\{F^{l_k} x_0\}$ is a U_n-Cauchy sequence. Since X is U_n-complete, $\{F^{l_k} x_0\}$ coverges to a point $z \in X$. Let $l_k \in \mathbb{N}$ be fixed. Then, by the lower semicontinuity of Γ, we have, for $l_2 > \cdots > l_{k-1} > l_k$

$$\begin{aligned}\Gamma(F^{l_k} x_0, F^{l_{k-1}} x_0, \ldots, F^{l_2} x_0, z) &\leq \liminf_{p \to \infty} \Gamma(F^{l_k} x_0, F^{l_{k-1}} x_0, \ldots, F^{l_2} x_0, F^p x_0) \\ &\leq \frac{r^{l_k}}{1-r} M.\end{aligned}$$

Assume that $Fz \neq z$. By (2.28), we have for $l > \cdots > m > k \in \mathbb{N}$

$$0 < \inf\{\Gamma(F^{l_k} x_0, F^{l_k+k} x_0, F^{l_k+m} x_0, \ldots, F^{l_k+l} x_0, y)\} \leq \inf \frac{r^{l_k}}{1-r} M = 0$$

which is a contradiction. Therefore, we have $Fz = z$. Now, if $Fw = w$, we have $\Gamma(w, \ldots, w) = \Gamma(Fw, F^2 w, \ldots, F^2 w, Fw) \leq r\Gamma(w, Fw, \ldots, Fw, w) = r\Gamma(w, \ldots, w)$ and so $\Gamma(w, \ldots, w) = 0$. □

3 Conclusion

In this work, we established some common fixed point results for two mapping $f, g : X \longrightarrow X$ satisfying contractive condition of types A_u and B_u. Also we studied some fixed point consequences for a self mapping in a complete U_n-metric space X under condition related to altering distance functions. Last result of our paper is a fixed point theorem involving Γ-distance.

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