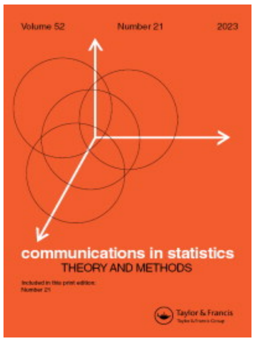

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Hermite-Hadamard and Fejér-type inequalities for generalized η-convex stochastic processes

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ABSTRACT

In this article, we introduce the concept of (η_1, η_2)-convex stochastic processes on coordinates and establish Hermite-Hadamard-type inequality for these stochastic processes. Moreover, we prove new integral inequality of Hermite-Hadamard-Fejér type for newly defined coordinated η-convex stochastic processes on a rectangle. The results presented in this article would provide extensions of those given in earlier works.

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1. Introduction

Convex sets and convex functions play an important role in applied mathematics, particularly in non linear programming and optimization theory. Many efforts have been made by researchers to generalize and extend the notion of convex functions. Gordji, Delavar, and Dragomir (2015) introduced the idea of η-convex functions as generalization of convex functions and investigated Hermite-Hadamard (H-H), Fejér, Jensen, and Slater-type inequalities for these functions. Yaldiz, Sarikaya, and Dahmani (2017) obtained new fractional H-H-Fejér-type integral inequalities for coordinated convex functions on a rectangle of $\mathbb{R}^2$. Further, Zaheer Ullah, Adil Khan, and Chu (2019) defined the generalized class of convex functions named as coordinate (η_1, η_2)-convex function and established H-H inequality for the class of these functions. They showed that every η-convex function defined on a rectangle is coordinated η-convex but the converse is not true in general. For more details on generalization of convexity, we can see Alomari and Darus (2009); Gordji, Delavar, and De La Sen (2016); Sharma, Bisht, and Mishra (2020); and Sharma et al. (2019).

Over the past decades, the study of stochastic processes is rapidly expanding, with increasing applications in numerous scientific fields. This subject has received enormous support outside of mathematics from such diverse fields as physics, control theory, information

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theory, biology, signal processing, statistics, computer science, telecommunications, and cryptography (see Allen 2010; Bhattacharya and Waymire 2009; Sobczyk 2001 and their references). Nikodem (1980) gave the concept of convex stochastic processes and showed that every measurable convex stochastic process is continuous. Further, many researchers investigated the properties of convex stochastic processes which generalize some known properties of convex functions (Skowroński 1992, 1995). Kotrys (2012) extended the classical H-H inequality to convex stochastic process. Let $X : K \times \Omega \to \mathbb{R}$ be a Jensen-convex, mean square continuous in interval $K \subseteq \mathbb{R}$, then

$$X\left(\frac{u+v}{2}.\right) \leq \frac{1}{v-u}\int_u^v X(x.)dx \leq \frac{X(u.)+X(v.)}{2} \quad (a.e.), \forall u, v \in K,$$

which is H-H inequality for convex stochastic process.

Maden, Tomar, and Set (2015) and Set, Tomar, and Maden (2014) presented s-convex stochastic processes and investigated relation between s-convex stochastic processes and convex stochastic processes. In 2014, Barráez et al. (2015) extended the class of h-convex functions to h-convex stochastic processs and presented Jensen-type inequality for these processes. Further, Set, Sarikaya, and Tomar (2015) presented convex stochastic processes on coordinates and proved H-H-type inequalities for coordinated convex stochastic processes. Karahan, Nurgül, and İ can (2018) considered convex stochastic processes on n-dimensional interval and dervied H-H-type inequality for convex stochastic processes on n-coordinates. For more results related to stochastic processes, we refer Fu et al. (2021); Kotrys (2015); Okur and Aliyev (2021); Okur, Iscan, and Usta (2018); Sharma, Mishra, and Hamdi (2022a) and their references.

Recently, Jung et al. (2021) introduced the notion of η-convex stochastic processes and derived H-H and Jensen-type inequality for these type of stochastic processes. Sharma, Mishra, and Hamdi (2022b) introduced the concept of strongly η-convex stochastic processes and proved the H-H and Ostrowski-type inequality for these generalized convex stochastic processes.

The above new developments in the field of stochastic convexity have inspired us to introduce a new class of convex stochastic processes named as (η_1, η_2)-convex stochastic processes on coordinates and obtained the generalization and extension of H-H-type inequality for these stochastic processes. Moreover, we derive H-H-Fejér-type inequality for coordinated η-convex stochastic processes. The results presented in this research article are the generalization and extension of earlier studies.

The presentation sequence of this article is as follows: Section 2 recall some basic definitions and results required for this paper. In Section 3, we define (η_1, η_2)-convex stochastic processes on coordinates and derive H-H inequality for these stochastic processes. We also obtain H-H-Fejér-type inequality for coordinated η-convex stochastic processes. Some special cases of our main results are also discussed in this section. In Section 4, conclusion of the proposed work is given.

2. Preliminaries

In this section, we collect some basic definitions and essential results required in the sequel of the paper.

Definition 1. Let (Ω, A, P) be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is A-measurable. A function $X : K \times \Omega \to \mathbb{R}$, where $K \subseteq \mathbb{R}$ is an interval, is a stochastic process if the function $X(x)$ is a random variable for every $x \in K$.

Definition 2. (Kotrys 2012). The stochastic process $X : K \times \Omega \to \mathbb{R}$ is called

- continuous in probability in Kif

$$P - \lim_{x \to x_0} X(x.) = X(x_0.),\ \forall\, x_0 \in K,$$

  where $P - \lim$ denotes the limit in probability;
- mean square continuous in K, if

$$\lim_{x \to x_0} E[(X(x.) - X(x_0.))^2] = 0,\ \forall\, x_0 \in K,$$

  where $E[X(x.)]$ denotes the expectation value of the random variable $X(x.)$.

Definition 3. (Kotrys 2012). Consider a stochastic process $X : K \times \Omega \to \mathbb{R}$ with $E[X(x)^2] < \infty$ for all $x \in K$ and $[r_1, s_1] \in K$. A random variable $Y : K \times \Omega \to \mathbb{R}$ is mean-square integral of the process X on $[r_1, s_1]$, if for each normal sequence of partitions of the interval $[r_1, s_1]$, $r_1 = x_0 < x_1 < x_2 < . < x_j = s_1$ and for each $\Theta_j \in [x_{j-1}, x_j],\ j = 1, 2.n$, we have

$$\lim_{n \to \infty} E\left[\left(\sum_{j=1}^{n} X(\Theta_j)\Delta(x_j - x_{j-1}) - Y\right)^2\right] = 0.$$

Then, we can write

$$Y(.) = \int_{r_1}^{s_1} X(y.)dy\ \ (a.e.).$$

Definition 4. (Nikodem 1980). Let (Ω, A, P) be a probability space and $K \subseteq \mathbb{R}$ be an interval. Then $X : K \times \Omega \to \mathbb{R}$ is said to be convex stochastic process if

$$X(\varrho u + (1 - \varrho)v.) \le \varrho X(u.) + (1 - \varrho)X(v.)\ (a.e.)\quad \forall u, v \in K,\ \varrho \in [0, 1].$$

Definition 5. (Set, Sarìkaya, and Tomar 2015). Let us consider a bidimensional interval $\Lambda = \Lambda_1 \times \Lambda_2$ in $\mathbb{R}^2$. Then $X : \Lambda \times \Omega \to \mathbb{R}$ is said to be convex stochastic process on Λ if

$$X((\varrho u + (1 - \varrho)w, \varrho v + (1 - \varrho)z).) \le \varrho X((u, v).) + (1 - \varrho)X((w, z).))\ (a.e.)$$

for all $(u, v), (w, z) \in \Lambda$ *and* $\varrho \in [0, 1]$.

Definition 6. (Set, Sarìkaya, and Tomar 2015). A stochastic process $X : \Lambda \times \Omega \to \mathbb{R}$ is said to be convex on coordinates on Λ if the partial mappings $X_y : \Lambda_1 \times \Omega \to \mathbb{R},\ X_y(u.) = X((u, y).)$ and $X_x : \Lambda_2 \times \Omega \to \mathbb{R},\ X_x(v.) = X((x, v).)$ are convex for all $x \in \Lambda_1, y \in \Lambda_2$.

Definition 7. (Jung et al. 2021). A stochastic process $X : K \times \Omega \to \mathbb{R}$ is said to be η-convex with respect to $\eta : X(K) \times X(K) \to \mathbb{R}$ if

$$X(\varrho u + (1 - \varrho)v.) \le X(v.) + \varrho\eta(X(u.), X(v.))\ (a.e.)\quad \forall u, v \in K,\ \varrho \in [0, 1].$$

Remark 1. If $\eta(X(u.),X(v.)) = X(u.) - X(v.)$, then η-convex stochastic process reduces to the definition of convex stochastic process.

Theorem 1. (Jung et al. 2021). *Suppose that $X:[r_1,s_1]\times\Omega\to\mathbb{R}$ is an η-convex stochastic process such that η is bounded above $X([r_1,s_1].)\times X([r_1,s_1].)$, then*

$$\begin{aligned} X\left(\frac{r_1+s_1}{2}.\right) - \frac{1}{2}M_\eta &\le \frac{1}{s_1-r_1}\int_{r_1}^{s_1} X(x.)dx \\ &\le \frac{X(r_1.)+X(s_1.)}{2} + \frac{1}{4}[\eta(X(r_1.),X(s_1.)) + \eta(X(s_1.),X(r_1.))] \\ &\le \frac{X(r_1.)+X(s_1.)}{2} + \frac{1}{2}M_\eta, \end{aligned} \tag{1}$$

where M_η is an upper bound of η.

Proposition 1. *Any η-convex stochastic process $X:[r_1,s_1]\times\Omega\to\mathbb{R}$ with respect to a bifuction η bounded from above on $X([r_1,s_1].)\times X([r_1,s_1].)$ has lower and upper bounds.*

Proof. Suppose that M_η is upper bound of η on $X([r_1,s_1].)\times X([r_1,s_1].)$. For any $x=\varrho r_1 + (1-\varrho)s_1 \in [r_1,s_1]$ and $\varrho\in[0,1]$, we have

$$\begin{aligned} X(x.) &= X(\varrho r_1 + (1-\varrho)s_1.) \\ &\le X(s_1.) + \varrho\eta(X(r_1.),X(s_1.)) \\ &\le max\{X(s_1.),\ X(s_1.)+\eta(X(r_1.),X(s_1.))\} \\ &\le max\{X(s_1.),\ X(s_1.)+M_\eta\}. \end{aligned}$$

Put $M = max\{X(s_1.),\ X(s_1.)+M_\eta\}$, we get

$$X(x.)\le M.$$

Therefore, X has upper bound. For lower bound of X, consider an arbitrary point in the form $\frac{r_1+s_1}{2}-\varepsilon\in[r_1,s_1]$. Then

$$\begin{aligned} X\left(\frac{r_1+s_1}{2}.\right) &= X\left(\left(\frac{r_1+s_1}{4}+\frac{\varepsilon}{2}+\frac{r_1+s_1}{4}-\frac{\varepsilon}{2}\right).\right) \\ &= X\left(\frac{1}{2}\left(\frac{r_1+s_1}{2}+\varepsilon\right)+\frac{1}{2}\left(\frac{r_1+s_1}{2}-\varepsilon\right).\right) \\ &\le X\left(\frac{r_1+s_1}{2}-\varepsilon,.\right)+\frac{1}{2}\eta\left(X\left(\frac{r_1+s_1}{2}+\varepsilon,.\right),X\left(\frac{r_1+s_1}{2}-\varepsilon,.\right)\right) \\ &\le X\left(\frac{r_1+s_1}{2}-\varepsilon.\right)+\frac{M_\eta}{2}. \end{aligned}$$

This implies

$$X\left(\frac{r_1+s_1}{2}.\right)-\frac{M_\eta}{2}\le X\left(\frac{r_1+s_1}{2}-\varepsilon.\right).$$

On putting $m = X\left(\frac{r_1+s_1}{2}.\right)-\frac{M_\eta}{2}$, we see that X has a lower bound.

Thus the statement is proved. □

3. Generalized η-convex stochastic processes

First, we give the definition of η-convex stochastic process defined on a rectangle. Throughout this section, let us consider the bi-dimensional interval $\Delta = [r_1, s_1] \times [r_2, s_2]$ in $\mathbb{R}^2$ with $r_1 < s_1$ and $r_2 < s_2$.

Definition 8. A stochastic process $X : \Delta \times \Omega \to \mathbb{R}$ is said to be η-convex with respect to $\eta : X(\Delta) \times X(\Delta) \to \mathbb{R}$ if the following inequality holds almost everywhere:

$$X((\varrho u + (1-\varrho)w, \varrho v + (1-\varrho)z).) \leq X((w,z).) + \varrho\eta(X((u,v).), X((w,z).))$$

for all $(u,v), (w,z) \in \Delta$ *and* $\varrho \in [0,1]$.

Remark 2. If $\eta(X((u,v).), X((w,z).)) = X((u,v).) - X((w,z).)$ in Definition 8, then X becomes convex stochastic process on Δ.

Example 1. Let $X : \Delta \times \Omega \to \mathbb{R}$, $\Delta = [0,\infty) \times [0,\infty)$ be a stochastic process defined as $X((x,y).) = 1 + x + y$ and $\eta : X(\Delta.) \times X(\Delta.) \to \mathbb{R}$, $\eta(X((x,y).), X((x',y').)) = X((x,y).) + X((x',y').)$. Then X is η-convex stochastic process on Δ.

Now, we present the definition of (η_1, η_2)-convex stochastic processes on coordinates.

Definition 9. A stochastic process $X : \Delta \times \Omega \to \mathbb{R}$ is said to be (η_1, η_2)-convex on coordinates on Δ if the partial mappings $X_y : [r_1, s_1] \times \Omega \to \mathbb{R}$, $X_y(u.) = X((u,y).)$ and $X_x : [r_2, s_2] \times \Omega \to \mathbb{R}$, $X_x(v.) = X((x,v).)$ are η_1- and η_2-convex stochastic process, respectiviely, for all $x \in [r_1, s_1], y \in [r_2, s_2]$.

Remark 3. If $\eta_1 = \eta_2 = \eta$, then X is called η-convex stochastic process on coordinates.

We give a formal definition of coordinated η-convex stochastic processes as follows:

Definition 10. A stochastic process $X : \Delta \times \Omega \to \mathbb{R}$ is said to be coordinated η-convex on Δ if for all $(u,v), (w,z) \in \Delta$ *and* $\varrho, \kappa \in [0,1]$, we have

$$\begin{aligned} &X((\varrho u + (1-\varrho)w, \kappa v + (1-\kappa)z).) \\ &\leq X((u,v).) + \varrho(1-\kappa)\eta(X((u,z).), X((u,v).)) \\ &\quad + \kappa(1-\varrho)\eta(X((w,v).), X((u,v).)) \\ &\quad + (1-\varrho)(1-\kappa)\eta(X((w,z).), X((u,v).)) \quad (a.e). \end{aligned}$$

Example 2. Let $X : \Delta \times \Omega \to \mathbb{R}$, $\Delta = [0,\infty) \times [0,\infty)$ be a stochastic process defined as $X((x,y).) = xy$ and $\eta : X(\Delta.) \times X(\Delta.) \to \mathbb{R}$, $\eta(X((x,y).), X((x',y').)) = 2X((x,y).) - X((x',y').)$. Then X is coordinated η-convex stochastic process on Δ.

Remark 4. If we put $\eta(X((x,y).), X((x',y').)) = X((x,y).) - X((x',y').)$ for all $(x,y), (x',y') \in \Delta$ in Definition 10, then X becomes coordinated convex stochastic processes (Set, Sarikaya, and Tomar 2015).

Theorem 2. *Every η-convex stochastic process $X : \Delta \times \Omega \to \mathbb{R}$ is η-convex on the coordinates on Δ.*

Proof. Let $(x, y), (u, v), (w, z) \in [r_1, s_1] \times [r_2, s_2]$. Then from the definition of η-convex stochastic process on $[r_1, s_1] \times [r_2, s_2]$ we have

$$\begin{aligned} X_x((\varrho v + (1-\varrho)z).) &= X((x, \varrho v + (1-\varrho)z).) \\ &= X((\varrho x + (1-\varrho)x, \varrho v + (1-\varrho)z).) \\ &\leq X((x,z).) + \varrho\eta(X((x,v).), X((x,z).)) \\ &= X_x(z.) + \varrho\eta(X_x(v.), X_x(z.)). \end{aligned} \tag{2}$$

Inequality (2) shows that X_x is η-convex stochastic process on interval $[r_2, s_2]$.

$$\begin{aligned} X_y((\varrho u + (1-\varrho)w).) &= X((y, \varrho u + (1-\varrho)w).) \\ &= X((\varrho y + (1-\varrho)y, \varrho u + (1-\varrho)w).) \\ &\leq X((y,w).) + \varrho\eta(X((y,u).), X((y,w).)) \\ &= X_y(w.) + \varrho\eta(X_y(u.), X_y(w.)). \end{aligned} \tag{3}$$

Inequality (3) shows that X_y is η-convex stochastic process on interval $[r_1, s_1]$.

Thus, X is η-convex stochastic process on the coordinates on Δ. □

Next we derive H-H-type inequality for (η_1, η_2)-convex stochastic processes on the coordinates.

Theorem 3. *Let $X : \Delta \times \Omega \to \mathbb{R}$ be a (η_1, η_2)-convex stochastic process on the coordinates and mean square integrable on Δ. Then, we have almost everywhere:*

$$\begin{aligned} & X\left(\left(\frac{r_1+s_1}{2}, \frac{r_2+s_2}{2}\right).\right) - \frac{M_{\eta_1}+M_{\eta_2}}{2} \\ & \leq \frac{1}{2}\left[\frac{1}{s_1-r_1}\int_{r_1}^{s_1} X\left(\left(x, \frac{r_2+s_2}{2}\right).\right)dx + \frac{1}{s_2-r_2}\int_{r_2}^{s_2} X\left(\left(\frac{r_1+s_1}{2}, y\right).\right)dy\right] \\ & \quad - \frac{M_{\eta_1}+M_{\eta_2}}{4} \\ & \leq \frac{1}{(s_1-r_1)(s_2-r_2)}\int_{r_1}^{s_1}\int_{r_2}^{s_2} X((x,y).)dydx \\ & \leq \frac{1}{4}\left[\frac{1}{(s_1-r_1)}\int_{r_1}^{s_1}\big[X((x,r_2).) + X((x,s_2).)\big]dx\right. \\ & \quad \left.+ \frac{1}{s_2-r_2}\int_{r_2}^{s_2}\big[X((r_1,y).) + X((s_1,y).)\big]dy + \frac{M_{\eta_1}+M_{\eta_2}}{4}\right. \\ & \leq \frac{1}{4}\big[X((r_1,r_2).) + X((s_1,r_2).) + X((r_1,s_2).) + X((s_1,s_2).)\big] + \frac{M_{\eta_1}+M_{\eta_2}}{2}, \end{aligned}$$

where M_{η_1} and M_{η_2} are upper bounds of η_1 and η_2, respectively.

Proof. Since stochastic process $X : \Delta \times \Omega \to \mathbb{R}$ is (η_1, η_2)-convex on the coordinates on Δ. Therefore, stochastic process $X_x : [r_2, s_2] \times \Omega \to \mathbb{R}$, $X_x(v.) = X((x, v).)$ is η_2-convex on $[r_2, s_2]$. It follows from Theorem 1 that

$$X_x\left(\frac{r_2+s_2}{2}.\right) - \frac{1}{2}M_{\eta_2} \leq \frac{1}{s_2-r_2}\int_{r_2}^{s_2} X_x(y.)dy$$
$$\leq \frac{1}{2}[X_x(r_2.) + X_x(s_2.)] + \frac{1}{2}M_{\eta_2}.$$

This implies

$$X\left(\left(x, \frac{r_2+s_2}{2}\right).\right) - \frac{1}{2}M_{\eta_2} \leq \frac{1}{s_2-r_2}\int_{r_2}^{s_2} X((x,y).)dy$$
$$\leq \frac{1}{2}[X((x,r_2).) + X((x,s_2).)] + \frac{1}{2}M_{\eta_2}. \tag{4}$$

On integrating (4) with respect to x over $[r_1, s_1]$, we get

$$\frac{1}{s_1-r_1}\int_{r_1}^{s_1} X\left(\left(x, \frac{r_2+s_2}{2}\right).\right) dx - \frac{1}{2}M_{\eta_2}$$
$$\leq \frac{1}{(s_1-r_1)(s_2-r_2)}\int_{r_1}^{s_1}\int_{r_2}^{s_2} X((x,y).)dydx$$
$$\leq \frac{1}{2(s_1-r_1)}\int_{r_1}^{s_1}[X((x,r_2).) + X((x,s_2).)]dx + \frac{1}{2}M_{\eta_2}. \tag{5}$$

Similarly, we can get the following inequality for $X_y(u.) = X((u,y).)$.

$$\frac{1}{s_2-r_2}\int_{r_2}^{s_2} X\left(\left(\frac{r_1+s_1}{2}, y\right).\right) dy - \frac{1}{2}M_{\eta_1}$$
$$\leq \frac{1}{(s_1-r_1)(s_2-r_2)}\int_{r_1}^{s_1}\int_{r_2}^{s_2} X((x,y).)dydx$$
$$\leq \frac{1}{2(s_2-r_2)}\int_{r_2}^{s_2}[X((r_1,y).) + X((s_1,y).)]dy + \frac{1}{2}M_{\eta_1}. \tag{6}$$

Adding (5) and (6), we have

$$\frac{1}{s_1-r_1}\int_{r_1}^{s_1} X\left(\left(x, \frac{r_2+s_2}{2}\right).\right) dx + \frac{1}{s_2-r_2}\int_{r_2}^{s_2} X\left(\left(\frac{r_1+s_1}{2}, y\right).\right) dy$$
$$- \frac{M_{\eta_1}+M_{\eta_2}}{2}$$
$$\leq \frac{2}{(s_1-r_1)(s_2-r_2)}\int_{r_1}^{s_1}\int_{r_2}^{s_2} X((x,y).)dydx$$
$$\leq \frac{1}{2(s_1-r_1)}\int_{r_1}^{s_1}[X((x,r_2).) + X((x,s_2).)]dx$$
$$+ \frac{1}{2(s_2-r_2)}\int_{r_2}^{s_2}[X((r_1,y).) + X((s_1,y).)]dy + \frac{M_{\eta_1}+M_{\eta_2}}{2}. \tag{7}$$

Now, using the (η_1, η_2)-convexity of X on the coordinates on $[r_1, s_1] \times [r_2, s_2]$ and Theorem 1, we get

$$X\left(\left(\frac{r_1+s_1}{2}, \frac{r_2+s_2}{2}\right).\right) - \frac{1}{2}M_{\eta_1} \leq \frac{1}{s_1-r_1}\int_{r_1}^{s_1} X\left(\left(x, \frac{r_2+s_2}{2}\right).\right)dx \tag{8}$$

and

$$X\left(\left(\frac{r_1+s_1}{2},\frac{r_2+s_2}{2}\right).\right)-\frac{1}{2}M_{\eta_2}\leq\frac{1}{s_2-r_2}\int_{r_2}^{s_2}X\left(\left(\frac{r_1+s_1}{2},y\right).\right)dy. \tag{9}$$

Adding (8) and (9), we find

$$\begin{aligned}&X\left(\left(\frac{r_1+s_1}{2},\frac{r_2+s_2}{2}\right).\right)-\frac{M_{\eta_1}+M_{\eta_2}}{2}\\&\leq\frac{1}{2}\left[\frac{1}{s_1-r_1}\int_{r_1}^{s_1}X\left(\left(x,\frac{r_2+s_2}{2}\right).\right)dx+\frac{1}{s_2-r_2}\int_{r_2}^{s_2}X\left(\left(\frac{r_1+s_1}{2},y\right).\right)dy\right]\\&-\frac{M_{\eta_1}+M_{\eta_2}}{4}.\end{aligned} \tag{10}$$

Again from Theorem 1, we obtain

$$\frac{1}{s_1-r_1}\int_{r_1}^{s_1}X((x,r_2).)dx\leq\frac{1}{2}[X((r_1,r_2).)+X((s_1,r_2).)]+\frac{M_{\eta_1}}{2} \tag{11}$$

$$\frac{1}{s_1-r_1}\int_{r_1}^{s_1}X((x,s_2).)dx\leq\frac{1}{2}[X((r_1,s_2).)+X((s_1,s_2).)]+\frac{M_{\eta_1}}{2} \tag{12}$$

$$\frac{1}{s_2-r_2}\int_{r_2}^{s_2}X((r_1,y).)dy\leq\frac{1}{2}[X((r_1,r_2).)+X((r_1,s_2).)]+\frac{M_{\eta_2}}{2} \tag{13}$$

$$\frac{1}{s_2-r_2}\int_{r_2}^{s_2}X((s_1,y).)dy\leq\frac{1}{2}[X((s_1,r_2).)+X((s_1,s_2).)]+\frac{M_{\eta_2}}{2} \tag{14}$$

Adding (11), (12), (13), and (14), we have

$$\begin{aligned}&\frac{1}{4(s_1-r_1)}\int_{r_1}^{s_1}[X((x,r_2).)+X((x,s_2).)]dx\\&+\frac{1}{4(s_2-r_2)}\int_{r_2}^{s_2}[X((r_1,y).)+X((s_1,y).)]dy\\&\leq\frac{1}{4}\big[X((r_1,r_2).)+X((s_1,r_2).)+X((r_1,s_2).)+X((s_1,s_2).)\big]+\frac{M_{\eta_1}+M_{\eta_2}}{4}.\end{aligned} \tag{15}$$

Adding $\frac{M_{\eta_1}+M_{\eta_2}}{4}$ on both sides of (15), we get

$$\begin{aligned}&\frac{1}{4}\left[\frac{1}{(s_1-r_1)}\int_{r_1}^{s_1}[X((x,r_2).)+X((x,s_2).)]dx\right.\\&\left.+\frac{1}{(s_2-r_2)}\int_{r_2}^{s_2}[X((r_1,y).)+X((s_1,y).)]dy\right]+\frac{M_{\eta_1}+M_{\eta_2}}{4}\\&\leq\frac{1}{4}\big[X((r_1,r_2).)+X((s_1,r_2).)+X((r_1,s_2).)+X((s_1,s_2).)\big]+\frac{M_{\eta_1}+M_{\eta_2}}{2}.\end{aligned} \tag{16}$$

From (7), (10), and (16), we obtain the desired result. □

Remark 5. If we take $\eta_1(X(x.),X(y.))=X(x.)-X(y.)$ and $\eta_2(X(x'.),X(y'.))=X(x'.)-X(y'.)$ for all $(x,y),(x',y')\in[r_1,s_1]\times[r_2,s_2]$ in Theorem 3, then we obtain H-H-type inequality for convex stochastic processes on the coordinates (Set, Sarikaya, and Tomar 2015).

Now we prove H-H-Fejér-type inequalities for coordinated η-convex stochastic processes.

Theorem 4. *Let $X : \Delta \times \Omega \to \mathbb{R}$ be a coordinated η-convex stochasitc process on Δ such that X is mean square integrable. If a stochastic process $Y : \Delta \times \Omega \to \mathbb{R}$ is non negative, mean–square integrable and symmetric with respect to $\frac{r_1+s_1}{2}$ and $\frac{r_2+s_2}{2}$ on coordinates. Then, we have almost everywhere:*

$$\begin{aligned}
&X\Big(\Big(\frac{r_1+s_1}{2},\frac{r_2+s_2}{2}\Big).\Big)\int_{r_1}^{s_1}\int_{r_2}^{s_2} Y((x,y).)dydx\\
&-\frac{1}{4}\int_{r_1}^{s_1}\int_{r_2}^{s_2}\big[\eta(X((x,r_2+s_2-y).),X((x,y).))\\
&+\eta(X((r_1+s_1-x,y).),X((x,y).))\\
&+\eta(X((r_1+s_1-x,r_2+s_2-y).),X((x,y).))\big]Y((x,y).)dydx\\
&\le\int_{r_1}^{s_1}\int_{r_2}^{s_2} X((x,y).)Y((x,y).)dydx\\
&\le\big[X((r_1,r_2).)+\frac{1}{4}\{\eta(X((r_1,s_2).),X((r_1,r_2).))+\eta(X((s_1,r_2).),X((r_1,r_2).))\\
&+\eta(X((s_1,s_2).),X((r_1,r_2).))\}\big]\int_{r_1}^{s_1}\int_{r_2}^{s_2} Y((x,y).)dydx.
\end{aligned}$$

Proof. Since X is a coordinated η-convex stochastic process on Δ, then for all $(\varrho,\kappa)\in[0,1]\times[0,1]$, we can write

$$\begin{aligned}
&X\Big(\Big(\frac{r_1+s_1}{2},\frac{r_2+s_2}{2}\Big).\Big)\\
&=X\Big(\Big(\frac{\varrho r_1+(1-\varrho)s_1+(1-\varrho)r_1+\varrho s_1}{2},\frac{\kappa r_2+(1-\kappa)s_2+(1-\kappa)r_2+\kappa s_2}{2}\Big).\Big)\\
&\le X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)\\
&+\frac{1}{4}\big[\eta(X((\varrho r_1+(1-\varrho)s_1,(1-\kappa)r_2+\kappa s_2).),X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).))\\
&+\eta(X((1-\varrho)r_1+\varrho s_1,\kappa r_2+(1-\kappa)s_2).),X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).))\\
&+\eta(X((1-\varrho)r_1+\varrho s_1,(1-\kappa)r_2+\kappa s_2).),((X(\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).))\big].
\end{aligned}\tag{17}$$

Multiplying both sides of (17) by $Y((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)$ and integrating the resultant with respect to (ϱ,κ) on $[0,1]\times[0,1]$, we get

$$\begin{aligned}
&X\Big(\Big(\frac{r_1+s_1}{2},\frac{r_2+s_2}{2}\Big).\Big)\int_0^1\int_0^1 Y((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)d\varrho d\kappa\\
&\le\int_0^1\int_0^1\big[X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)\\
&+\frac{1}{4}\{\eta(X((\varrho r_1+(1-\varrho)s_1,(1-\kappa)r_2+\kappa s_2).),X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).))\\
&+\eta(X((1-\varrho)r_1+\varrho s_1,\kappa r_2+(1-\kappa)s_2).),X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).))
\end{aligned}$$

$$+\eta(X((1-\varrho)r_1+\varrho s_1,(1-\kappa)r_2+\kappa s_2).),X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).))\}]$$
$$\times Y((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)d\varrho d\kappa. \tag{18}$$

Substituting $x=\varrho r_1+(1-\varrho)s_1, y=\kappa r_2+(1-\kappa)s_2$ in (3), then

$$\begin{aligned}
&X\left(\left(\frac{r_1+s_1}{2},\frac{r_2+s_2}{2}\right).\right)\int_{r_1}^{s_1}\int_{r_2}^{s_2}Y((x,y).)dydx\\
&\le\int_{r_1}^{s_1}\int_{r_2}^{s_2}X(x,y).)Y((x,y).)dydx\\
&\quad+\frac{1}{4}\left[\int_{r_1}^{s_1}\int_{r_2}^{s_2}\eta(X((x,r_2+s_2-y).),X((x,y).))Y((x,y).)dydx\right.\\
&\quad+\int_{r_1}^{s_1}\int_{r_2}^{s_2}\eta(X((r_1+s_1-x,y).),X((x,y).))Y((x,y).)dydx\\
&\quad\left.+\int_{r_1}^{s_1}\int_{r_2}^{s_2}\eta(X((r_1+s_1-x,r_2+s_2-y).),X((x,y).))Y((x,y).)dydx\right].
\end{aligned}\tag{19}$$

Since X is η-convex stochastic process on coordinates on $[r_1,s_1]\times[r_2,s_2]$, then for all (ϱ,κ), we have

$$\begin{aligned}
&X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).))\\
&\quad+X((\varrho r_1+(1-\varrho)s_1,(1-\kappa)r_2+\kappa s_2).))\\
&\quad+X((1-\varrho)r_1+\varrho s_1,(1-\kappa)r_2+\kappa s_2).))\\
&\quad+X((1-\varrho)r_1+\varrho s_1,\kappa r_2+(1-\kappa)s_2).))\\
&\le 4X((r_1,r_2).)+\eta(X((r_1,s_2).),X((r_1,r_2).))+\eta(X((s_1,r_2).),X((r_1,r_2).))\\
&\quad+\eta(X((s_1,s_2).),X((r_1,r_2).)).
\end{aligned}\tag{20}$$

Multiplying inequality (20) by $Y((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)$ and integrating the resultant with respect to (ϱ,κ) on $[0,1]\times[0,1]$, we get

$$\begin{aligned}
&\int_0^1\int_0^1\left[X((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)\right.\\
&\quad+X((\varrho r_1+(1-\varrho)s_1,(1-\kappa)r_2+\kappa s_2).)\\
&\quad+X((1-\varrho)r_1+\varrho s_1,(1-\kappa)r_2+\kappa s_2).)\\
&\quad\left.+X((1-\varrho)r_1+\varrho s_1,\kappa r_2+(1-\kappa)s_2).)\right]\\
&\quad\times Y((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)d\varrho d\kappa\\
&\le\left[4X((r_1,r_2).)+\eta(X((r_1,s_2).),X((r_1,r_2).))\right.\\
&\quad\left.+\eta(X((s_1,r_2).),X((r_1,r_2).))+\eta(X((s_1,s_2).),X((r_1,r_2).))\right]\\
&\quad\times\int_0^1\int_0^1 Y((\varrho r_1+(1-\varrho)s_1,\kappa r_2+(1-\kappa)s_2).)d\varrho d\kappa.
\end{aligned}$$

Using the change of variable and the fact that Y is symmetric with respect to $\frac{r_1+s_1}{2}$ and $\frac{r_2+s_2}{2}$, we have

$$\int_{r_1}^{s_1}\int_{r_2}^{s_2} X((x,y).))Y((x,y).)dydx$$
$$\leq \big[X((r_1,r_2).) + \frac{1}{4}\{\eta(X((r_1,s_2).),X((r_1,r_2).)) + \eta(X((s_1,r_2).),X((r_1,r_2).))$$
$$+ \eta(X((s_1,s_2).),X((r_1,r_2).))\}\big]\int_{r_1}^{s_1}\int_{r_2}^{s_2} Y((x,y).)dydx. \tag{21}$$

From (19) and (21), we get the desired result.

□

Corollary 1. *If we take $\eta(X((x,y).),X((x',y').)) = X((x,y).) - X((x',y').)$ for all $(x,y),(x',y') \in [r_1,s_1]\times[r_2,s_2]$ in above theorem, then we obtain H-H-Fejér-type inequality for coordinated convex stochastic process as follows:*

$$X\left(\left(\frac{r_1+s_1}{2},\frac{r_2+s_2}{2}\right).\right)\int_{r_1}^{s_1}\int_{r_2}^{s_2} Y((x,y).)dydx$$
$$\leq \int_{r_1}^{s_1}\int_{r_2}^{s_2} X((x,y).)Y((x,y).)dydx$$
$$\leq \frac{1}{4}\big[X((r_1,r_2).) + X((r_1,s_2).) + X((s_1,r_2).) + X((s_1,s_2).)\big]$$
$$\times \int_{r_1}^{s_1}\int_{r_2}^{s_2} Y((x,y).)dydx \ \ (a.e.).$$

4. Conclusion

In this article, we have defined (η_1,η_2)-convex stochastic process on the coordinates which is the generalization of convex stochastic process on coordinates. We have proved H-H inequality for this convex stochastic process. We have also showed that every η-convex stochastic process on Δ is η-convex on the coordinates. Further, we have established H-H-Fejér inequality for coordinated η-convex stochastic process. In the similar way, readers can generalize the concept of other type of stochastic convexity on coordinates.

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